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Momentum, Heat, and Mass Transfer

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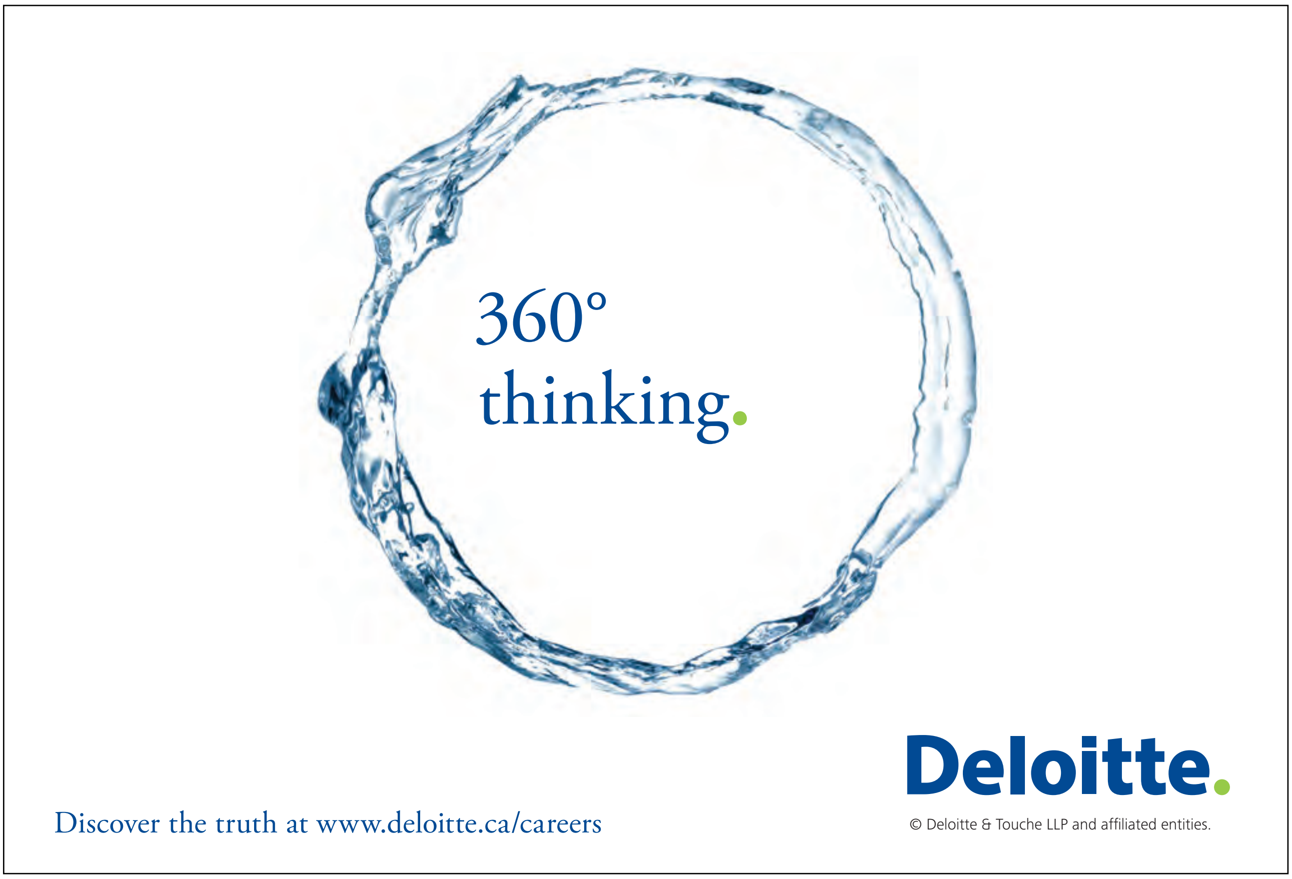
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1 Introduction

In this book, we will examine the transport of momentum, heat, and mass through a system. Thermodynamics deals with systems at equilibrium and transitions between equilibrium states. It can tell us things like the properties of various states or the criteria for equilibrium. For example using thermodynamics, we can prove that, for all positions $\mathbf{r}$ and $\mathbf{r}'$ in a system, equilibrium is given by (N.B.: The following equations neglect the presence of gravity)

$$
\begin{aligned}
p(\mathbf{r}) &= p(\mathbf{r}') \\
T(\mathbf{r}) &= T(\mathbf{r}') \\
\mu_\alpha(\mathbf{r}) &= \mu_\alpha(\mathbf{r}')
\end{aligned}
$$

where $p(\mathbf{r})$ is the pressure, $T(\mathbf{r})$ is the temperature, and $\mu_\alpha(\mathbf{r})$ is the chemical potential of species α at position $\mathbf{r}$. Mathematically, this tells us that the pressure, temperature, and chemical potentials are uniform in a system at equilibrium. If there is a gradient in any of these quantities, then the system is out of equilibrium. As a consequence, momentum, energy, and mass will flow through the system to try to bring it to equilibrium.

Despite its usefulness, thermodynamics tells us nothing about the rate of changes between states or the rate of approach to equilibrium. It does not even apply to steady state processes, where the properties of the system are independent of time, but it is still not in equilibrium. Most processes that are of practical interest are not in equilibrium and never truly achieve equilibrium. In order to describe these systems, we need to study fluid mechanics, heat transfer, and mass transport, which are also known collectively as non-equilibrium thermodynamics or transport phenomena.

1.1 Balance equations

In order to quantitatively handle transport phenomena, we must first develop a mathematical description for the motion of momentum, energy, and mass through a system. The first equation we will develop is the balance equation. We will begin by examining the balance equation for a general quantity B. This quantity can be anything at all. For example, B can be energy, the momentum in the x-direction, apples, people, oxygen molecules, etc. The balance equation for B can be written as

$$\text{accumulation of } B = (\text{influx of } B) - (\text{outflux of } B) + (\text{generation of } B) \tag{1.1}$$

Equation (1.1) is applicable to any system; however, it is not always convenient to use, especially in cases where we are interested in the variation of a property across a system. In order to study these situations, we need to develop differential equations.

1.1.1 Flux

Before we can develop a differential equation to describe the balance of *B*, we need to introduce the concept of *flux*. The flux of a general property *B*, denoted by $\mathbf{J}_B$, describes the flow of *B* through a system. The flux is a vector quantity which can vary with position and time. The units of $\mathbf{J}_B$ are equal to the units of *B* per unit time per unit area.

To demonstrate the physical meaning of $\mathbf{J}_B$, let's place a small plane at a position **r** in the system. The area of the plane is small enough such that the flux of *B* can be considered constant across the surface of the plane. The orientation of the plane is given by the unit vector $\hat{\mathbf{n}}$, which is normal to the surface of the plane. The rate of *B* that crosses the plane is given by

$$\text{rate of } B \text{ crossing plane} = \mathbf{J}_B \cdot \hat{\mathbf{n}}\,\hat{}\ \ (\text{area of plane}) \tag{1.2}$$

If the plane is perpendicular to the x-direction, the rate of *B* that crosses the plane, per unit area, is equal to $J_{B,x}$, the x-component of the flux vector.

In general, the flux of property *B* is due to two factors: convection and diffusion.

$$\mathbf{J}_B = \mathbf{J}_B^{\text{conv}} + \mathbf{J}_B^{\text{diff}} \tag{1.3}$$

The convective flux is due to the macroscopic flow of the fluid which carry with it the property B into and out of the control volume. This is of the form:

$$\mathbf{J}_B^{\text{conv}} = c_B \mathbf{v} \tag{1.4}$$

where c_B is the concentration of *B*, and **v** is the velocity of the fluid.

The diffusive flux is due to molecular motion. This flux leads to irreversibility and generates entropy. It is assumed that the fluxes are proportional to the thermodynamic driving forces.

1.1.2 Differential approach: One-dimensional balances

We first consider a system where the property B only varies in one direction, which we choose to be the x-direction. For this one-dimensional problem, we can perform a balance around a differential element (which is referred to as a *control volume*) located between the positions x and $x + \Delta x$, with a cross-subsectional area *A*. The width Δx of the control volume is chosen to be small enough such that the concentration of *B* (i.e., the amount of B per unit volume) in the element $c_B(x,t)$ can be considered uniform.

The total amount of *B* currently within the control volume is $c_B(x,t)A\Delta x.$. The rate of production of *B* per unit volume is $\sigma_B(x,t)$; therefore, the net rate of B that is produced in the system is $\sigma_B(x,t)A\Delta x$. *B* enters the control volume from the left side at a rate of $J_{B,x}(x,t)A$, and exits the control volume from the right side at a rate of $J_{B,x}(x+\Delta x,t)A$. Putting this all together in Eq. (1.1), we find

$$\frac{\partial}{\partial t}[c_B(x,t)A\Delta x] = AJ_{B,x}(x,t) - AJ_{B,x}(x+\Delta x,t) + \sigma_B(x,t)A\Delta x$$

If we divide both sides of the equation by $A\Delta x$, we find

$$\frac{\partial c_B(x,t)}{\partial t} = -\frac{J_{B,x}(x+\Delta x,t) - J_{B,x}(x,t)}{\Delta x} + \sigma_B(x,t)$$

Taking the limit as Δx goes to zero, the first term on the right side of the equation becomes a derivative, and so

$$\frac{\partial c_B(x,t)}{\partial t} = -\frac{\partial J_{B,x}(x,t)}{\partial x} + \sigma_B(x,t) \tag{1.5}$$

The balance equation for any one-dimensional problem can always be written in the form given in Eq. (1.5).

1.1.3 Differential approach: Three-dimensional balances

In this subsection, we consider a system where the property *B* varies in all directions, x, y, and z. We choose as our control volume a small rectangular prism of dimensions Δx, $\Delta\, y$, and Δz. These dimensions are chosen to be small enough so that the concentration of *B* within the control volume c_B is approximately uniform.

For a three-dimensional differential volume element, we have:

$$\begin{aligned}\frac{\partial}{\partial t}(c_B\Delta x\Delta y\Delta z) = &-\Delta y\Delta z J_{B,x}(x+\Delta x,y,z,t) + \Delta y\Delta z J_{B,x}(x,y,z,t)\\ &-\Delta x\Delta z J_{B,y}(x,y+\Delta y,z,t) + \Delta x\Delta z J_{B,y}(x,y,z,t)\\ &-\Delta x\Delta y J_{B,z}(x,y,z+\Delta z,t) + \Delta x\Delta y J_{B,z}(x,y,z,t)\\ &+(\sigma_B\Delta x\Delta y\Delta z)\end{aligned}$$

$$\begin{aligned}\frac{\partial c_B}{\partial t} = &-\left[\frac{J_{B,x}(x+\Delta x,y,z,t) - J_{B,x}(x,y,z,t)}{\Delta x}\right]\\ &-\left[\frac{J_{B,y}(x,y+\Delta y,z,t) - J_{B,x}(x,y,z,t)}{\Delta y}\right]\\ &-\left[\frac{J_{B,z}(x,y,z+\Delta z,t) - J_{B,x}(x,y,z,t)}{\Delta z}\right]\\ &+\sigma_B\end{aligned} \tag{1.6}$$

Taking the limit Δx, $\Delta\, y$, $\Delta z \to 0$, we find

$$\frac{\partial c_B}{\partial t} = -\frac{\partial J_{B,x}(x,y,z,t)}{\partial x} - \frac{\partial J_{B,y}(x,y,z,t)}{\partial y} - \frac{\partial J_{B,z}(x,y,z,t)}{\partial z} + \sigma_B \tag{1.7}$$

This can be written in vector form as

$$\frac{\partial c_B}{\partial t} = -\nabla \cdot \mathbf{J}_B + \sigma_B \tag{1.8}$$

where ∇ is the gradient operator, defined as

$$\nabla \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{1.9}$$

where $\hat{\mathbf{e}}_x$ is a unit vector pointing in the positive x-direction, $\hat{\mathbf{e}}_y$ is a unit vector pointing in the positive y-direction, and $\hat{\mathbf{e}}_z$ is a unit vector pointing in the positive z-direction.

1.2 Index notation

Before proceeding on to analyze the balance equations for mass, momentum, and energy, we will introduce the use of *index notation*, which will help in keeping the equations more compact and clear, as well as facilitating their manipulation. In index notation, vectors and matrices are explicitly referred to by their various components. The expression v_i denotes the ith component of the vector **v** where i can be either x, y, or z. The expression A_{ij} denotes the ijth element of the matrix A.

If an index is repeated in a term, then this implies that the index is summed over all components.

$$u_i v_i \rightarrow u_x v_x + u_y v_y + u_z v_z$$

This is referred to as the *summation convention*. For example, the dot product between two vectors **u** and **v** is given by

$$\mathbf{u} \cdot \mathbf{v} \rightarrow u_i v_i$$

The multiplication of a matrix τ by a vector can be written as

$$\tau \cdot \mathbf{v} \rightarrow \tau_{ij} v_j$$

The multiplication of two matricies is given by

$$A \cdot B \rightarrow A_{ij} B_{jk}$$

The general balance equation, Eq. (1.7) or (1.8), can then be written as

$$\frac{\partial c_B}{\partial t} = -\frac{\partial J_{B,i}}{\partial x_i} + \sigma_B \tag{1.10}$$

where x_i is the ith component of the position vector (i.e., $x_x = x$, $x_y = y$, and $x_z = z$), and $J_{B,i}$ is the ith component of the flux vector $\mathbf{J}_B$. Note the use of the summation convention in the first term on the right side of the equation.

1.3 Mass balance

The first balance equation that we will consider is for the overall mass. The concentration of mass is simply the density ρ. Mass cannot be created or destroyed, so there is no generation term in the balance equation. Finally, the only flux present for the overall mass is the convective flux. Combining all these facts together with Eq. (1.8), we arrive at

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}$$

This is typically referred to as the *continuity equation* and is usually written in the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{1.11}$$

Using index notation, we can rewrite Eq. (1.11) as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{1.12}$$

For an incompressible fluid, the density remains constant. Mathematically, this means that all of the derivatives of the density are equal to zero. With this assumption, we find that the continuity equation reduces to

$$\nabla \cdot \mathbf{v} = 0 \tag{1.13}$$

Note that this is only true for an incompressible fluid.

2 Momentum transport

2.1 Introduction

In this section, we will consider the momentum balance equation. Unlike the previous quantities that we have considered, which were all scalars, momentum is a *vector*. This means that we need to consider the balance equation for each of its three components.

The "concentration" of momentum is the amount of momentum contained in the system per unit volume. If we recall that the momentum of a particle is equal to $m\mathbf{v}$, where m is the mass of the particle and $\mathbf{v}$ is the velocity of the particle, then the "concentration" of momentum is given by $\rho\mathbf{v}$. Note that because momentum is a vector, there are three types of momentum concentration: ρv_x, ρv_y, and ρv_z.

The next term to consider is generation of momentum. So what is responsible for the creation of momentum? The answer to this question is given by Newton's second law:

$$\frac{\partial}{\partial t}(m\mathbf{v}) = \mathbf{F} \tag{2.1}$$

where m is the mass, and $\mathbf{F}$ is the force acting on the mass. Therefore, we see that forces generate momentum.

Let's consider the possible forces that act on a cubic element of a fluid which has dimensions Δx, Δy, and Δz (i.e., the control volume). The first force we consider is the pressure. The pressure always acts inwards, compressing the cube from all six faces. The total force acting on the cube due to pressure is, therefore, the sum of six terms (one corresponding to each face of the cube):

$$\begin{aligned}\text{pressure force} = \ & [p(x,y,z)\Delta y\Delta z - p(x+\Delta x,y,z)\Delta y\Delta z]\,\hat{\mathbf{e}}_x \\ & + [p(x,y,z)\Delta x\Delta z - p(x,y+\Delta y,z)\Delta x\Delta z]\,\hat{\mathbf{e}}_y \\ & + [p(x,y,z)\Delta x\Delta y - p(x,y,z+\Delta z)\Delta x\Delta y]\,\hat{\mathbf{e}}_z\end{aligned}$$

The generation of momentum per unit volume due to the pressure force is just the pressure force acting on the cube, divided by the volume of the cube (which is $\Delta x\,\Delta y,\Delta z$). In the limit where the dimensions of the cube are very small, this becomes $-\nabla p$.

Another source of momentum is the force of gravity. The gravitational force acting on a mass m is given by $m\mathbf{g}$. The generation of momentum per unit volume is then given by ρg.

Combining these two terms, we find that the generation of momentum can be written as:

$$\sigma_{\text{momentum}} = -\nabla p + \rho\mathbf{g} \tag{2.2}$$

Note that because momentum is a vector quantity, the generation of momentum σ_{momentum} is also a vector.

The final term we need to discuss is the momentum flux $\mathbf{J}_{\text{momentum}}$, which represents transport of momentum across the boundaries of the control volume. The momentum flux consists of two contributions: the convective flux $\mathbf{J}^{\text{conv}}_{\text{momentum}}$, and the diffusive flux $\mathbf{J}^{\text{diff}}_{\text{momentum}}$. The convective contribution is given by

$$\mathbf{J}^{\text{conv}}_{\text{momentum}} = (\rho\mathbf{v})\mathbf{v} \tag{2.3}$$

The convective flux represents reversible transport of momentum across the surface of the control volume. Before we discuss the form of the diffusive flux, let's first consider the terms we have so far.

2.2 Bernoulli's equation

If we neglect the contribution of the diffusive flux, then the momentum balance can be written as:

$$\begin{aligned}\frac{\partial c_{\text{momentum}}}{\partial t} &= -\nabla\cdot\mathbf{J}_{\text{momentum}} + \sigma_{\text{momentum}} \\ \frac{\partial \rho\mathbf{v}}{\partial t} &= -\nabla\cdot(\rho\mathbf{v}\mathbf{v}) - \nabla p + \rho\mathbf{g}\end{aligned} \tag{2.4}$$

With the help of the continuity equation, Eq. (1.11), we can rearrange this relation to

$$\begin{aligned}
\rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} &= -\mathbf{v}\nabla\cdot(\rho\mathbf{v}) - \rho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla p + \rho\mathbf{g} \\
\rho\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] - \rho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla p + \rho\mathbf{g} \\
\rho\frac{\partial \mathbf{v}}{\partial t} &= -\rho\mathbf{v}\cdot\nabla\mathbf{v} - \nabla p + \rho\mathbf{g} \\
\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v}\cdot\nabla\mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{g}
\end{aligned}$$

If we assume that the fluid is incompressible (i.e. ρ is a constant) and that the flow is irrotational (i.e. $\nabla \times \mathbf{v} = 0$), then we find

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\left(\frac{1}{2}v^2 + \frac{p}{\rho} - \mathbf{g}\cdot\mathbf{r}\right) \tag{2.5}$$

For a system at steady state conditions, all derivatives with respect to time vanish, and so the right side of the above equation can be integrated to yield

$$\frac{1}{2}v^2 + \frac{p}{\rho} - \mathbf{g}\cdot\mathbf{r} = \text{const} \tag{2.6}$$

This is just Bernoulli's equation. So for an incompressible fluid at steady state, the momentum equation reduces to Bernoulli's equation when the diffusive momentum flux can be neglected.

2.3 Diffusive momentum flux: Newton's law of viscosity

In a fluid, there is a tendency to try to make all parts of the fluid move at the same velocity. Consider the situation shown in Fig. 2.1. In this case, the upper plane of fluid *A* is travelling with a greater velocity than the fluid in plane *B*, which is travelling with a greater velocity than the fluid in plane *C*. Because the fluid in plane *A* is travelling faster than the fluid in plane *B*, it exerts a force on plane *B* in the positive x-direction; on the other hand, the plane *B* exerts a force of equal magnitude on plane *A* in the negative x-direction. In a similar manner, forces are exerted between planes *B* and *C*, and between planes *C* and *D*. These forces, which are often referred to as a viscous forces, are proportional to the area of contact between the adjacent planes and are related to the velocity difference (or gradient) between the planes.

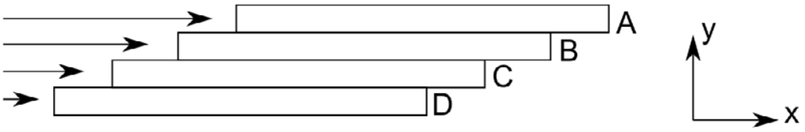


Figure 2.1: Planes of fluid traveling at differing velocities.

Now, let's consider a fluid that is confined between two plates, separated by a distance *H* (see Fig. 2.2(a)). Inititially, the fluid and both plates are motionless. At a certain time, the upper plate is accelerated to a velocity *V* and is kept moving at a constant velocity. The fluid near a solid surface typically travels at the same velocity as the surface. This is known as the *no slip boundary condition*. For the situations depicted in Figs. 2.2(b) and (c), the fluid at the upper surface has a velocity *V*, and the fluid at the lower surface is motionless.

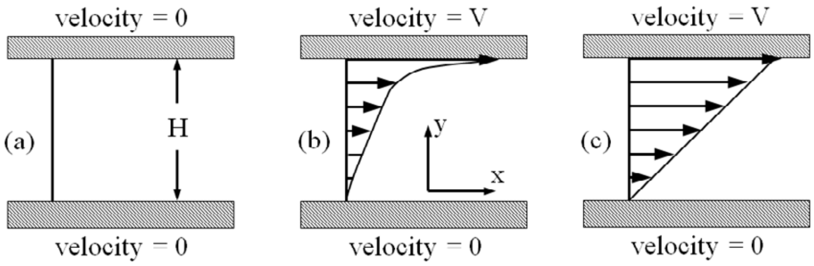


Figure 2.2: Flow of a fluid between a stationary and moving plate.

Immediately after the upper plate is accelerated, only a small subsection of fluid that is very near to the plate actually moves. The rest of the fluid remains motionless. However, as time passes, the viscous forces between the fluid begins to accelerate the lower parts of the fluid (see Fig. 2.2(b)). This can be interpreted as the gradual "diffusion" momentum from the upper part to the lower part of the fluid. At very long times, the system reaches steady state (see Fig. 2.2(c)).

In order to maintain the constant velocity V of the upper plate, a force F needs to be applied. This is due to the fact that the viscous forces in the fluid tend to decelerate the plate. We define the *viscosity*

μ of a fluid by the ratio:

$$\mu = \frac{F/A}{V/H} \tag{2.7}$$

where A is the cross-subsectional area of the plates. Note that the units of viscosity are pressure multiplied by time (e.g., Pa s). The more viscous a fluid, the more force it takes to keep the upper plate moving at the same speed. In addition, for a given applied force, the less viscous a fluid, the faster the upper plate will move. This is all consistent with our intuitive idea of "viscosity".

Now let's consider a two-dimensional element of fluid in a general flow field. The stresses due to viscous forces that act on this particular element of fluid are shown in Fig. 2.3. The upper and lower surfaces are referred to as y-faces, because they are perpendicular to the y-direction; the right and left surfaces are referred to as x-faces, because they are perpendicular to the x-direction. The stress (force per unit area) acting on the upper y-faces in the x-direction is referred to as τ_{yx}. On the upper surface, τ_{yx} points in the positive x-direction, while on the lower surface, it points in the negative x-direction. The stress acting on the x-faces in the y-direction is referred to as τ_{xy}. This points in the positive y-direction on the right face, and in the negative y-direction on the left face.

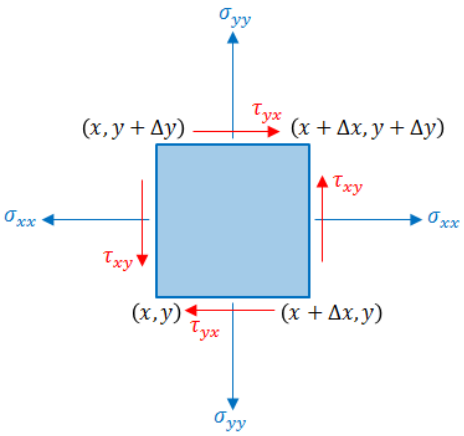


Figure 2.3: Stresses acting on a two-dimensional control volume. Note that $\sigma_{xx} = -p + \tau_{xx}$, etc.

The diffusive momentum flux $\mathbf{J}^{\text{diff}}_{\text{momentum}}$ is due to these stresses, which are caused by viscous forces. For the general three-dimensional situation, the viscous stresses acting on an element of fluid are shown in Fig. 2.4. By summing all the viscous forces that act on an element of fluid, we find that the diffusive momentum flux is given by

$$\mathbf{J}^{\text{diff}}_{\text{momentum}} = -\boldsymbol{\tau} \tag{2.8}$$

where τ is the stress tensor

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{2.9}$$

The stress tensor τ is a 3×3 matrix. The first index of the matrix refers to the face on which the stress acts, while the second index of the matrix indicates the direction in which the stress acts.

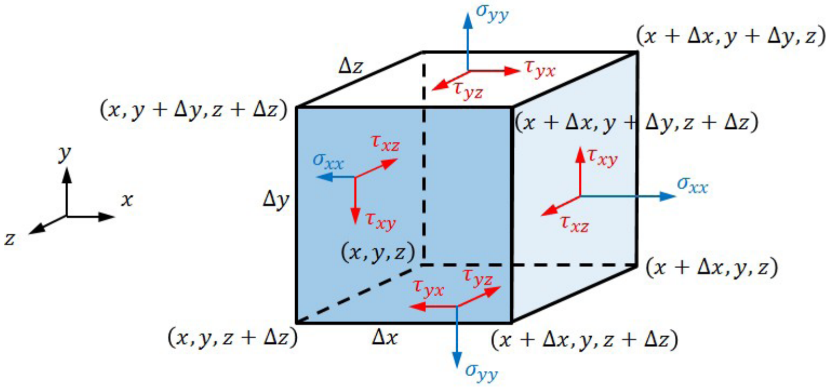


Figure 2.4: Stresses acting on a three-dimensional control volume. Note that $\sigma_{xx} = -p + \tau_{xx}$, etc.

Adding the contribution of diffusive flux to the momentum equation, we find

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \nabla \cdot \boldsymbol{\tau} - \nabla p + \rho \mathbf{g} \tag{2.10}$$

Using index notation, the momentum equation (see Eq. (2.10)) can be rewritten as:

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \tag{2.11}$$

Using the continuity equation, the momentum equation can also be written as

$$\begin{aligned} \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t} &= -\rho v_j \frac{\partial v_i}{\partial x_j} - v_i \frac{\partial \rho v_j}{\partial x_j} + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \\ \rho \frac{\partial v_i}{\partial t} &= -\rho v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \end{aligned} \tag{2.12}$$

2.4 Newton's law of viscosity

Let's consider the flow shown in Fig. 2.2(c) and focus on an element of fluid of height $\Delta\, y$ located at a distance y above the bottom plate. The bottom surface of this element of fluid is travelling at velocity $v_x(y)$, while the upper surface of this element of fluid is travelling at velocity $v_x(y+\Delta y) \approx v_x(y) + \Delta y \partial v_x/\partial y$. The difference of these two velocities is $\Delta y \partial v_x/\partial y$. Comparing this situation with Eq. (2.7) where $F/A \sim \tau_{yx}$, $H \sim \Delta y$, and $V \sim \Delta y \partial v_x/\partial y$, we find

$$\begin{aligned} \mu &= \frac{\tau_{yx}}{\Delta y \partial v_x/\partial y/\Delta y} \\ \tau_{yx} &= \mu \frac{\partial v_x}{\partial y} \end{aligned} \tag{2.13}$$

This relates the viscous stress to the gradients in the velocity. A fluid which obeys this relation is known as a *Newtonian* fluid. Note that this relation was derived in the special case where there is only a velocity gradient in the y-direction. For a general three-dimensional flow, Eq. (2.13) generalizes

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \cdot \mathbf{v} \tag{2.14}$$

where i and j are indices that represent the x, y, or z-directions, δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$), and λ is the longitudinal viscosity. For most fluids, $\lambda = -2\mu/3$.

Substituting the constituitive relations Eq. (2.14) into the momentum balance, Eq. (2.10) or (2.11), we find

$$\begin{aligned} \frac{\partial \rho v_i}{\partial t} &= -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] + \lambda \frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \\ &= -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial v_i}{\partial x_j}\right] + \frac{\partial}{\partial x_i}\left[(\mu + \lambda)\frac{\partial v_j}{\partial x_j}\right] - \frac{\partial p}{\partial x_i} + \rho g_i \end{aligned} \tag{2.15}$$

For an incompressible fluid (where $\nabla \cdot \mathbf{v} = 0$) with a constant viscosity, this reduces to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\mu}{\rho} \nabla^2 \mathbf{v} - \frac{1}{\rho} \nabla p + \mathbf{g} \tag{2.16}$$

This is known as the Navier-Stokes equation and is the starting point for analyzing the flow behavior of incompressible, Newtonian fluids (e.g., water).

3 Laminar flow problems

3.1 Introduction

In the previous sections, we derived the general mass and momentum balance equations. In this section, we will apply this method to examine two specific examples: flow of a falling fluid film and flow of a fluid through a cylindrical pipe.

3.2 Flow of a falling film

Let's consider the problem of a film of water flowing down an inclined plane. A schematic drawing of the system is given in Fig. 3.1. The thickness of the film is δ. The angle of the plane from vertical is θ.

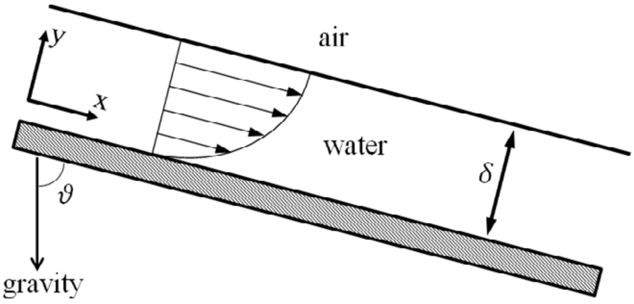


Figure 3.1: Flow of a film of liquid down the surface of an inclined plane.

At the bottom of the film, the water is in direct contact with a solid wall, which is motionless. At this surface, we employ *non-slip* boundary conditions, where the velocity of the fluid is assumed to be the same as the velocity of solid. At the top surface of the film, the water is in contact with air.

Now let's focus our attention on a small cube of fluid, of dimensions dx, dy, and dz, located at position **r**. Because the system is at steady state, there is no accumulation, and the forces that are acting on the system should be balanced.

We assume that the fluid flows only in the x-direction

$$0 = \text{gravitational force} + \text{net viscous force} \tag{3.1}$$

The gravitational force acting on the fluid in the x-direction is given by

$$\text{gravitational force} = (dxdydz)\rho g \cos\theta \tag{3.2}$$

The viscous force is given by

$$\text{net viscous force} = \tau_{yx}(x, y + dy, z)dxdz - \tau_{yx}(x, y, z)dxdz \tag{3.3}$$

Inserting the expressions for the gravitational and viscous forces into the momentum balance equation, we find

$$\begin{aligned} 0 &= (dxdydz)\rho g \cos\theta + \tau_{yx}(x, y + dy, z)dxdz - \tau_{yx}(x, y, z)dxdz \\ 0 &= \rho g \cos\theta + \frac{\tau_{yx}(x, y + dy, z) - \tau_{yx}(x, y, z)}{dy} \end{aligned} \tag{3.4}$$

If we take the limit $dy \to 0$, then

$$0 = \rho g \cos\theta + \frac{\partial \tau_{yx}}{\partial y} \tag{3.5}$$

Another more rapid method to arrive at Eq. (3.5) is to start from the general momentum balance equation, Eq. (2.12). If we consider only the x-component of the equation, we find

$$\rho\frac{\partial v_x}{\partial t} = -\rho v_j \frac{\partial v_x}{\partial x_j} + \frac{\partial \tau_{jx}}{\partial x_j} - \frac{\partial p}{\partial x} + \rho g_x$$

Since the system is at steady state, the left side of the equation is equal to zero. Also, if we note that there is no flow in the y and z-directions (i.e., $v_y = v_z = 0$) and that v_x only varies in the y-direction (so that $\partial v_x / \partial x = 0$), then the first term on the right side of the equation vanishes. Neglecting the variation of pressure in the x-direction (i.e., $\partial p / \partial x = 0$) and noting that $g_x = g\cos\theta$, we arrive at Eq. (3.5).

We can integrate Eq. (3.5) with respect to y, which gives

$$\begin{aligned} 0 &= y\rho g\cos\theta + A + \tau_{yx}(y) \\ \tau_{yx}(y) &= -y\rho g\cos\theta - A \end{aligned} \tag{3.6}$$

where A is an unknown integration constant. We can determine the integration constant by using the boundary condition at the top surface of the fluid, at $y = \delta$ where $\tau_{yx}(\delta) = 0$.

$$\begin{aligned} \tau_{yx}(\delta) &= -\delta\rho g\cos\theta - A = 0 \\ A &= -\delta\rho g\cos\theta \end{aligned} \tag{3.7}$$

Therefore, the stress in the fluid is given by

$$\tau_{yx}(y) = \delta\rho g\cos\theta\left(1 - \frac{y}{\delta}\right) \tag{3.8}$$

The stress induced by the plate on the bottom of the fluid is

$$\tau_{yx}(0) = \delta\rho g\cos\theta \tag{3.9}$$

The stress induced on the plate by the fluid is precisely the same magnitude as $\tau_{yx}(0)$ but in the opposite direction.

To determine the velocity profile, we use the relationship between stress and the velocity gradient for a Newtonian fluid:

$$\tau_{yx} = \mu\frac{\partial v_x}{\partial y} \tag{3.10}$$

Substituting the expression for the stress, we find

$$\begin{aligned} \delta\rho g\cos\theta\left(1 - \frac{y}{\delta}\right) &= \mu\frac{\partial v_x}{\partial y} \\ \frac{\partial v_x}{\partial y} &= \frac{\delta\rho g\cos\theta}{\mu}\left(1 - \frac{y}{\delta}\right) \end{aligned} \tag{3.11}$$

If we integrate this equation with respect to y, we find

$$\begin{aligned} v_x(y) &= \frac{\delta\rho g\cos\theta}{\mu}\left(y - \frac{y^2}{2\delta}\right) + B \\ &= \frac{\delta^2\rho g\cos\theta}{\mu}\left(\frac{y}{\delta}\right)\left(1 - \frac{y}{2\delta}\right) + B \end{aligned} \tag{3.12}$$

where B is an unknown integration constant. The value of this integration constant can be determined by using the boundary condition at the bottom surface, at $y = 0$, where we have

$$v_x(0) = 0 \tag{3.13}$$

This gives us $B = 0$, and, therefore,

$$v_x(y) = \frac{\delta^2 \rho g \cos\theta}{\mu} \left(\frac{y}{\delta}\right) \left(1 - \frac{y}{2\delta}\right) \tag{3.14}$$

The total volumetic flowrate $\dot{Q}$ of water flowing down the plane can be obtained by integrating the velocity profile. If the width of the film is l, then

$$\begin{aligned}
\dot{Q} &= \int_0^\delta l dy v_x(y) \\
&= \int_0^\delta l dy \frac{\delta^2 \rho g \cos\theta}{2\mu} \frac{y}{\delta} \left[2 - \frac{y}{\delta}\right] \\
&= l \frac{\delta^3 \rho g \cos\theta}{2\mu} \int_0^1 d\xi \xi (2 - \xi) \\
&= l \frac{\delta^3 \rho g \cos\theta}{3\mu}
\end{aligned} \tag{3.15}$$

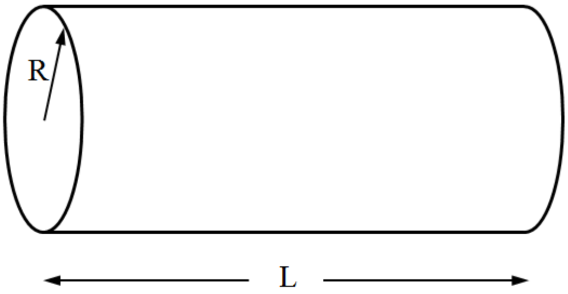


Figure 3.2: Flow through a horizontal pipe of radius *R* and length *L*.

The average velocity $\bar{v}_x$ of the fluid is given by

$$\begin{aligned}\bar{v}_x =& \frac{\dot{Q}}{A} = \frac{\dot{Q}}{l\delta} \\ =& \frac{\delta^2 \rho g \cos\theta}{3\mu}\end{aligned} \tag{3.16}$$

3.3 Flow through a circular tube

Now let's consider steady-state flow through a circular pipe of radius *R*. The length of the pipe is *L*, and the pressure drop across the length of the pipe is Δ*p*. A schematic drawing of the system is given in Fig. 3.2

Let's focus on a cylindrical shell of fluid of radius r and thickness dr. We will work in cylindrical coordinates. We assume that there is no flow in the radial and angular directions (i.e., $v_r = v_\phi = 0$), so we only need to consider the momentum balance in the z-direction. Steady state flow implies that there is no accumulation in the system. Therefore, the various forces in the system must be balanced.

$$0 = \text{net pressure force} + \text{net viscous force} \tag{3.17}$$

The net pressure force acting on the shell is given by

$$\begin{aligned}\text{net pressure force} &= p(0)2\pi r dr - p(L)2\pi r dr \\ &= [p(0) - p(L)]2\pi r dr \\ &= \Delta p\, 2\pi r dr\end{aligned} \tag{3.18}$$

The net viscous forces acting on the cylindrical shell are

$$\text{net viscous forces} = \tau_{rz}(r+dr)2\pi(r+dr)L - \tau_{rz}(r)2\pi r L \tag{3.19}$$

Putting these forces together, we find:

$$0 = \Delta p\, 2\pi r dr + \tau_{rz}(r+dr)2\pi(r+dr)L - \tau_{rz}(r)2\pi r L \tag{3.20}$$

Dividing by the volume of the cylindrical shell $2\pi r dr$

$$0 = \frac{\Delta p}{L} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) \tag{3.21}$$

If we substitute the constitutive relation $\tau_{rz} = \mu \partial v_z / \partial r$, we find

$$0 = \frac{\Delta p}{L} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu \frac{\partial v_z}{\partial r}\right) \tag{3.22}$$

which can be rearranged to give

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{\Delta p}{\mu L} r \tag{3.23}$$

Integrating this equation twice with respect to r, we find

$$v_z(r) = -\frac{\Delta p}{4\mu L} r^2 + A \ln r + B \tag{3.24}$$

where A and B are unknown constants of integration. From the physical requirement that the velocity of the fluid should remain finite at the center of the pipe (i.e., at $r = 0$), we find that $A = 0$. We can determine the value of B by using the *no-slip* boundary condition: at the surface of the pipe (i.e., $r = R$) the velocity of the fluid should be zero

$$\begin{aligned} v_z(0) &= -\frac{\Delta p}{4\mu L}R^2 + B = 0 \\ B &= \frac{\Delta p}{4\mu L}R^2 \end{aligned}$$

Finally, we find

$$v_z(r) = \frac{\Delta p R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{3.25}$$

The volumetric flowrate $\dot{Q}$ is given by

$$\begin{aligned} \dot{Q} &= \int_0^R 2\pi r dr v_z(r) \\ &= \int_0^R 2\pi r dr \frac{\Delta p R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] \\ &= 2\pi \frac{\Delta p R^4}{4\mu L}\int_0^1 x dx (1 - x^2) \\ &= \pi \frac{\Delta p R^4}{8\mu L} \end{aligned} \tag{3.26}$$

The average velocity $\bar{v}_z$ of the fluid is

$$\begin{aligned} \bar{v}_z &= \frac{\dot{Q}}{A} = \frac{\dot{Q}}{\pi R^2} \\ &= \frac{\Delta p R^2}{8\mu L} \end{aligned} \tag{3.27}$$

The Reynolds number is given by

$$\mathrm{Re} = \frac{D\bar{v}_z\rho}{\mu} \tag{3.28}$$

where $D = 2R$ is the diameter of the pipe. The friction factor is defined as

$$f = \frac{\Delta p}{2\rho\bar{v}_z^2}\frac{D}{L} \tag{3.29}$$

Examining Eq. (3.27), we find that the relation between the Reynolds number and the friction factor is given by

$$f = \frac{16}{\mathrm{Re}} \tag{3.30}$$

4 Dimensional analysis

4.1 Introduction

Units play a crucial role in engineering. They allow us to quantitatively communicate the values of the physical properties of a system. Despite their importance to us as engineers, however, units are entirely artificial things. Nature does not care about units. That is, the behavior of a system is entirely independent of the units that we use. This fairly obvious idea is extremely powerful. In fact, it allows us to extract a lot of information about a system, without having to write down or solve any equations.

To illustrate this, let's examine a pendulum consisting of a mass M tied to the end of a string of length L (see Figure 4.1). For this system, the acceleration due to gravity is g.

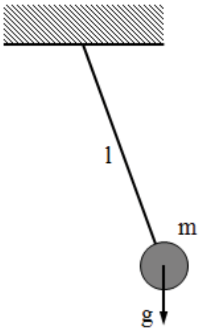


Figure 4.1: Schematic of a swinging pendulum.

How does the period T of the pendulum depend on its physical properties (i.e., L, m, and g)? Mathematically, this question means that we want to determine the function θ such that

$$T = \theta(L, m, g) \tag{4.1}$$

Although Eq. (4.1) makes mathematical sense, it does not really make physical sense. This is because the numerical value of the function θ depends on the particular choice of units that are used to measure time. If we use seconds to measure time instead of hours, then the function increases by a factor of 3600. Also, the precise form of the function will change, depending on the units that are used to report the quantities L, m, and g. Because we know that the behavior of a system should not depend on the units that are used, we should expect that the function relating the period of oscillation of a pendulum to its properties should also be independent of the choice of units.

We can remedy this situation by only using dimensionless quantities. If we choose to measure time in units of t (e.g., seconds, hours, etc.), mass in units of m (e.g., kilograms, pounds, etc.), and length in units of l (e.g., meter, feet, etc.), then we write down the following expression:

$$\frac{T}{t} = \Theta\left(\frac{L}{l}, \frac{M}{m}, \frac{gt^2}{l}\right) \tag{4.2}$$

where Θ the function that we want to determine. Note that all the quantities in this equation, including the function Θ, are dimensionless. Therefore, Eq. (4.2) is completely independent of the units that are used, which is what we want.

We can push the analysis even further, however, based on the fact that we are free to choose whatever units we want. We choose as our unit of length, the length of the pendulum (i.e., $l = L$). As our unit of mass, we choose the mass of the pendulum (i.e., $m = M$). As our unit of time, we choose $t = \sqrt{L/g}$. Given this system of units, we find that Eq. (4.2) reduces to

$$\frac{T}{\sqrt{L/g}} = \Theta\left(1, 1, 1\right) \tag{4.3}$$

Note is that the right side of Eq. (4.3 is simply a constant. We no longer need to determine a function of three arguments – all we need is a single number $\Theta(1, 1, 1)$. This can either be determined by experiment or by a more careful theoretical analysis. Thus, without setting up or solving any differential equations, we were able to determine the mathematical relationship between the period of a pendulum and its properties.

In the previous section, we examined two examples of setting up the differential equations that govern a particular flow and then solving those equations. In many cases, however, we can greatly simplify a problem by simply examining its dimensions.

For the pipe flow problem, if we are only interested in how the pressure drop across a length of pipe varies as a function of the flow rate, then we can use dimensional analysis to greatly simplify the problem. In general, the pressure drop per unit length $\Delta p/L$ depends on all the properties of the system. That is

$$\frac{\Delta p}{L} = \theta(D, \rho, \mu, \bar{v}_z) \tag{4.4}$$

where θ denotes a general function. So at first glance, it seems that the pressure drop is a function of four different variables.

One thing that is peculiar about the function given in Eq. (4.4), however, is that it depends on the units in which the physical quantities are measured. For example, if we measure pressure in units of bars instead of pascals, then the function θ must change by a factor of 10^5. If we measure length in meters instead of feet, then the dependence of the function θ on its various parameters needs to change. We know that the physics of a system should not depend on the units that we use. This fact should be reflected in the properties of the function θ.

If we measure mass in terms of units of m, length in terms of l, and time in terms of t, we find that the function for the pressure drop can be written as

$$\frac{\Delta p}{L}\frac{l^2t^2}{m} = \Theta\left(\frac{D}{l}, \frac{\rho l^3}{m}, \frac{\mu l t}{m}, \frac{\bar{v}_z t}{l}\right) \tag{4.5}$$

We are completely free to choose the values of m, l, and t. For example, in the mks system of units, m = 1 kg, l = 1 m, and t = 1 s. Notice that all the arguments of the function Θ, as well as its value, are dimensionless. This means that the function Θ does not depend on the units that are used.

We can simplify Eq. (4.5) by choosing a set of units based on the properties of the system. We choose to measure mass in units of $m = \rho D^3$, length in units of $l = D$, and time in units of $t = D/\bar{v}_z$. With this choice of units, we find Eq. (4.5) can be written as

$$\begin{aligned}\frac{\Delta p}{L}\frac{D^2(D/\bar{v}_z)^2}{\rho D^3} &= \Theta\left(\frac{D}{D}, \frac{\rho D^3}{\rho D^3}, \frac{\mu D(D/\bar{v}_z)}{\rho D^3}, \frac{\bar{v}_z(D/\bar{v}_z)}{D}\right)\\ \frac{\Delta p}{\rho \bar{v}_z^2}\frac{D}{L} &= \Theta\left(1, 1, \frac{\mu}{D\rho\bar{v}_z}, 1\right)\\ f &= \bar{f}(\mathrm{Re})\end{aligned} \tag{4.6}$$

While in Eq. (4.4), the function θ depends on four variables, the function $\bar{f}$ depends on only one variable. This is a tremendous simplification of the problem!

5 Energy transport

5.1 Energy balance

In this section, we derive the balance equation for energy. The total energy of the system is the sum of three terms: (i) the internal energy of the system, (ii) the kinetic energy of the system, and (iii) the potential energy of the system. The total energy per unit mass e can then be written as:

$$e = U + \frac{1}{2}v^2 + \phi(\mathbf{r}) \tag{5.1}$$

where U is the internal energy per unit mass of the system, and ϕ is the potential energy per unit mass of the system, which is, in general, a function of position. Examples of potential energy are gravitational energy (where $\phi(\mathbf{r}) = -\mathbf{g} \cdot \mathbf{r}$) or electrostatic energy (e.g., a charged system in the presence of an electric potential).

Now let's perform an energy balance on a small subsection of a system. Our control volume will be a small rectangular box of dimensions dx, dy, and dz. The concentration of energy inside the box is equal to ρe. Thus, the rate of accumulation of energy in the control volume is

$$\text{accumulation} = \frac{\partial}{\partial t}(\rho e dx dy dz)$$

The rate of accumulation per unit volume is then

$$\frac{\text{accumulation}}{dxdydz} = \frac{\partial}{\partial t}(\rho e)$$

From the first law of thermodynamics, we know that energy is conserved. Thus, any accumulation of energy in our control volume must be due to flow of energy from other parts of the system. This can occur through several mechanisms. In the following, we will consider convection, conduction, viscous work, and pressure work.

5.2 Convection

When material enters (or leaves) the control volume due to the flow of the fluid, it carries with it energy. This mode of energy transport is known as convection. The convective flux of energy is then given by

$$\mathbf{J}^{\text{conv}} = (\rho e)\mathbf{v} \tag{5.2}$$

For our control volume, there are six different surfaces

$$\begin{aligned}\text{convection} = {} & [J_x^{\text{conv}}(x,y,z) - J_x^{\text{conv}}(x+dx,y,z)]dydz \\ & + [J_y^{\text{conv}}(x,y,z) - J_y^{\text{conv}}(x,y+dy,z)]dxdz \\ & + [J_z^{\text{conv}}(x,y,z) - J_z^{\text{conv}}(x,y,z+dz)]dxdy\end{aligned}$$

Dividing the above equation by the total volume of the system and taking the limit that *dx*, *dy*, and *dz* → 0 yields

$$\begin{aligned}\frac{\text{convection}}{dxdydz} &= -\frac{\partial J_x^{\text{conv}}}{\partial x} - \frac{\partial J_y^{\text{conv}}}{\partial y} - \frac{\partial J_z^{\text{conv}}}{\partial z} \\ &= -\nabla \cdot \mathbf{J}^{\text{conv}} \\ &= -\frac{\partial J_i^{\text{conv}}}{\partial x_i}\end{aligned}$$

5.3 Diffusive transport

In addition to convection, energy can also flow into the control volume through diffusive transport. Diffusive energy (heat) flux **q** is caused by gradients in temperature and chemical potential. We will only consider heat flux which is driven by temperature differences, which is known as conduction. One simple relationship between heat flux and the temperature gradient is given by Fourier's law of conduction

$$\mathbf{q} = -k\nabla T \tag{5.3}$$

where k is the thermal conductivity.

There is yet another type of flux for energy, which has no analogy in the transport of other quantities (such as mass and momentum). This is transport of energy due to radiation. We will not consider this mode of heat transfer in this module.

5.4 Viscous work

Because the material outside the control volume moves at a different velocity than the material inside, viscous stresses arise on the surfaces of the control volume. These viscous stresses lead to the conversion of kinetic energy into internal energy in the system. The rate of conversion on a particular surface is given by the force acting on the surface multiplied by the velocity at that surface. The total rate at which viscous forces perform work on the system is then given by the sum of the work performed on each of the surfaces:

$$\begin{aligned}
\text{viscous work} =&[\tau_{xx}(x+dx,y,z)v_x(x+dx,y,z)-\tau_{xx}(x,y,z)v_x(x,y,z)]dydz\\
&+[\tau_{xy}(x+dx,y,z)v_y(x+dx,y,z)-\tau_{xy}(x,y,z)v_y(x,y,z)]dydz\\
&+[\tau_{xz}(x+dx,y,z)v_z(x+dx,y,z)-\tau_{xz}(x,y,z)v_z(x,y,z)]dydz\\
&+[\tau_{yx}(x,y+dy,z)v_x(x,y+dy,z)-\tau_{yx}(x,y,z)v_x(x,y,z)]dxdz\\
&+[\tau_{yy}(x,y+dy,z)v_y(x,y+dy,z)-\tau_{yy}(x,y,z)v_y(x,y,z)]dxdz\\
&+[\tau_{yz}(x,y+dy,z)v_z(x,y+dy,z)-\tau_{yz}(x,y,z)v_z(x,y,z)]dxdz\\
&+[\tau_{zx}(x,y,z+dz)v_x(x,y,z+dz)-\tau_{zx}(x,y,z)v_x(x,y,z)]dxdy\\
&+[\tau_{zy}(x,y,z+dz)v_y(x,y,z+dz)-\tau_{zy}(x,y,z)v_y(x,y,z)]dxdy\\
&+[\tau_{zz}(x,y,z+dz)v_z(x,y,z+dz)-\tau_{zz}(x,y,z)v_z(x,y,z)]dxdy
\end{aligned}$$

The viscous work per unit volume is then

$$\begin{aligned}\frac{\text{viscous work}}{dxdydz} =& \frac{\partial}{\partial x}(\tau_{xx}v_x) + \frac{\partial}{\partial x}(\tau_{xy}v_y) + \frac{\partial}{\partial x}(\tau_{xz}v_z) \\ &+ \frac{\partial}{\partial y}(\tau_{yx}v_x) + \frac{\partial}{\partial y}(\tau_{yy}v_y) + \frac{\partial}{\partial y}(\tau_{yz}v_z) \\ &+ \frac{\partial}{\partial z}(\tau_{zx}v_x) + \frac{\partial}{\partial z}(\tau_{zy}v_y) + \frac{\partial}{\partial z}(\tau_{zz}v_z) \\ =& \frac{\partial}{\partial x_i}(\tau_{ij}v_j)\end{aligned}$$

5.5 Pressure work

Now let's consider the work due to pressure. The rate at which work is performed by pressure forces on the surface is equal to the magnitude of the force acting on the surface multiplied by the velocity of the system at the surface. Therefore, the total work due to pressure is

$$\begin{aligned}\text{pressure work} =& [-p(x+dx,y,z)v_x(x+dx,y,z) + p(x,y,z)v_x(x,y,z)]dydz \\ &+ [-p(x,y+dy,z)v_y(x,y+dy,z) + p(x,y,z)v_y(x,y,z)]dxdz \\ &+ [-p(x,y,z+dz)v_z(x,y,z+dz) + p(x,y,z)v_z(x,y,z)]dxdy\end{aligned}$$

The rate at which the pressure forces do work on the system per unit volume is then:

$$\frac{\text{pressure work}}{dxdydz} = -\frac{\partial}{\partial x_i}(pv_i)$$

5.6 Finishing touches

Applying the general balance equation to energy, we find

$$\frac{\partial \rho e}{\partial t} = -\frac{\partial}{\partial x_i}(\rho e v_i) - \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}(\tau_{ij}v_j) - \frac{\partial}{\partial x_i}(pv_i) \tag{5.4}$$

Using the continuity equation, this can also be written as

$$\rho\frac{\partial e}{\partial t} = -\rho v_i\frac{\partial e}{\partial x_i} - \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}(\tau_{ij}v_j) - \frac{\partial}{\partial x_i}(pv_i) \tag{5.5}$$

Now let's look at some simple limits of this equation. If the system is entirely motionless and there is no potential energy (e.g., gravitational forces are not important), then $e = U$, $\mathbf{v} = \mathbf{0}$, and the energy balance equation is given by

$$\rho\frac{\partial U}{\partial t} = -\nabla\cdot\mathbf{q} \tag{5.6}$$

If we make the further assumption that the thermal conductivity of the system is constant, and we make use of the thermodynamic relation $dU = C_v dT$ at constant volume (density), where C_v is the isochoric heat capacity, then

$$\rho C_v \frac{\partial T}{\partial t} = k\nabla^2 T \tag{5.7}$$

which is known as the heat or diffusion equation.

We will now rearrange the energy equation (see Eq. (5.5)) in a more convenient form for later work. The first thing we need to do is to develop a balance equation for the kinetic energy. By multiplying Eq. (2.12) by v_i, we find

$$\begin{aligned}
\rho v_i \frac{\partial v_i}{\partial t} &= -\rho v_j v_i \frac{\partial v_i}{\partial x_j} + v_i \frac{\partial \tau_{ji}}{\partial x_j} - v_i \frac{\partial p}{\partial x_i} + \rho g_i v_i \\
\frac{1}{2}\rho \frac{\partial v^2}{\partial t} &= -\frac{1}{2}\rho v_j \frac{\partial v^2}{\partial x_j} + v_i \frac{\partial \tau_{ji}}{\partial x_j} - v_i \frac{\partial p}{\partial x_i} + \rho g_i v_i \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= -\frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) + v_i \frac{\partial \tau_{ji}}{\partial x_j} - v_i \frac{\partial p}{\partial x_i} + \rho g_i v_i
\end{aligned} \tag{5.8}$$

Subtracting Eq. (5.8) from Eq. (5.5), we find

$$\begin{aligned}
\rho \frac{\partial}{\partial t}\left(U + \frac{1}{2}v^2 - g_i x_i\right) &= -\rho v_j \frac{\partial}{\partial x_j}\left(U + \frac{1}{2}v^2 - g_i x_i\right) - \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}(\tau_{ij} v_j) - \frac{\partial}{\partial x_i}(p v_i) \\
\rho \frac{\partial}{\partial t}(U - g_i x_i) &= -\rho v_j \frac{\partial}{\partial x_j}(U - g_i x_i) - \frac{\partial q_i}{\partial x_i} + \tau_{ji}\frac{\partial v_i}{\partial x_j} - p\frac{\partial v_i}{\partial x_i} - \rho g_i v_i \\
\rho \frac{\partial U}{\partial t} &= -\rho v_j \frac{\partial U}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \tau_{ji}\frac{\partial v_i}{\partial x_j} - p\frac{\partial v_i}{\partial x_i}
\end{aligned}$$

If we assume that the material in the control volume is in local thermodynamic equilibrium, we can relate the internal energy to the local temperature and pressure of the system

$$\begin{aligned}
dU &= \left(\frac{\partial U}{\partial T}\right)_p dT + \left(\frac{\partial U}{\partial p}\right)_T dp \\
&= \left[T\left(\frac{\partial S}{\partial T}\right)_p - p\left(\frac{\partial V}{\partial T}\right)_p\right] dT + \left[T\left(\frac{\partial S}{\partial p}\right)_T - p\left(\frac{\partial V}{\partial p}\right)_T\right] dp \\
&= \left[T\left(\frac{\partial S}{\partial T}\right)_p - p\left(\frac{\partial V}{\partial T}\right)_p\right] dT + \left[-T\left(\frac{\partial V}{\partial T}\right)_p - p\left(\frac{\partial V}{\partial p}\right)_T\right] dp \\
&= \left(C_p - \frac{\alpha_p p}{\rho}\right) dT - (\alpha_p T - \kappa_T p)\frac{dp}{\rho}
\end{aligned} \tag{5.9}$$

where α_p is the thermal expansivity of the material, and κ_T is the isothermal compressibility of the material.

For a liquid or a solid, the pressure dependence of the internal energy is quite weak (i.e. second term in Eq. (5.9) is small). In this case, we find that the energy balance can be written as:

$$\rho C_v \frac{\partial T}{\partial t} = -\rho C_v v_j \frac{\partial T}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \tau_{ji} \frac{\partial v_i}{\partial x_j} - p \frac{\partial v_i}{\partial x_i} \tag{5.10}$$

6 Multicomponent mass transfer

6.1 Introduction

In many chemical engineering unit operations, the key element is the transport of a particular chemical species from one location to another. For example, in the case of a chemical reactor, we need the reactants to mix and diffuse together in order for them to react to form the product we desire. In a liquid-liquid extractor, we want a chemical species to transfer from one liquid phase to another. In a gas absorption unit, we want a particular species in the gas phase to diffuse into the contacting liquid phase. The key to designing these processes is understanding mass transfer.

In mass transfer problems, we generally deal with multicomponent systems. Based on the general balance equations that we developed in the previous Sec. 1, deriving the balance equation for a species in a multicomponent system is fairly straightforward. For a particular molecular species α, we have (see Eq. (1.8))

$$\frac{\partial c_\alpha}{\partial t} = -\nabla \cdot \mathbf{N}_\alpha + \sigma_\alpha \tag{6.1}$$

where c_α is the molar concentration of α, $\mathbf{N}\alpha$ is the absolute molar flux of species α, and σ_α is the rate of generation of α per unit volume (e.g., by chemical reaction).

In a multicomponent system, different species generally travel at different velocities. Therefore, for each component α that is present in the system, there is a velocity $\mathbf{v}_\alpha$. The absolute molar flux of α is given by

$$\mathbf{N}_\alpha = c_\alpha \mathbf{v}_\alpha \tag{6.2}$$

Now let's consider what happens when a drop of ink is placed in a moving stream of water. The molecules of ink move as a result of two processes. First, the drop of ink moves along with the general flow of the stream. This contribution to the motion of the ink is referred to as convective flux. In addition to moving along with the flow of the stream, the ink also spreads outwards. This contribution to the motion is referred to as diffusive flux.

Before we can define a diffusive flux, we must first define a reference velocity with respect to which diffusion occurs. This choice of this reference velocity $\mathbf{v}_\alpha$. is quite arbitrary, but, in general, takes the form

$$\mathbf{v}^a = \sum_\alpha a_\alpha \mathbf{v}_\alpha \tag{6.3}$$

where a_α are arbitrary weighting functions that must satisfy the normalization condition

$$\sum_\alpha a_\alpha = 1 \tag{6.4}$$

If we let $a_\alpha = w_\alpha$, the mass fraction of component *α* in the system, then $\mathbf{v}^a = \mathbf{v}$, the center-of-mass velocity of the system:

$$\mathbf{v} \equiv \sum_\alpha w_\alpha \mathbf{v}_\alpha \tag{6.5}$$

This is the velocity that appeared in the continuity equation, momentum balance equation, and the energy balance equation that were developed in the previous section.

If we let $a_\alpha = x_\alpha$, the mole fraction of component *α* in the system, then $\mathbf{v}^a = \mathbf{v}^*$, the molar-average velocity:

$$\mathbf{v}^* = \sum_\alpha x_\alpha \mathbf{v}_\alpha \tag{6.6}$$

In what follows, we will use $\mathbf{v}^*$ as our reference velocity.

With the definition of the reference velocity, the convective flux is given by $c_\alpha \mathbf{v}^*$, and the diffusive flux $\mathbf{J}_\alpha$ is defined as

$$\mathbf{J}_\alpha = c_\alpha(\mathbf{v}_\alpha - \mathbf{v}^*) \tag{6.7}$$

To finish off this subsection, we demonstrate how the absolute flux can be written in terms of the diffusive flux and the convective flux.

$$\begin{aligned}\mathbf{J}_\alpha &= c_\alpha(\mathbf{v}_\alpha - \mathbf{v}^*)\\ &= \mathbf{N}_\alpha - c_\alpha \sum_{\alpha'} x_{\alpha'} \mathbf{v}_{\alpha'}\\ &= \mathbf{N}_\alpha - x_\alpha \sum_{\alpha'} \mathbf{N}_{\alpha'}\end{aligned}$$

and so we find

$$\mathbf{N}_\alpha = \mathbf{J}_\alpha + x_\alpha \sum_{\alpha'} \mathbf{N}_\alpha \tag{6.8}$$

The first term on the right side of the equation represents the diffusive flux, while the second term represents the convective flux.

6.2 Diffusive flux

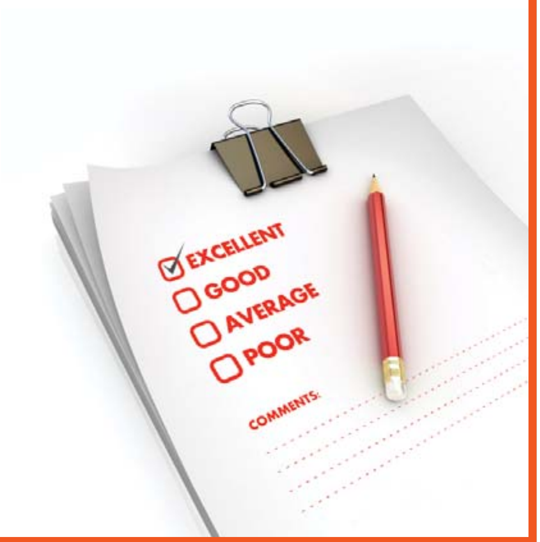
In order to complete the description of the transport of species α, we need an expression for the diffusive flux $\mathbf{J}_\alpha$. We need a way to relate the diffusive flux to the various driving forces in the system. In this subsection, we will develop this expression for binary mixtures, composed of species A and B.

The main driving forces for diffusion are gradients in the chemical potentials of the various species in the system. Let us consider a binary mixture consisting of molecules of type A and type B. A given species A will tend to move from areas where its chemical potential μA is high to areas where its chemical potential is low. The effective force felt by a molecule A due to nonuniformities in its chemical potentials is $-\nabla\mu_A$.

The A molecules, however, are impeded from freely moving (to even out their chemical potential) by the presence of the B molecules. The B molecules get in the way. The force exerted on the A molecules by the B molecules is given by

$$\frac{RT}{\mathcal{D}_{AB}} x_B (\mathbf{v}_A - \mathbf{v}_B)$$

where R is the gas constant, T is the absolute temperature, and $\mathcal{D}_{AB}$ is the diffusion coefficient. This “drag” force of the B molecules on the A molecules is proportional to the amount of B molecules (therefore the x_B factor) and to the relative velocities between A and B molecules.



The diffusion coefficient $\mathcal{D}_{AB}$ is an empirical coefficient that describes diffusion, like the thermal conductivity describes heat conduction and the viscosity describes momentum diffusion. The diffusion coefficient has the units of length squared over time (e.g., m^2/s). The larger the diffusion coefficient, the weaker the drag of *B* on *A*, and as a result, the *A* molecules can diffuse more quickly. The smaller the diffusion coefficient, the slower the *A* molecules diffuse.

If we balance the driving force for motion of species *A* against the drag exerted by species *B*, which opposes the motion, we find

$$-\nabla\mu_A = \frac{RT}{\mathcal{D}_{AB}} x_B(\mathbf{v}_A - \mathbf{v}_B) \tag{6.9}$$

This equation can be manipulated to yield

$$\begin{aligned} -\nabla\mu_A &= \frac{RT}{\mathcal{D}_{AB}}[\mathbf{v}_A - x_A\mathbf{v}_A - x_B\mathbf{v}_B] \\ &= \frac{RT}{\mathcal{D}_{AB}}[\mathbf{v}_A - \mathbf{v}] \\ &= \frac{RT}{c_A\mathcal{D}_{AB}} c_A[\mathbf{v}_A - \mathbf{v}] \\ \mathbf{J}_A &= -\frac{c_A\mathcal{D}_{AB}}{RT}\nabla\mu_A \end{aligned} \tag{6.10}$$

where c_A is the molar concentration of *A*. Equation (6.10) is known as the Maxwell-Stefan equation. It shows that the diffusive flux is proportional to chemical potential gradients.

From thermodynamics, we known that the chemical potential of a species *α* can be written as

$$\mu_\alpha = \mu_\alpha^\circ(T,p) + RT\ln x_\alpha\gamma_\alpha \tag{6.11}$$

If we substitute this into the Maxwell-Stefan equation, we find

$$\mathbf{J}_A = -\frac{c_A\mathcal{D}_{AB}}{RT}\nabla[\mu_A^\circ(T,p) + RT\ln x_A\gamma_A] \tag{6.12}$$

If we assume that the temperature and pressure of the system are uniform (i.e., constant with respect to position), the Maxwell-Stefan equation reduces to

$$\mathbf{J}_A = -c\mathcal{D}_{AB}\left[1 + x_A\frac{\partial\ln\gamma_A}{\partial x_A}\right]\nabla x_A \tag{6.13}$$

where *c* is the overall molar concentration of molecules. If we further assume that the system behaves as an ideal mixture (i.e., $\gamma_A = 1$), we have

$$\mathbf{J}_A = -c\mathcal{D}_{AB}\nabla x_A \tag{6.14}$$

This is known as Fick's law.

7 Examples of steady diffusion

7.1 Introduction

In this section, we will go over some simple examples of how to apply Fick's law to various mass transport problems.

7.2 Arnold diffusion cell

In this example, we consider a cylinder containing pure liquid *A* (see Fig. 7.1). Immediately above the liquid surface, the mole fraction of *A* is y_{A0}. The distance between the surface of the liquid and the top of the cylinder is *H*, and this region is filled with a stagnant gas *B*. Outside the cylinder, there is a fast moving stream of gas, which maintains the mole fraction of *A* at y_{AH}. We assume that the system is at steady state.

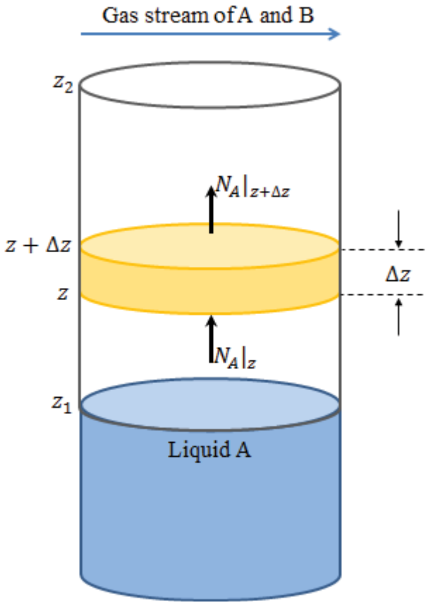


Figure 7.1: Arnold diffusion cell.

We begin this problem, as with all diffusion problems, by performing a balance on species *A* on a thin slab of gas located within the cylinder. The location of the bottom surface of the slab is z, the thickness of the slab is dz, and its cross-subsectional area is *S*. Because we are dealing with a steady-state problem, there should be no accumulation of *A*. In addition, because there are no chemical reactions occuring in the system, there should be no generation/consumption of *A*. From the symmetry of the system, we assume that motion only occurs in the z-direction. Therefore, the influx of *A* through the bottom surface of the slab should be equal to the outflux of *A* through the upper surface

$$0 = (N_{A,z}S)_z - (N_{A,z}S)_{z+dz}$$

Dividing by the volume of the slab Sdz,

$$0 = -\frac{1}{S}\left[\frac{(N_{A,z}S)_{z+dz} - (N_{A,z}S)_z}{dz}\right]$$

Taking the limit $dz \to 0$,

$$\frac{\partial}{\partial z}[N_{A,z}(z)S(z)] = 0$$

where we have multiplied both sides of the equation by S.

This equation can be easily integrated to yield

$$N_{A,z}(z)S = w_A \tag{7.1}$$

where w_A is an unknown integration constant, which represents the total molar flowrate of A up the cylinder.

The absolute flux of A in the system is given by

$$N_{A,z} = J_{A,z} + y_A(N_{A,z} + N_{B,z}) \tag{7.2}$$

The first term on the right side of the equation is the flux of A due to diffusion; the second term is the flux of A due to the overall motion of the system (i.e., convection). In this example, the A molecules will diffuse through a stagnant layer of B. The B molecules do not move. Therefore, we have $\mathbf{N}_B = 0$. The expression for the absolute flux of A then becomes

$$\begin{aligned} N_{A,z} &= J_{A,z} + y_A(N_{A,z} + 0) \\ &= \frac{1}{1 - y_A} J_{A,z} \end{aligned} \tag{7.3}$$

If we assume that the gas mixture of A and B behaves as an ideal gas and that the temperature and pressure are uniform along the height of the cylinder, the diffusive flux follows Fick's law (see Eq. (6.14))

$$J_{A,z} = -c\mathcal{D}_{AB}\frac{\partial y_A}{\partial z} \tag{7.4}$$

and, consequently, the absolute flux of A is given by

$$N_{A,z} = -\frac{c\mathcal{D}_{AB}}{1 - y_A}\frac{\partial y_A}{\partial z} \tag{7.5}$$

Substituting this expression into Eq. (7.1), we find

$$\begin{aligned} -\frac{c\mathcal{D}_{AB}}{1 - y_A}\frac{\partial y_A}{\partial z} S &= w_A \\ \ln(1 - y_A) &= \frac{w_A}{Sc\mathcal{D}_{AB}} z + C \end{aligned} \tag{7.6}$$

Substituting the boundary conditions (i.e., $y_A = y_{A0}$ at $z = 0$, and $y_A = y_{AH}$ at $z = H$) into Eq. (7.6), we have

$$\begin{aligned} \ln(1 - y_{A0}) &= C \\ \ln(1 - y_{AH}) &= \frac{w_A}{Sc\mathcal{D}_{AB}} H + C \end{aligned} \tag{7.7}$$

which can be solved for the integration constants w_A and C

$$\begin{aligned} C &= \ln(1 - y_{A0}) \\ w_A &= \frac{Sc\mathcal{D}_{AB}}{H} \ln\frac{1 - y_{AH}}{1 - y_{A0}} \end{aligned} \tag{7.8}$$

Substituting these expressions for the integration constants into Eq. (7.6), we find

$$\begin{aligned}\ln\frac{1-y_A}{1-y_{A0}} &= \frac{z}{H}\ln\frac{1-y_{AH}}{1-y_{A0}}\\ \frac{1-y_A}{1-y_{A0}} &= \left[\frac{1-y_{AH}}{1-y_{A0}}\right]^{z/H}\end{aligned} \tag{7.9}$$

7.3 Heterogeneous combustion

In this example, we will examine the combustion of a spherical coal particle of radius R. Very far from the coal particle, the mole fraction of oxygen (which we label as species *A*) is given by $y_A = y_{A\infty} = 0.21$. On the surface of the coal particle, the oxygen reacts with carbon to form carbon monoxide

$$2\text{C} + \text{O}_2 \rightarrow 2\text{CO}$$

We assume that this reaction proceeds so quickly that the concentration of oxygen on the surface of the particle is zero (i.e., yA = 0 at r = *R*). In order for the coal particle to burn, oxygen needs to diffuse from far from the particle to the surface of the particle. In this problem, the rate of the reaction is controlled by the transport of oxygen to the coal particle.

For every oxygen molecule that reacts, two molecules of CO are created. These CO molecules diffuse in the opposite direction as the oxygen molecules. Therefore, we have $\mathbf{N}_{\text{CO}} = -2\mathbf{N}_A$. In addition, we assume that the nitrogen in the atmosphere is stationary (i.e., $\mathbf{N}_{\text{N}_2} = 0$).

We start the problem by performing a balance on oxygen molecules *A* within a spherical shell of inner radius r and thickness dr. Because the system is at steady state, there is no accumulation in the shell. In addition, because the only chemical reaction that occurs is on the surface of the coal particle, there is no creation/consumption of oxygen in the shell. We make the assumption that transport of species only occurs in the *r*-direction. So in this case, the influx of *A* at the inner surface, which has an area $S(r) = 4\pi r^2$ (the surface area of a sphere of radius r) must be balanced by the outflux of *A* from the outer surface:

$$0 = (N_{A,r}S)_r - (N_{A,r}S)_{r+dr}$$

If we divide by the volume of the spherical shell $S(r)dr = 4\pi r^2 dr$, we find

$$0 = -\frac{1}{S}\left[\frac{(N_{A,r}S)_{r+dr} - (N_{A,r}S)_r}{dr}\right]$$

If we take the limit $dr \to 0$, we arrive at the following differential equation

$$\frac{\partial}{\partial r}[N_{A,r}(r)S(r)] = 0 \tag{7.10}$$

where we have multiplied both sides of the equation by S. This equation can be integrated to obtain

$$N_{A,r}(r)S(r) = w_A \tag{7.11}$$

where w_A is an integration constant. The physical meaning of w_A is the total moles of oxygen that diffuse toward the coal particle per unit time.

The absolute flux of oxygen can be written as

$$\begin{aligned} N_{A,r} &= J_{A,r} + y_A(N_{A,r} + N_{\mathrm{N_2},r} + N_{\mathrm{CO},r}) \\ &= J_{A,r} + y_A(N_{A,r} - 2N_{A,r}) \\ &= \frac{1}{1+y_A} J_{A,r} \end{aligned} \tag{7.12}$$

According to Fick's law

$$J_{A,r} = -c\mathcal{D}_{AB}\frac{\partial y_A}{\partial r} \tag{7.13}$$

If we substitute Fick's law into Eq. (7.12) and then insert the resulting expression into Eq. (7.11), we find

$$\begin{aligned} -\frac{c\mathcal{D}_{AB}}{1+y_A}\frac{\partial y_A}{\partial r}4\pi r^2 &= w_A \\ -\frac{c\mathcal{D}_{AB}}{1+y_A}\frac{\partial y_A}{\partial r} &= \frac{w_A}{4\pi r^2} \\ -c\mathcal{D}_{AB}\ln(1+y_A) &= -\frac{w_A}{4\pi r}+C \end{aligned} \tag{7.14}$$

where we have used the fact that $S(r) = 4\pi r^2$.

We can determine the unknown integration constants w_A and *C* by substituting the boundary conditions into the expression for y_A. By doing this, we find

$$\begin{aligned} -c\mathcal{D}_{AB}\ln(1+y_{A\infty}) &= C \\ 0 &= -\frac{w_A}{4\pi R}+C, \end{aligned} \tag{7.15}$$

which can be solved to give

$$C = -c\mathcal{D}_{AB}\ln(1+y_{A\infty}) \tag{7.16}$$

$$w_A = -4\pi Rc\mathcal{D}_{AB}\ln(1+y_{A\infty}) \tag{7.17}$$

Note that *wA* represents the rate at which oxygen is diffusing towards the coal particle. This is the same rate at which oxygen is being consumed by the combustion reaction.

By substituting the expressions for the integration constants back into the original expression for y_A, we get

$$\begin{aligned} c\mathcal{D}_{AB}\ln\frac{1+y_A}{1+y_{A\infty}} &= -\frac{4\pi Rc\mathcal{D}_{AB}}{4\pi r}\ln(1+y_{A\infty}) \\ \ln\frac{1+y_A}{1+y_{A\infty}} &= -\frac{R}{r}\ln(1+y_{A\infty}) \end{aligned} \tag{7.18}$$

From the solution of this problem, we can also estimate how long it will take for the coal particle to completely react. Equation (7.17) tells us how quickly oxygen reacts to form CO. This is precisely twice the rate at which carbon atoms (coal) are consumed at the surface of the particle. Therefore, we have $w_C = 2w_A$.

To determine the rate at which the coal particle is shrinking, we perform a mole balance for carbon. For a particle of radius *R*, the amount of carbon present is given by $4\pi R^3 c_C/3$, the volume of the particle times the molar density of carbon. The rate at which the particle shrinks is then given by:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{4\pi R^3}{3}c_C\right) =& w_C \\ \frac{\partial}{\partial t}\left(\frac{4\pi R^3}{3}c_C\right) =& -8\pi R c\mathcal{D}_{AB}\ln(1+y_{A\infty}) \\ \frac{c_C}{2}R\frac{\partial R}{\partial t} =& -c\mathcal{D}_{AB}\ln(1+y_{A\infty}) \\ R^2 =& -\frac{4c\mathcal{D}_{AB}t}{c_C}\ln(1+y_{A\infty})+R_0^2 \end{aligned} \tag{7.19}$$

where R_0 is the initial radius of the particle. The time t required for the particle to disappear is

$$t = \frac{c_C R_0^2}{4c\mathcal{D}_{AB}\ln(1+y_{A\infty})} \tag{7.20}$$

7.4 Diffusion with homogeneous reaction

In this example, we consider the absorption of species *A* from a gas phase into a liquid phase composed mostly of *B* (see Fig. 7.2). The mole fraction of *A* just inside the liquid phase (i.e., at z = 0) is equal to y_{A0}. At a distance δ beneath the surface of the liquid layer, there is an impenetrable solid wall. In addition to the diffusion of *A* in the liquid phase, there is also a first order reaction that converts species *A* to species *C*. The kinetics of this reaction are given by

$$\begin{aligned} \sigma_A &= -k_1 c_A \\ &= -k_1 c y_A \end{aligned} \tag{7.21}$$

where σ_A is the rate of generation of *A* per unit volume, k_1 is a kinetic constant, and *c* is the overall molar concentration of the liquid.

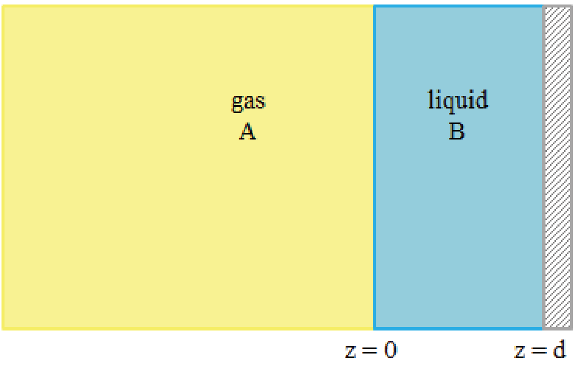


Figure 7.2: Absorption of *A* from a gas phase to a liquid phase.

To begin the problem, we perform a species balance for A on a thin slab in the liquid. Because we are dealing with a steady-state problem, there is no accumulation in the slab. From the geometry of the problem, we assume there is only motion in the z-direction. So there is an influx of A from the left side of the slab and an outflux of A from the right side of the slab. Unlike the previous examples we considered in this handout, there is a chemical reaction that consumes A. Therefore, the balance equation is given by

$$\begin{aligned} 0 &= (N_{A,z}S)_z - (N_{A,z}S)_{z+dz} - (k_1 c y_A) S dz \\ 0 &= -\frac{1}{S}\left[\frac{(N_{A,z}S)_{z+dz} - (N_{A,z}S)_z}{dz}\right] - (k_1 c y_A) \end{aligned} \tag{7.22}$$

Taking the limit $dz \to 0$,

$$\begin{aligned} \frac{1}{S}\frac{\partial}{\partial z}(N_{A,z}S) &= -(k_1 c y_A) \\ \frac{\partial N_{A,z}}{\partial z} &= -k_1 c y_A \end{aligned} \tag{7.23}$$

We assume that *B* is stationary (i.e., $\mathbf{N}_B = 0$). In addition, we assume that species *C* flows in the opposite direction of species *A* (i.e., $\mathbf{N}_C = -\mathbf{N}_A$). The absolute flux is then given by

$$\begin{aligned} N_{A,z} &= J_{A,z} + y_A(N_{A,z} + N_{B,z} + N_{C,z}) \\ &= J_{A,z} + y_A(N_{A,z} + 0 - N_{A,z}) \\ &= J_{A,z} \\ &= -c\mathcal{D}_{AB}\frac{\partial y_A}{\partial z} \end{aligned} \tag{7.24}$$

Putting this expression into Eq. (7.23)

$$\begin{aligned} -\frac{\partial}{\partial z}c\mathcal{D}_{AB}\frac{\partial y_A}{\partial z} &= -k_1 c y_A \\ \frac{\partial^2 y_A}{\partial z^2} &= y_A/l^2 \end{aligned} \tag{7.25}$$

where $l = (\mathcal{D}_{AB}/k_1)^{1/2}$. This is a second-order, linear differential equation with constant coefficients. The general solution to this equation is given by

$$y_A(z) = C_1 e^{z/l} + C_2 e^{-z/l} \tag{7.26}$$

where C_1 and C_2 are constants, which for the moment are unknown. The flux can be determined by substituting Eq. (7.26) into Eq. (7.24):

$$\begin{aligned} N_{A,z} &= -c\mathcal{D}_{AB}\frac{\partial y_A}{\partial z} \\ &= -\frac{c\mathcal{D}_{AB}}{l}\left(C_1 e^{z/l} - C_2 e^{-z/l}\right) \end{aligned} \tag{7.27}$$

We can determine the constants C_1 and C_2 by using the boundary conditions. At z = 0, we have $y_A = y_{A0}$, which leads to

$$y_{A0} = C_1 + C_2 \tag{7.28}$$

Since the wall is impermeable to species *A*, the flux of *A* at z = δ must be zero:

$$\begin{aligned} N_{A,z}(\delta) &= -\frac{c\mathcal{D}_{AB}}{l}\left(C_1 e^{\delta/l} - C_2 e^{-\delta/l}\right) \\ 0 &= C_1 e^{\delta/l} - C_2 e^{-\delta/l} \end{aligned} \tag{7.29}$$

Equations (7.28) and (7.29) can be solved to give the coefficients C_1 and C_2

$$C_1 = y_{A0}\frac{e^{-\delta/l}}{e^{\delta/l}+e^{-\delta/l}} \tag{7.30}$$

$$C_2 = y_{A0}\frac{e^{\delta/l}}{e^{\delta/l}+e^{-\delta/l}} \tag{7.31}$$

Substituting these coefficients into Eq. (7.26), we find

$$\begin{aligned} y_A(z) &= y_{A0}\left[\left(\frac{e^{-(\delta-z)/l}}{e^{\delta/l}+e^{-\delta/l}}\right)+\left(\frac{e^{(\delta-z)/l}}{e^{\delta/l}+e^{-\delta/l}}\right)\right] \\ &= y_{A0}\left[\frac{e^{-(\delta-z)/l}+e^{(\delta-z)/l}}{e^{\delta/l}+e^{-\delta/l}}\right] = y_{A0}\frac{\cosh(\delta-z)/l}{\cosh\delta/l} \end{aligned} \tag{7.32}$$

To determine the rate at which *A* is absorbed from the gas phase, we need to determine the flux of *A* at the surface of the liquid. The flux is given by

$$\begin{aligned} N_{A,z}(z) &= -\,c\mathcal{D}_{AB}\frac{\partial y_A}{\partial z} \\ &= -\,c\mathcal{D}_{AB}\frac{y_{A0}}{l}\left[\frac{e^{-(\delta-z)/l}-e^{(\delta-z)/l}}{e^{\delta/l}+e^{-\delta/l}}\right] \\ &= \frac{c_{A0}\mathcal{D}_{AB}}{l}\left[\frac{e^{(\delta-z)/l}-e^{-(\delta-z)/l}}{e^{\delta/l}+e^{-\delta/l}}\right] \\ &= \frac{c_{A0}\mathcal{D}_{AB}}{l}\frac{\sinh((\delta-z)/l)}{\cosh(\delta/l)} \end{aligned} \tag{7.33}$$

where c_{A0} is the concentration of *A* at the left surface of the liquid at z = 0. The flux of *A* at the surface is given by

$$N_{A,z}(z=0) = \frac{c_{A0}\mathcal{D}_{AB}}{l}\frac{\sinh(\delta/l)}{\cosh(\delta/l)} \tag{7.34}$$

$$\begin{aligned} &= \frac{c_{A0}\mathcal{D}_{AB}}{(\mathcal{D}_{AB}/k_1)^{1/2}}\frac{\sinh(\delta/l)}{\cosh(\delta/l)} \\ &= c_{A0}\left(k_1\mathcal{D}_{AB}\right)^{1/2}\tanh(\delta/l) \end{aligned} \tag{7.35}$$

This is the rate (per unit area of liquid) at which *A* is absorbed from the gas phase.

8 Introduction to turbulence and non-linear dynamics

8.1 Introduction

The equations that govern the flow of a fluid were developed in the previous sections. These are the continuity equation and the momentum balance equation.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i) \tag{8.1}$$

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \tag{8.2}$$

In most situations that we will be interested in, the flows are much slower than the speed of sound. In these cases, the fluid can be considered incompressible (i.e., the density ρ is independent of position and time), even for a gas. For an incompressible fluid, the above equations reduce to:

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{8.3}$$

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \tag{8.4}$$

In addition, for an incompressible Newtonian fluid, the shear stress is related to the velocity gradients by

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{8.5}$$

where μ is the viscosity of the fluid. Equations (8.3), (8.4), and (8.5) provide the exact description of the flow of incompressible fluids.

Let's consider the flow of water through a pipe. If we examine, experimentally, the pressure drop as a function of the flowrate for this system, we find that at low flowrates (for Reynolds numbers less than 2300), the flowrate is proportional to the pressure drop. That is, if we double the pressure drop in the pipe, we double the flowrate of water. We have already solved this problem in the Sec. 3, and this is exactly the result that we expect.

However, at high flowrates (with Reynolds numbers greater than about 2300), the flowrate is proportional to the square root of the pressure drop. That is, if we double the pressure drop in the pipe, the flowrate only increases by a factor of about 1.4. So a significant portion of the work that we are using to pump the water is not being converted to flow; it is lost. This is not the result that we expected from our solution of the Navier-Stokes equation.

What is going on? Are the Navier-Stokes equations that we solved in Sec. 3 incorrect? To get a better idea of what is happening, we consider Reynolds' experiment. We inject a thin stream of dye into the center of a clear pipe, as shown in Fig. 8.1.

When we are at low flow rates, the stream of dye remains intact, in the same radial and angular position in the pipe as to where it was injected (see Fig. 8.1a). This is the behavior that we expect from our solution of the Navier-Stokes equation. The stream lines of the flow are well defined, and the dye traces one of these stream lines. This type of flow is known as laminar flow.

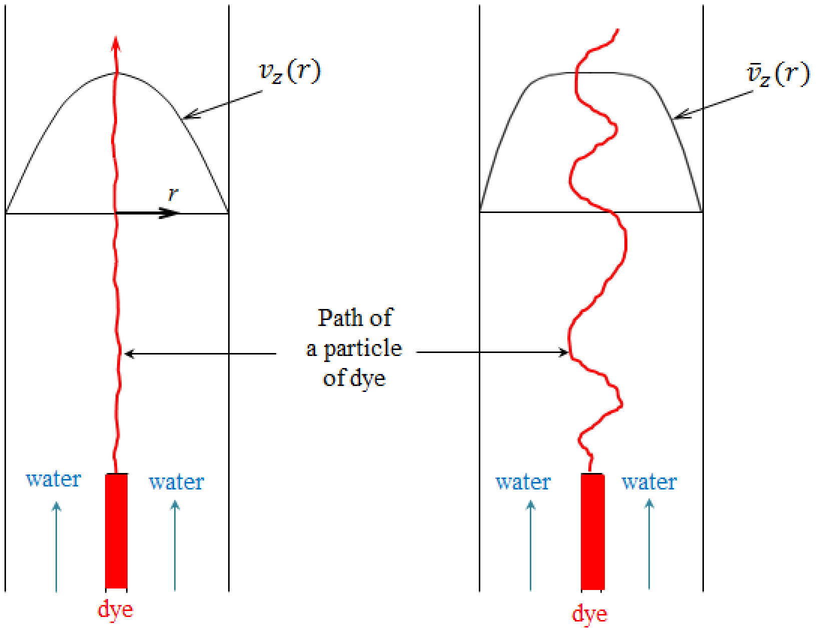


Figure 8.1: Reynolds experiment: streamline in (a) laminar flow and (b) turbulent flow. Note that *r* is the radial distance from the center-line of the pipe, $v_z(r)$ is the fluid velocity at r, and $\bar{v}_z(r)$ is the time averaged velocity of the fluid.

When we are at high flow rates, the stream of dye behaves erratically (see Fig. 8.1b). Rather than remaining in the same radial and angular position within the pipe, it travels in a seemingly random path. In addition, the dye mixes with the rest of the water in the pipe much more rapidly than at low flow rates. So the "missing energy" (the pumping work that is not going into water flow) is going into the transient, "random" motion of the flow. This type of flow is known as turbulent flow.

The flow of Newtonian fluids, such as water or air, is governed by a set of differential equations. Therefore, these systems are deterministic. That means that if we know the state of the system at some point in time (i.e., the velocity and pressure throughout the system), then we can, in principle, predict the future behavior of the system at any time by solving the differential equations. In principle, this means that we could accurately predict the weather.

However, when we have turbulent flows, the system seems to behave randomly, which implies that we cannot predict the future behavior of the system. How do we resolve this contradiction?

8.2 The logistic map

Part of the answer to this paradox is that non-linear equations can do very strange things! Actually, even extremely simple non-linear equations, much less complicated than the Navier-Stokes equations, can exhibit quite complex behavior.

Let's consider the dynamics of a very simple system: the logistic map. The logistic map was developed as a very simple model for growth (or shrinkage) of a population with time. It is a simple algebraic equation that relates x_n, the population in generation n, to x_{n+1}, the population in generation $n+1$:

$$x_{n+1} = rx_n(1 - x_n) \tag{8.6}$$

The parameter r characterizes the growth rate. The logistic map balances the growth of a population against the competition between members of the population for resources. Given the initial size of the population (i.e., a value for x_0), Eq. (8.6) can be used to predict the size of any future generation.

Now, let's examine the behavior of logistic map. In our analysis, the initial size of the population x_0 will be restricted to a value between 0 and 1. As we will see, the dynamics of the logistic map depends crucially on the value of the parameter r.

For $0 < r < 1$, x_n eventually approaches zero for any value of x_0 between 0 and 1. Examples of the evolution of the logistic map with r = 0.8 are shown in Fig. 8.2. The particular variation of x_n with n depends on x_0, howeve the value of x_n as $n \to \infty$ is always 0.

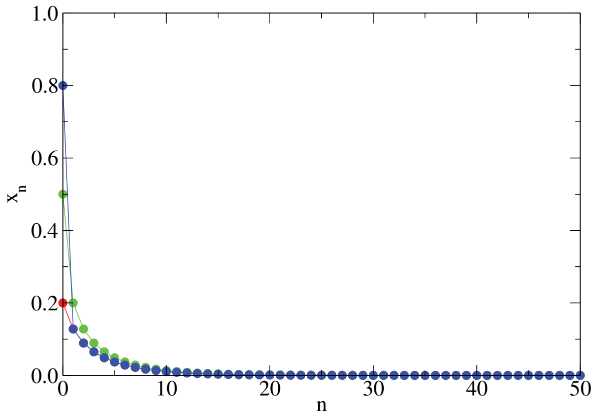


Figure 8.2: Evolution of the logistic map for r = 0.8 and (i) x_0 = 0.2 (red circles), (ii) x_0 = 0.5 (green circles), and (iii) x_0 = 0.8 (blue circles).

In order to determine the possible steady-state values of the logistic map, we need to solve the equation

$$x^* = rx^*(1 - x^*) \tag{8.7}$$

This is a quadratic equation with two roots: $x^* = 0$ and $x^* = (r - 1)/r$.

When $1 < r < 3$, x_n no longer vanishes to 0 but instead eventually approaches the value $(r - 1)/r$, independently of x_0. For values of r between 1 and 2, x_n approaches the asymptotic value monotonically. For values of r between 2 and 3, x_n oscillates around its final asymptotic value, with the amplitude of the oscillations decaying with n. As the value of r becomes closer to 3, the oscillations decay more slowly. Examples of the evolution of the logistic map with r = 2.8 are shown in Fig. 8.3.

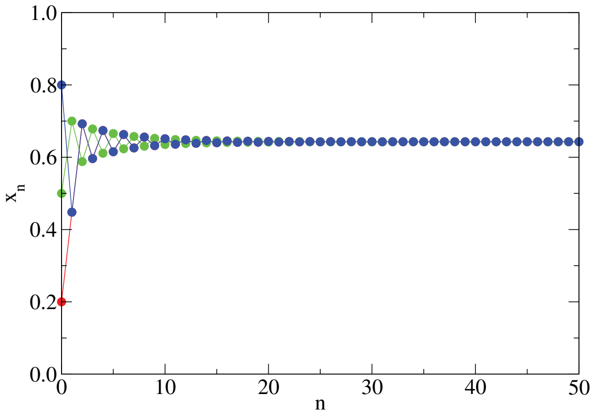


Figure 8.3: Dynamics of the logistic map for r = 2.8: (i) x_0 = 0.2 (red circles), (ii) x_0 = 0.5 (green circles), and (iii) x_0 = 0.8 (blue circles).

As the value of r is increased to beyond 3, the logistic map begins to display a different type of steadystate behavior. Instead of settling to the single value $(r-1)/r$ at long times, it alternates between two distinct values. These values are dependent on the particular value of r, but they are independent of the initial condition (i.e., x_0). Some examples of the evolution of the logistic map with $r = 3.3$ is given in Fig. 8.4. This type of behavior persists for $3 < r < 1+\sqrt{6} \approx 3.449$.

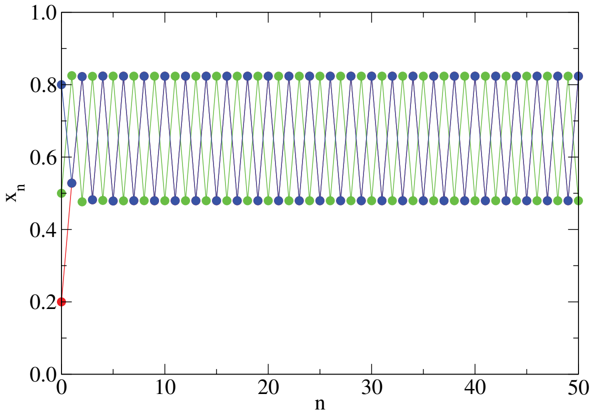


Figure 8.4: Dynamics of the logistic map for $r = 3.3$: (i) $x_0 = 0.2$ (red circles), (ii) $x_0 = 0.5$ (green circles), and (iii) $x_0 = 0.8$ (blue circles).

What has happened to the steady state value at $(r-1)/r$? This value is still a fixed point of the map. That is, if we start the system exactly at the value $(r-1)/r$, then it will remain at that value. For $r < 3$, we see that the system is attracted to the value $(r-1)/r$: for nearly all starting points, the system tends toward that value. This is what we call an attractive fixed point. However, when $r > 3$, we find that $(r-1)/r$ becomes a repulsive fixed point. Unless the system is precisely at the value $(r-1)/r$, it will eventually drift away from that point. The closer the system is to $(r-1)/r$, the longer it will stay near that value.

To illustrate this point, we consider a system that is very close to this unstable fixed point. In Fig. 8.5, the dynamics of the logistic map with $r = 3.3$ is shown. The fixed point of this map is $x^* = 2.3/3.3$. If we start the system with a value close to 2.3/3.3, then we see that the system appears to remain at that value. However, eventually it will drift to the stable period-2 oscillation which we observed previously. The closer the system is to $x^* = 2.3/3.3$, the longer it takes for it to go to the steady-state oscillations.

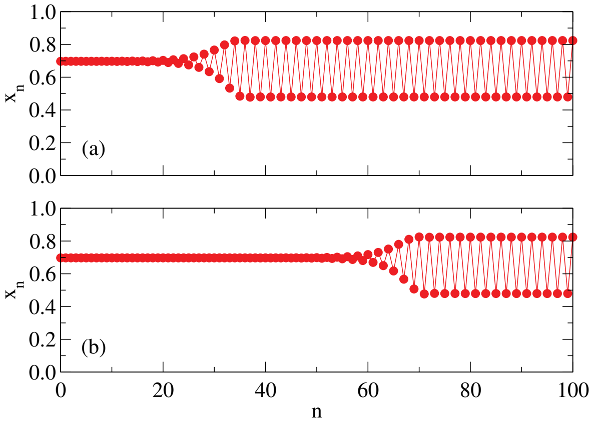


Figure 8.5: Dynamics of the logistic map for $r = 3.3$: (a) $x_0 = 0.697$ and (b) $x_0 = 0.6969697$.

Increasing r to values greater than $1+\sqrt{6}$, the logistic map no longer settles to a steady state that oscillates between two values. Instead, the steady state oscillates between four values. This is referred to as period doubling. In Fig. 8.6, examples of the dynamics of the logistic map with $r = 3.5$ are shown.

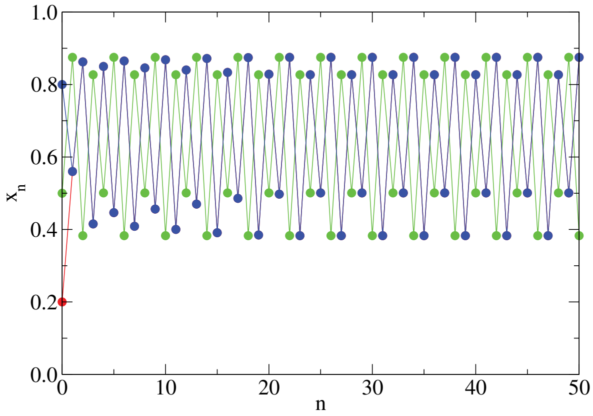


Figure 8.6: Dynamics of the logistic map for $r = 3.5$: (i) $x_0 = 0.2$ (red circles), (ii) $x_0 = 0.5$ (green circles), and (iii) $x_0 = 0.8$) (blue circles).

As the value of r increases, the logistic map goes through a series of period doubling transitions. The rate of period doubling with r increases. A summary of the values where period doubling occurs is given in Table 1.

Once r increases beyond a value of 3.569946 . . . , the logistic map is no longer periodic. In this case it becomes chaotic. The dynamics of the logistic map with $r = 4$ is shown in Fig. 8.7. The dynamics of the system appears to be completely random.

The long time steady-state behavior that we have observed for the logistic map can be summarized in a bifurcation diagram, which can be seen on http://en.wikipedia.org/wiki/File:Logistic_Bifurcation_map_High_Resolution.png. The bifurcation diagram displays the values the logistic map attains at long times.

r	period
$r_1 = 3$	2
$r_2 = 3.449\ldots$	4
$r_3 = 3.54409\ldots$	8
$r_4 = 3.5644\ldots$	16
$r_5 = 3.568759\ldots$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946\ldots$	∞

Table 1: Parameter values for period doubling of the logistic map.

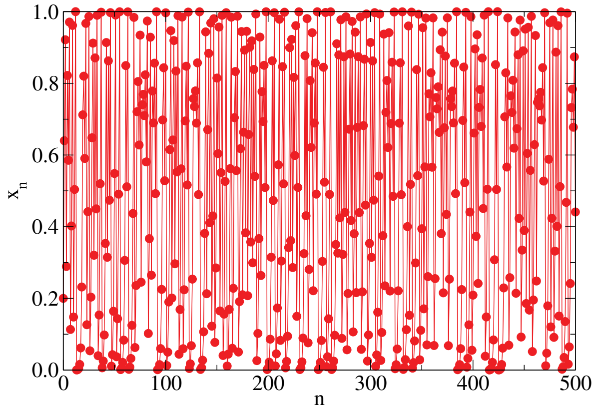


Figure 8.7: Dynamics of the logistic map for *r* = 4 and *x*0 = 0.2.

The time evolution of a system that is chaotic is extremely sensitive to the initial conditions. This makes the prediction of the precise trajectory of a chaotic system practically impossible. However, even though we cannot make statements about the precise time evolution of a system, in many cases we can make accurate predictions of the statistics of the system's time evolution.

For the logistic map with $r = 4$, the fraction of time that the system has a particular value is shown in Fig. 8.8. This system is chaotic, but it can be shown that the distribution of the value of x_n is given by the Beta distribution, which is defined as

$$p(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{8.8}$$

with the parameter values $\alpha = 1/2$ and $\beta = 1/2$, and where Γ is the Gamma function. Therefore, even though we can not precisely describe the dynamics of a chaotic system, we can make predictions about its statistical behavior. This is an important point in trying to model turbulent flows.

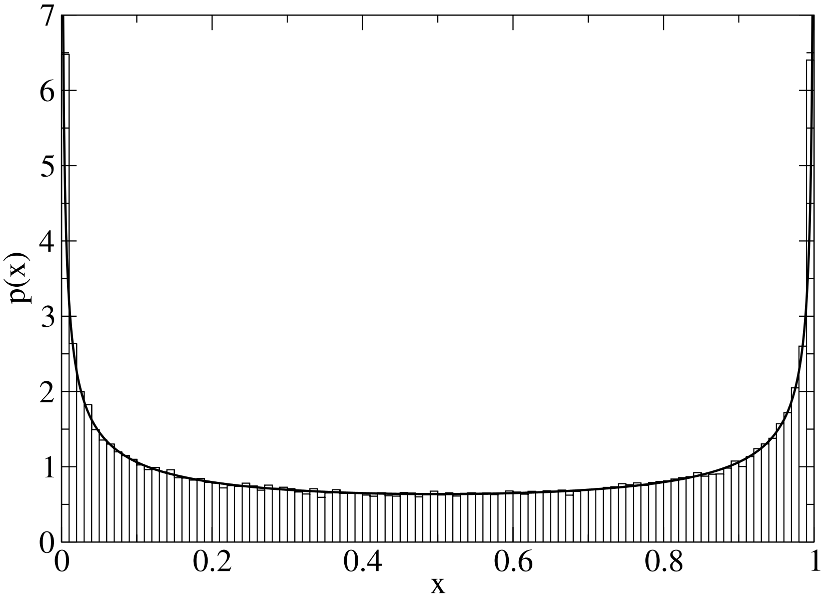
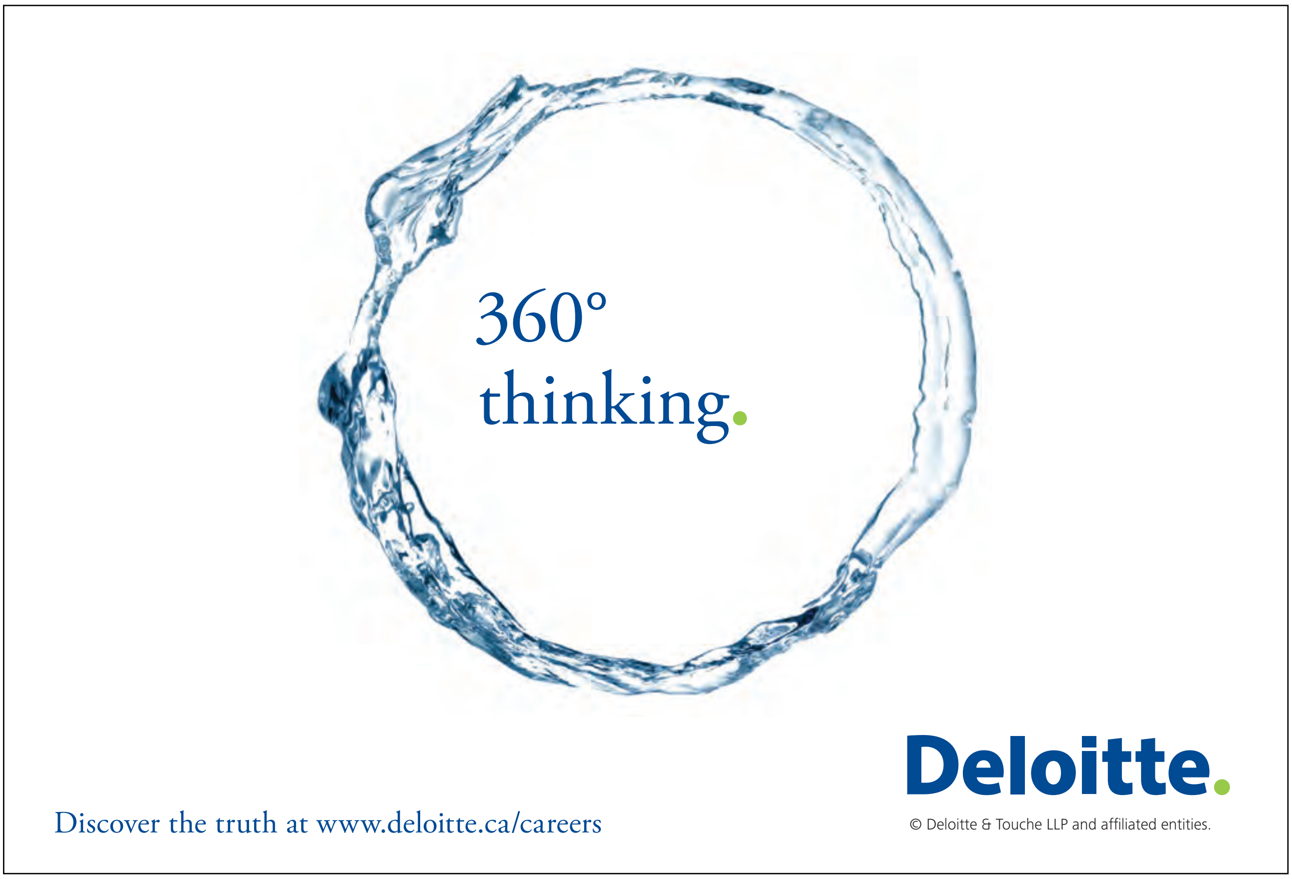


Figure 8.8: Probability distribution of observing a particular value of x_n in the logistic map with $r = 4$.

9 Statistical treatment of turbulence

9.1 Introduction

In turbulent flows, the various properties of the fluid, such as the velocity, pressure, etc., vary in a very complicated manner with position and time. The instantaneous values of these properties appear to behave almost randomly with time. Luckily, however, we are generally not interested in the instantaneous properties of a fluid flow; rather, we want information on how the average properties vary. These average properties vary in a regular manner and can be modeled (at least approximately) even for a turbulent flow.



9.2 Brief overview of statistics

Before we can proceed, we need to define precisely what is meant by an "average property." Consider the situation where we are trying to perform measurements on a turbulent flow. In order to obtain statistics for the measurement, we repeat the experiment N times. The time variation of the property f that we are measuring (e.g., pressure, velocity, etc.) will be different each time we measure it, due do slight differences in the initial conditions of the experiments. The average variation of the property f with time, which we denote as $\langle f(t)\rangle$, is defined as

$$\langle f(t)\rangle \equiv \frac{1}{N}\sum_{i=1}^{N} f_i(t) \tag{9.1}$$

where $f_i(t)$ is the time dependence of f measured during the ith experiment.

For a given experiment, the quantity f can always be written as a sum of two terms:

$$f(t) = \langle f(t)\rangle + f'(t) \tag{9.2}$$

where the first term is the average value of f, and the second term is the deviation of the quantity from its average value (note that in this case, f' is *not* the derivative of f). If we take the average of both sides of the equation, we find

$$\begin{aligned} &\langle f(t)\rangle = \langle f(t)\rangle + \langle f'(t)\rangle \\ &\langle f'(t)\rangle = 0 \end{aligned} \tag{9.3}$$

which is just a statement of the fact that the average deviation of a property from its mean is zero. We will make frequent use of this fact shortly. Although the average of f' is zero, the average of f'^2 is generally not (i.e., $\langle f'^2\rangle \neq 0$). The larger the value of $\langle f'^2\rangle \neq 0$, the larger the fluctuations in the property.

9.3 Reynold stresses

The various properties of a fluid can be written as the sum of an average value and an instantaneous fluctuation from the average

$$v_i = \langle v_i\rangle + v_i' \tag{9.4}$$

$$\tau_{ij} = \langle \tau_{ij}\rangle + \tau_{ij}' \tag{9.5}$$

$$p = \langle p\rangle + p' \tag{9.6}$$

Note that for a laminar flow, the fluctuations of the various properties are identically equal to zero (i.e., $v_i' = \tau_{ij}' = p' = 0$).

We are not directly interested in the instantaneous values v_i, τ_{ij}, and p of the flow, which are governed by the Navier-Stokes equation (see Eqs. (8.3)–(8.5)). We are only concerned in their average values (i.e., $\langle v_i \rangle$, $\langle \tau_{ij} \rangle$, and $\langle p \rangle$). So we need to develop the equations that govern the average properties. To do this, we just insert Eqs. (9.4)–(9.6) into the Navier-Stokes equations.

For the continuity equation, this becomes:

$$\begin{aligned} \frac{\partial v_i}{\partial x_i} &= 0 \\ \frac{\partial}{\partial x_i}(\langle v_i \rangle + v_i') &= 0 \end{aligned}$$

Taking the average of both sides of the equation, we find

$$\frac{\partial \langle v_i \rangle}{\partial x_i} = 0 \tag{9.7}$$

where we have used the fact that $\langle v_i' \rangle = 0$. This is the same as the original equation, but now involves the average velocity rather than the instantaneous velocity.

The relationship between the stress and the velocity gradients becomes:

$$\begin{aligned} \tau_{ij} &= \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \\ \langle \tau_{ij} \rangle + \tau_{ij}' &= \mu \left(\frac{\partial}{\partial x_j}(\langle v_i \rangle + v_i') + \frac{\partial}{\partial x_i}(\langle v_j \rangle + v_j') \right) \end{aligned}$$

Taking the average of both sides of this equation, we find

$$\langle \tau_{ij} \rangle = \mu \left(\frac{\partial \langle v_i \rangle}{\partial x_j} + \frac{\partial \langle v_j \rangle}{\partial x_i} \right) \tag{9.8}$$

where we have used the fact that $\langle f' \rangle = 0$. Again, we find that this is the same as the original equation, but with the instantaneous values replaced by their averages.

For the momentum equation, we find

$$\begin{aligned} \frac{\partial \rho v_i}{\partial t} &= -\frac{\partial}{\partial x_j}(\rho v_j v_i) + \frac{\partial \tau_{ji}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \\ \frac{\partial}{\partial t}\rho(\langle v_i \rangle + v_i') &= -\frac{\partial}{\partial x_j}\rho(\langle v_j \rangle + v_j')(\langle v_i \rangle + v_i') + \frac{\partial}{\partial x_j}(\langle \tau_{ji} \rangle + \tau_{ji}') - \frac{\partial}{\partial x_i}(\langle p \rangle + p') + \rho g_i \\ \frac{\partial}{\partial t}\rho(\langle v_i \rangle + v_i') &= -\frac{\partial}{\partial x_j}\rho(\langle v_j \rangle \langle v_i \rangle + v_j' \langle v_i \rangle + \langle v_j \rangle v_i' + v_j' v_i') \\ &\quad + \frac{\partial}{\partial x_j}(\langle \tau_{ji} \rangle + \tau_{ji}') - \frac{\partial}{\partial x_i}(\langle p \rangle + p') + \rho g_i \end{aligned} \tag{9.9}$$

Taking the time average of both sides of the equation, we find

$$\begin{aligned}\frac{\partial \rho\langle v_i\rangle}{\partial t} &= -\frac{\partial}{\partial x_j}\rho(\langle v_j\rangle\langle v_i\rangle + \langle v_j' v_i'\rangle) + \frac{\partial\langle\tau_{ji}\rangle}{\partial x_j} - \frac{\partial\langle p\rangle}{\partial x_i} + \rho g_i \\ &= -\frac{\partial}{\partial x_j}\rho\langle v_j\rangle\langle v_i\rangle + \frac{\partial}{\partial x_j}(\langle\tau_{ji}\rangle - \rho\langle v_j' v_i'\rangle) - \frac{\partial\langle p\rangle}{\partial x_i} + \rho g_i \\ &= -\frac{\partial}{\partial x_j}\rho\langle v_j\rangle\langle v_i\rangle + \frac{\partial}{\partial x_j}(\langle\tau_{ji}\rangle + \tau_{ji}^{(t)}) - \frac{\partial\langle p\rangle}{\partial x_i} + \rho g_i \qquad (9.10)\end{aligned}$$

where $\tau_{ij}^{(t)}$ is known as the Reynolds stress and is given by

$$\tau_{ij}^{(t)} = -\rho\langle v_i' v_j'\rangle \qquad (9.11)$$

For the momentum equation, unlike the previous two equations, we find that instantaneous properties are not simply replaced by their average values. We actually have an extra stress term. This additional stress $\tau_{ij}^{(t)}$ is due to the turbulent motion of the fluid.

The average total stress τ_{ij} within a fluid can, therefore, be decomposed into two contributions: (i) viscous forces $\langle\tau_{ij}\rangle$ and (ii) turbulent eddies $\tau_{ij}^{(t)}$

$$\tau_{ij}^{\text{total}} = \langle\tau_{ij}\rangle + \tau_{ij}^{(t)} \qquad (9.12)$$

When the flow is laminar, then $\tau_{ij}^{(t)} = 0$.

In Fig. 9.1, we plot a typical velocity profile of a fluid in turbulent flow near a wall. The flow can be divided into three general regions: (i) the laminar sublayer (or viscous sublayer), (ii) the buffer layer, and (iii) the turbulent core. In the laminar sublayer, the flow is relatively slow, due to the no-slip boundary conditions imposed by the nearby wall. In this region of the flow, the level of turbulence is extremely low, and viscous stress is much greater than turbulent stress.

Away from the wall, however, the Reynolds stress dominates over viscous stress. This region is known as the turbulent core. In the region between the laminar sublayer and the turbulent core, viscous transport and turbulent transport of momentum play roughly equal roles. This region of the flow is known as the buffer layer. In general for a system in turbulent flow, most of the fluid is in the turbulent core. The laminar sublayer is typically only a thin slab near the wall; the thickness of this layer varies inversely with the Reynolds number.

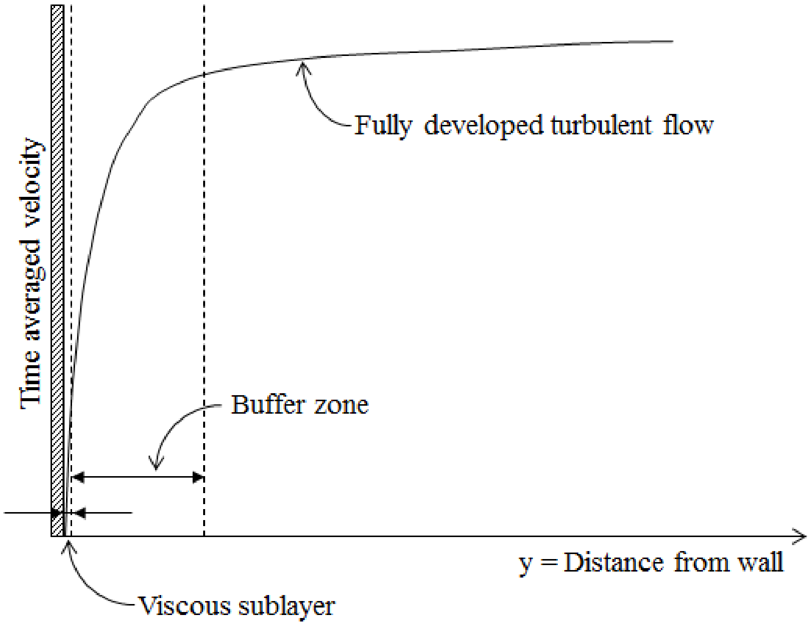


Figure 9.1: Typical velocity profile for a system in turbulent flow conditions.

In order to get a feel for the relative contributions of the viscous and Reynolds stresses, we show a typical distribution of stress for the turbulent flow of a fluid through a pipe of radius R in Fig. 9.2. Near the wall of the pipe (located at $r = R$), the viscous stresses dominate the flow. It is only when we move away to the pipe wall, do the turbulent stresses begin to make a significant contribution. The size of the region where the Reynolds stresses are significant depends on Reynolds number of the flow. The higher the Reynolds number, the large the region. For low Reynolds number where the flow is laminar, the Reynolds stresses make no contribution.

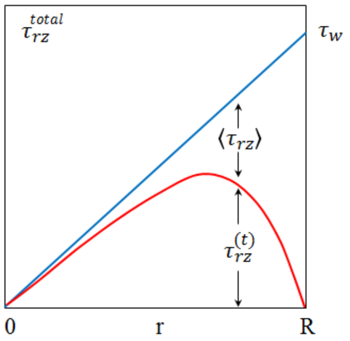


Figure 9.2: Relative contribution of viscous $\langle \tau_{rz} \rangle$ and Reynolds $\tau_{rz}^{(t)}$ stresses as a function of radial position in a pipe.

In order to complete the description of turbulent flow, we need to relate the Reynold stress to other properties of the flow (e.g., velocity gradients, etc.). Unfortunately, this is not a simple task. For over a century, many attempts have been made to model turbulent flows; however, no completely satisfactory description of turbulence is currently available. In the next section, we will discuss a particularly simplistic, yet extremely useful, model of turbulence.

10 Approximate models for turbulence

10.1 Introduction

In this section, we discuss some approximate methods to make predictions for turbulent flows.

10.2 Boussinesq hypothesis or the mean-velocity field closure

According to Newton's law of the viscosity, the shear stress is related to the velocity gradients

$$\langle \tau_{ij} \rangle = \mu \left(\frac{\partial \langle v_i \rangle}{\partial x_j} + \frac{\partial \langle v_j \rangle}{\partial x_i} \right)$$

Boussinesq suggested that the Reynolds stress can be described in an analogous manner to the viscous stresses:

$$\tau_{ij}^{(t)} \approx \mu^{(t)} \left(\frac{\partial \langle v_i \rangle}{\partial x_j} + \frac{\partial \langle v_j \rangle}{\partial x_i} \right) \tag{10.1}$$

where $\mu^{(t)}$ is the eddy viscosity, which plays an analogous role to the viscosity μ for viscous stresses. The relation given in Eq. (10.1) is not exact. In many cases it is known to be a poor approximation; however, in general, it provides a reasonable description of turbulence and is the starting point of many theories.

Unlike the viscosity, the eddy viscosity is not simply a property of the fluid. That is, given the temperature and pressure of a fluid, one cannot specify the value of the eddy viscosity, as is the case for viscosity. The eddy viscosity is, in fact, dependent on the local flow conditions, and, consequently, its value varies with the position in the flow.

If we consider flow in a circular pipe, then the total stress in the fluid can be written as:

$$\begin{aligned} \langle \tau_{rz} \rangle + \tau_{rz}^{(t)} &= \tau_{rz}^{\text{total}} \\ \mu \frac{\partial \langle v_z \rangle}{\partial r} + \mu^{(t)} \frac{\partial \langle v_z \rangle}{\partial r} &= \tau_{rz}^{\text{total}} \\ (\mu + \mu^{(t)}) \frac{\partial \langle v_z \rangle}{\partial r} &= \tau_{rz}^{\text{total}} \end{aligned} \tag{10.2}$$

where we have used the Boussinesq approximation for the Reynold stress. From this equation, we can see that the ratio of the eddy viscosity to the molecular viscosity is a measure of the relative contribution of viscous transport and turbulent transport of momentum. In Fig. 10.1, we plot $\mu^{(t)}/\mu$ as a function of radial position for turbulent flow in a circular pipe of radius *R*. From this plot we again see that viscous stresses dominate near the wall, while turbulent stresses dominate in most of the region away from the wall.

The Boussinesq hypothesis has essentially shifted the problem of determining the Reynolds stress $\tau_{ij}^{(t)}$ to determining the eddy viscosity $\mu^{(t)}$, which is a somewhat simpler problem since the eddy viscosity is a scalar quantity while the Reynolds stress is a tensor (matrix). However, the eddy viscosity is still an undetermined function of position, and so our description of turbulence is still incomplete.

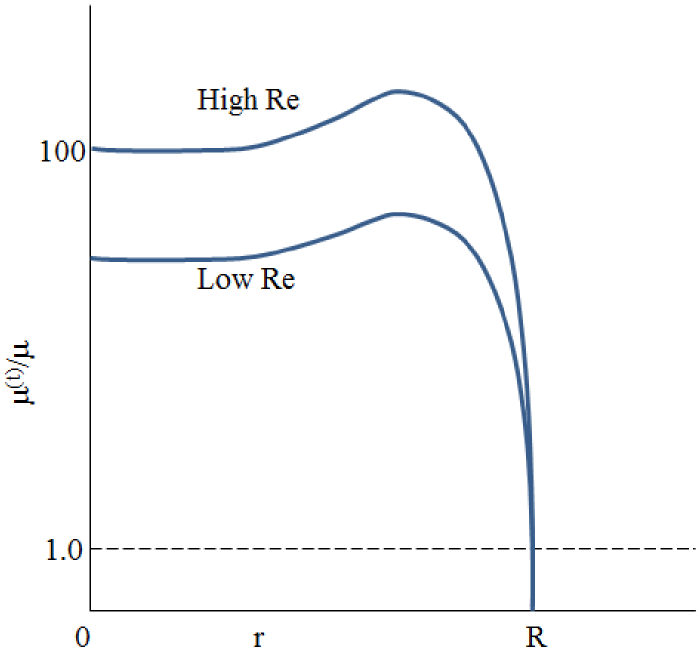


Figure 10.1: Ratio of the eddy viscosity to the molecular viscosity as a function of radial position for a fluid in turbulent flow in a pipe of radius *R*.

10.3 Prandtl mixing-length theory and von Kármán similarity hypothesis

In order to develop a simple model for the Reynolds stresses, let's examine a turbulent flow with an average flow in the x-direction (see Fig. 10.2). In turbulent flow, the main structures are swirling eddies that spontaneously appear and then disappear due to dissipation. Prandtl assumed that these eddies would transport a subsection of fluid in the flow a distance *l*, known as the mixing length, before they disappear.

For example, a piece of fluid that was located at position $y + l$, with a velocity given by the average velocity of the flow $\langle v_x(y+l)\rangle$ would be transported by the eddy to a position y, where the average velocity of the fluid is $\langle v_x(y)\rangle$. Immediately after being transported, the velocity of this piece of fluid would differ from the average velocity of the surrounding fluid at position y. From this physical argument, we would expect the order of magnitude of the fluctuations in the x-component of the velocity v'_x to be

$$\begin{aligned} v'_x &\sim \langle v_x(y+l)\rangle - \langle v_x(y)\rangle \\ &\sim l\frac{\partial\langle v_x(y)\rangle}{\partial y} \end{aligned} \tag{10.3}$$

where we have used the Taylor series expansion. The fluctuations in the y-component of the velocity

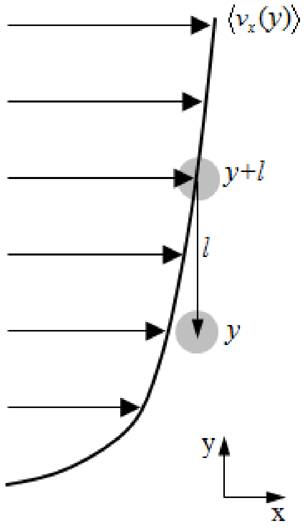


Figure 10.2: Illustration of Prandtl's mixing length theory.

are expected to be of the same order of magnitude (i.e., $v'_y \sim v'_x$).

Therefore, we expect

$$\langle v'_y v'_x\rangle \sim l^2\left(\frac{\partial v_x}{\partial y}\right)^2$$

and so Prandtl's expression for the Reynolds stress is

$$\begin{aligned} \tau^{(t)}_{yx} &= -\rho\langle v'_y v'_x\rangle \\ &= \rho l^2\left|\frac{\partial\langle v_x(y)\rangle}{\partial y}\right|\frac{\partial\langle v_x(y)\rangle}{\partial y} \end{aligned} \tag{10.4}$$

Comparing this to the Boussinesq expression for the Reynolds stress, we can make the identification:

$$\mu^{(t)} = \rho l^2 \left| \frac{\partial \langle v_x \rangle}{\partial y} \right| \tag{10.5}$$

We still have the difficulty of determining how the mixing length l depends on position. This mixing length physically represent, more or less, the size of the turbulent eddies. Away from the walls, we expect these can be fairly large; near a wall, however, these eddies are limited in size. von Kármán suggested that the mixing length is proportional to the distance y from a surface.

$$l = \kappa y \tag{10.6}$$

where κ is an empirical constant, which has been found to be approximately equal to 0.4. Note that the constant κ is independent of the geometry of the flow. With this relation, we now have a complete (although fairly crude) description of turbulent flow.

10.4 Three-region model for momentum transport (universal velocity profile)

In this subsection, we will discuss the "universality" of the velocity profile of a turbulent flow. What we mean by this is that the velocity profile is to a large extent *independent* of the geometry of the flow (e.g., pipe diameter) and only depends on a few properties of the fluid.

To begin, we examine the velocity profile of a fluid confined between two planes separated by a distance H. The bottom plane is stationary, while the upper plane is moving at a speed U in the x direction. In order to maintain the motion of the upper plane, a stress τ_w must be applied. For this system, the flow is driven entirely by the motion of the upper plane, and there is no pressure gradient.

We perform a force (momentum) balance on a small slab of fluid of thickness dy, located at y above the bottom plane. The system is at steady state, so the forces acting on the slab must sum to zero.

$$\begin{aligned} 0 &= \tau_{yx}^{\text{total}}(y+dy)A - \tau_{yx}^{\text{total}}(y)A \\ 0 &= \frac{1}{A}\left[\frac{\tau_{yx}^{\text{total}}(y+dy)A - \tau_{yx}^{\text{total}}(y)A}{dy}\right] \end{aligned} \tag{10.7}$$

In the limit that $dy \to 0$, we find

$$\begin{aligned} \frac{1}{A}\frac{\partial \tau_{yx}^{\text{total}}A}{\partial y} &= 0 \\ \frac{\partial \tau_{yx}^{\text{total}}}{\partial y} &= 0 \end{aligned} \tag{10.8}$$

where we have made use of the fact that the cross-subsectional area A is constant. This equation can be integrated to give

$$\tau_{yx}^{\text{total}} = \tau_w$$

where the integration constant τ_w is the shear stress at the wall. This equation states that the total shear stress in the problem is independent of position. For a system in turbulent flow condtions, the total shear stress is made up of two contributions: (i) a laminar contribution $\langle \tau_{yx} \rangle$, and (ii) a “turbulent” contribution $\tau_{yx}^{(t)}$, the Reynold stresses. Therefore, we have

$$\langle \tau_{yx} \rangle + \tau_{yx}^{(t)} = \tau_w \tag{10.9}$$

Before we proceed any further in analyzing this problem, we need to identify the characteristic scales in the system. These characteristic scales determine how large or small a quantity is, with respect to the system. For example, we say that we are at a position “far from the wall”, this means that the distance from the wall is much greater than the characteristic length of the system.

The characteristic scales are based upon the physical properties of the system; in this case, these are the wall shear stress τ_w, the fluid density ρ, and the fluid viscosity μ. From these properties, we find that the characteristic velocity $\bar{V}$ is

$$\bar{V} = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

and the characteristic length $\bar{l}$ is

$$\bar{l} = \frac{\mu}{\sqrt{\tau_w \rho}}$$

Very close to a wall in the laminar sublayer (when the distance from the wall is much less than the characteristic length), viscous transport of momentum dominates over turbulent transport. In this case, the force balance simplifies to:

$$\begin{aligned} \langle \tau_{yx} \rangle &\approx \tau_w \\ \mu \frac{\partial \langle v_x \rangle}{\partial y} &\approx \tau_w \end{aligned} \tag{10.10}$$

Integrating this equation, we find

$$\langle v_x \rangle \approx \frac{\tau_w}{\mu} y \tag{10.11}$$

where the integration constant is equal to zero, due to the fact that the velocity must vanish at the wall. Introducing the dimensionless velocity $\langle v_x^+ \rangle = \langle v_x \rangle / \bar{V}$ and the dimensionless distance $y^+ = y/\bar{l}$, we find

$$\langle v_x^+ \rangle \approx y^+ \tag{10.12}$$

In this form, we see that the velocity profile in the laminar sublayer is actually independent on the nature of the fluid (i.e., the properties of the fluid) and the nature of the flow.

In the turbulent core, far from any walls, the Reynolds stress is much larger than the viscous stress (i.e., $\tau_{yx}^{(t)} \gg \tau_{yx}$).

Therefore, the viscous stress can be neglected:

$$\tau_{yx}^{(t)} \approx \tau_w$$

If we use the Boussinesq hypothesis along with the Prandtl mixing-length expression for the eddy viscosity, we find

$$\begin{aligned}
\mu^{(t)} \frac{\partial \langle v_x \rangle}{\partial y} &\approx \tau_w \\
\rho l^2 \left(\frac{\partial \langle v_x \rangle}{\partial y} \right)^2 &\approx \tau_w \\
\rho (\kappa y)^2 \left(\frac{\partial \langle v_x \rangle}{\partial y} \right)^2 &\approx \tau_w \\
\frac{\partial \langle v_x \rangle}{\partial y} &\approx \frac{1}{\kappa y} \left(\frac{\tau_w}{\rho} \right)^{1/2} \\
\langle v_x \rangle &\approx \bar{V} \left(\frac{1}{\kappa} \ln y + C^+ \right) \\
\langle v_x^+ \rangle &\approx \frac{1}{\kappa} \ln y + C^+ \qquad (10.13)
\end{aligned}$$

where C^+ is an integration constant. Experimentally, we find $\kappa \approx 0.4$ and $C^+ \approx 5.5$. Again, we find that the velocity profile is independent of the nature of the fluid and the nature of the flow.

In summary, there are three different regions in turbulent flow past a surface. These are the laminar sublayer, the turbulent core, and the buffer layer (which interpolates between the previous two regions). The velocity profiles in each of these layers can be written in a "universal form," which is independent of the nature of the fluid and the details of the flow:

laminar sublayer	$v^+ = y^+$	$0 < y^+ < 5$
buffer layer	$v^+ = 5 \ln y^+ - 3.05$	$5 < y^+ < 30$
turbulent core	$v^+ = 2.5 \ln y^+ + 5.5$	$30 < y^+$

10.5 Turbulent flow in a pipe

In this subsection, we will use the "universal velocity profile" developed in the previous subsection to get a relation between the friction factor and the Reynolds number for turbulent flow in a circular pipe with radius *R*. The thickness of the laminar sublayer is inversely proportional to the Reynolds number. Therefore, at very high flowrates, nearly all of the flow in a pipe should be occupied by the turbulent core. This means we can use the expression for the velocity profile obtained for the turbulent core as the velocity profile for the entire flow in the pipe.

$$\begin{aligned}\bar{v}_x &= \frac{1}{\pi R^2}\int_0^R 2\pi r dr \langle v_x(r)\rangle \\ &= \frac{2}{R^2}\int_0^R r dr \bar{V}\left(\frac{1}{\kappa}\ln\frac{y}{\bar{l}} + C^+\right)\end{aligned} \tag{10.14}$$

The coordinate *y* represents the distance from the wall. This is related to the coordinate *r* in the pipe by the relation $y = R - r$. Substituting the variable y for the variable y, we find

$$\begin{aligned}\bar{v}_x &= \frac{2\bar{V}}{R^2}\int_R^0 (R-y)(-dy)\left(\frac{1}{\kappa}\ln\frac{y}{\bar{l}} + C^+\right) \\ &= 2\bar{V}\int_0^1 (1-\xi)d\xi\left(\frac{1}{\kappa}\ln\frac{\xi R}{\bar{l}} + C^+\right) \\ &= \bar{V}\left(\frac{1}{\kappa}\ln\frac{R}{\bar{l}} + C^+ - \frac{3/2}{\kappa}\right)\end{aligned} \tag{10.15}$$

This expression can be written in a more familiar form by introducing the friction factor *f* and the Reynolds number Re. The Reynolds number is defined as

$$\mathrm{Re} \equiv \frac{D\bar{v}_x\rho}{\mu} = \frac{2R\bar{v}_x\rho}{\mu} \tag{10.16}$$

where *D* = 2*R* is the diameter of the pipe.

The friction factor is defined as

$$f \equiv \frac{\Delta p}{2\rho\bar{v}_x^2}\frac{D}{L} \tag{10.17}$$

This can be expressed in terms of the shear stress at the wall τ_w. At steady state, the pressure force acting on the fluid should be precisely equal to the shear stress on the fluid due to the wall of the pipe.

$$\begin{aligned} \Delta p \frac{\pi D^2}{4} &= \pi D L \tau_w \\ \Delta p &= 4\left(\frac{L}{D}\right)\tau_w \end{aligned} \tag{10.18}$$

Substituting this into the definition of the friction factor, we get

$$f = \frac{2\tau_w}{\rho \bar{v}_x^2} \tag{10.19}$$

If we insert the definition of the friction factor and the Reynolds number into Eq. (10.15), we find

$$\begin{aligned} \frac{1}{\sqrt{f/2}} &= \frac{1}{\kappa}\ln \frac{\mathrm{Re}}{2}\sqrt{\frac{f}{2}} + C^+ - \frac{3/2}{\kappa} \\ \frac{1}{\sqrt{f}} &= 4.06 \log \mathrm{Re}\sqrt{f} - 0.60 \end{aligned} \tag{10.20}$$

This relation was first derived by von Kármán.

Nikuradse empirically fit data from a series of flow experiments and found the following relation

$$\frac{1}{\sqrt{f}} = 4.0 \log \mathrm{Re}\sqrt{f} - 0.40$$

So we see that the mixing length theory of Prandtl provides a reasonable description of turbulent pipe flow.

11 Turbulence in energy and mass transport

11.1 Introduction

Thus far, we have only considered the effects of turbulence on momentum transport. However, turbulence also has a significant effect on the transport of energy and mass. We can derive the required relations for energy and mass transport in turbulent systems by following the same procedure we used for momentum transport.

11.2 Energy transport

For an incompressible fluid, the energy balance equation can be written as

$$\rho C_v \frac{\partial T}{\partial t} = -\rho C_v v_j \frac{\partial T}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \tau_{ji} \frac{\partial v_i}{\partial x_j}. \tag{11.1}$$

If we write all properties in terms of a sum of an average value and a fluctuation, we find

$$\begin{aligned}\rho C_v \frac{\partial}{\partial t}(\langle T\rangle + T') = -\rho C_v(\langle v_j\rangle + v_j')\frac{\partial}{\partial x_j}(\langle T\rangle + T') - \frac{\partial}{\partial x_i}(\langle q_i\rangle + q_i') \\ + (\langle \tau_{ji}\rangle + \tau_{ji}')\frac{\partial}{\partial x_j}(\langle v_i\rangle + v_i').\end{aligned} \tag{11.2}$$

Taking the time average of both sides of this equation, we find

$$\begin{aligned}\rho C_v \frac{\partial \langle T\rangle}{\partial t} &= -\rho C_v \langle v_i\rangle \frac{\partial \langle T\rangle}{\partial x_i} - \rho C_v \left\langle v_i' \frac{\partial T'}{\partial x_i}\right\rangle - \frac{\partial \langle q_i\rangle}{\partial x_i} + \langle \tau_{ji}\rangle \frac{\partial \langle v_i\rangle}{\partial x_j} + \left\langle \tau_{ji}' \frac{\partial v_i'}{\partial x_j}\right\rangle \\ &= -\rho C_v \langle v_i\rangle \frac{\partial \langle T\rangle}{\partial x_i} - \frac{\partial}{\partial x_i}\left[\langle q_i\rangle + \rho C_v \langle v_i' T'\rangle\right] + \langle \tau_{ji}\rangle \frac{\partial \langle v_i\rangle}{\partial x_j} + \left\langle \tau_{ji}' \frac{\partial v_i'}{\partial x_j}\right\rangle \\ &= -\rho C_v \langle v_i\rangle \frac{\partial \langle T\rangle}{\partial x_i} - \frac{\partial}{\partial x_i}\left[\langle q_i\rangle + q_i^{(t)}\right] + \langle \tau_{ji}\rangle \frac{\partial \langle v_i\rangle}{\partial x_j} + \left\langle \tau_{ji}' \frac{\partial v_i'}{\partial x_j}\right\rangle\end{aligned} \tag{11.3}$$

The first term on the right side of the equation represents the transport of energy due to convection. The second term represents transport due to "conduction." We will discuss this term later in more detail. The third term represents the conversion of kinetic energy to thermal energy (dissipation) due to the average flow of the fluid. The fourth and final term represents the conversion of the swirling motion of the turbulent eddies into thermal energy. The third and fourth terms in the energy balance equation are usually much smaller in magnitude to the other terms and, therefore, are neglected.

In this case, we see that we have an additional contribution to the heat conduction term (i.e., the second term), due to turbulent mixing:

$$q_i^{(t)} = \rho C_v \langle v_i' T' \rangle \tag{11.4}$$

In the case of laminar flow, this contribution vanishes (i.e., $q_i^{(t)} = 0$).

We can estimate the order of magnitude of the temperature fluctuations T' by using the same arguments as Prandtl did for momentum transport. In this case, we find

$$\begin{aligned} T' &\sim \langle T(y+l) \rangle - \langle T(y) \rangle \\ &\sim l \frac{\partial \langle T \rangle}{\partial y} \end{aligned} \tag{11.5}$$

Therefore, we would estimate that the turbulent contribution to heat conduction is given by

$$q_y^{(t)} = \rho C_v l^2 \left| \frac{\partial \langle v_x \rangle}{\partial y} \right| \frac{\partial \langle T \rangle}{\partial y} \tag{11.6}$$

where l is the Prandtl mixing length. As before, we have $l = \kappa y$.

11.3 Mass transport

As with momentum and energy transport, we can also develop an equation for the turbulent transport of mass. The balance equation for a particular species *α* in a multicomponent system is given by

$$\begin{aligned} \frac{\partial c_\alpha}{\partial t} &= -\frac{\partial}{\partial x_i}(c_\alpha v_{\alpha,i}) + \sigma_\alpha \\ &= -\frac{\partial}{\partial x_i}[c_\alpha (v_{\alpha,i} - v_i)] - \frac{\partial}{\partial x_i}(c_\alpha v_i) + \sigma_\alpha \\ &= -\frac{\partial J_{\alpha,i}}{\partial x_i} - \frac{\partial}{\partial x_i}(c_\alpha v_i) + \sigma_\alpha \end{aligned}$$

where $J_{\alpha,i}$ is the *i*th component ($i = x$, y, or z) of the diffusive flux for component *α*. Writing all properties in terms of an average value and a fluctuation, we find

$$\frac{\partial}{\partial t}(\langle c_\alpha \rangle + c_\alpha') = -\frac{\partial}{\partial x_i}(\langle J_{\alpha,i} \rangle + J_{\alpha,i}') - \frac{\partial}{\partial x_i}[(\langle c_\alpha \rangle + c_\alpha')(\langle v_{\alpha,i} \rangle + v_{\alpha,i}')] + \sigma_\alpha$$

Taking the average of both sides of the equation, we find

$$\frac{\partial \langle c_\alpha \rangle}{\partial t} = -\frac{\partial}{\partial x_i}(\langle c_\alpha \rangle \langle v_{\alpha,i} \rangle) - \frac{\partial}{\partial x_i}\left(\langle J_{\alpha,i} \rangle + \langle c_\alpha' v_{\alpha,i}' \rangle\right) + \langle \sigma_\alpha \rangle$$

If we make the additional assumption that the overall molar concentration *c* is constant, then we have:

$$\frac{\partial\langle c_\alpha\rangle}{\partial t} = -\frac{\partial}{\partial x_i}(c\langle y_\alpha\rangle\langle v_{\alpha,i}\rangle) - \frac{\partial}{\partial x_i}\left(\langle J_{\alpha,i}\rangle + c\langle y'_\alpha v'_{\alpha,i}\rangle\right) + \langle\sigma_\alpha\rangle \tag{11.7}$$

where y_α is the mole fraction of species *α*.

As in the case of momentum and energy transport, an additional term appears in this equation that is related to the transport of species *α* due to turbulent fluctuations in the fluid:

$$J^{(t)}_{\alpha,i} = c\langle y'_\alpha v'_{\alpha,i}\rangle \tag{11.8}$$

The order of magnitude of the concentration fluctuations y'_α can be estimated by using the same arguments used to obtain the velocity and temperature fluctuations:

$$\begin{aligned} y'_\alpha &\sim \langle y_\alpha(y+l)\rangle - \langle y_\alpha(y)\rangle \\ &\sim l\frac{\partial\langle y_\alpha\rangle}{\partial y} \end{aligned} \tag{11.9}$$

With this result, we can write

$$J^{(t)}_y = cl^2\left|\frac{\partial\langle v_x\rangle}{\partial y}\right|\frac{\partial\langle y_\alpha\rangle}{\partial y} \tag{11.10}$$

12 Boundary layer theory

12.1 Flowpast objects: Boundary layers

In this section, we will study the transfer of momentum, heat, and mass from an object immersed in a flowing fluid. To begin, we will first focus on the flow of a fluid past a stationary object. Consider a stationary object that is submerged in a moving fluid. Far from the object, the fluid has a uniform velocity U, which is constant in the direction perpendicular to the surface of the object; the velocity of the fluid parallel to the surface of the object may vary (i.e., U is a function of x and independent of y). In this region, viscosity plays very little role in the flow of the fluid.

At the surface of the object, the velocity of the fluid must be zero, due to the no-slip boundary condition. Near the surface of the object, viscosity plays a major role in determining the flow of the fluid. As a result, the object influences the flow of the fluid.

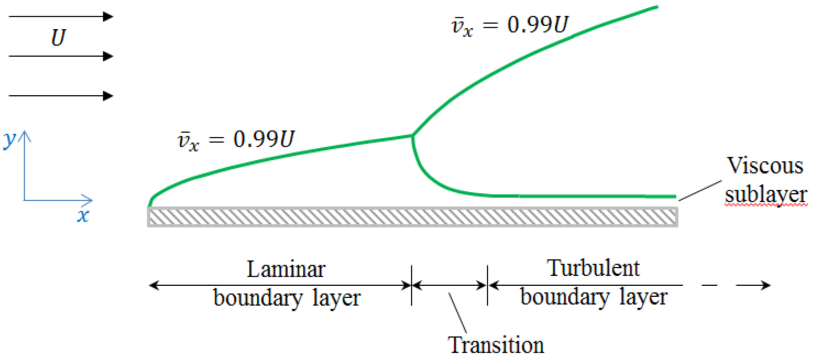


Figure 12.1: Boundary layer for flow past a flat plate.

We define a boundary layer, which divides the flow into an inner and outer region. At the surface of the boundary layer, the velocity of the fluid is 99% of the velocity of the fluid far from the object (i.e., at the edge of the boundary layer, $v_x = 0.99U$). Outside the boundary layer, the object has very little influence on the flow of the fluid; viscous forces are insignificant. Inside the boundary layer, viscous forces play a significant role, and the flow of the fluid is strongly influenced by the presence of the object.

A schematic drawing of the flow past a flat plate is shown in Fig. 12.1. At the leading edge of the plate, the boundary layer thickness is zero. The boundary layer thickness increases as we head further into the plate. In this part of the boundary layer, the flow is laminar. However, as we pass a critical distance into the plate, the flow inside the boundary layer suddenly becomes turbulent. This transition to turbulent flow is accompanied by a rapid increase in the thickness of the boundary layer. This general type of behavior is observed in flow past objects of almost any shape.

In this section, we will develop approximate methods for determining the flow past objects. In addition, we will also study the transfer of heat and mass from objects submerged in a flowing fluid.

12.2 Boundary layer theory

The general equations that govern the steady-state, two-dimensional flow of an incompressible fluid are (these can be derived from equations from the previous sections)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{12.1}$$

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) - \frac{\partial p}{\partial x} \tag{12.2}$$

$$\rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} = \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) - \frac{\partial p}{\partial y} \tag{12.3}$$

Equation (12.1) describes the conservation of mass, Eq. (12.2) is the balance equation for the y component of the momentum of the fluid, and Eq. (12.3) is the balance equation for the y-component of the momentum.

These equations are very difficult to solve, and, in general, the solution cannot be written in closed, analytical form. In most cases, "brute force" numerical methods must be used. However, if we use some physical insight, we can simplify these equations and rearrange them in a form where we can obtain accurate, analytical approximations to the problem.

12.2.1 Outer flow

Very far from the surface of the object, the velocity of the fluid does not vary in the y-direction. As a result, the viscous stresses in the fluid do not play an important role in determining the flow; consequently, the viscous terms in the equation of motion can be neglected. The resulting equation that governs the outer flow then simplifies to

$$\rho U \frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x} \tag{12.4}$$

12.2.2 Inner flow

Inside the boundary layer, viscous forces play a significant role. In order to simplify the equations of motion in this region, we need to determine the relative contribution of each of the terms in Eqs. (1.11)–(12.3). The relevant properties of the flow inside the boundary layer are: the bulk velocity of the fluid U, the fluid viscosity μ, the fluid density ρ. Another property is the thickness δ of the boundary layer, which sets the characteristic length in the y-direction. In addition, we assume there is a characteristic length L in the x-direction (e.g., length of the plate). We make the assumption that $\delta/L \ll 1$.

In order to determine the relative magnitudes of the properties of the fluid, it is convenient to work in dimensionless variables. We define dimensionless variables by scaling by the characteristic properties:

$$\begin{aligned} x^+ &= \frac{x}{L} \\ y^+ &= \frac{y}{\delta} \\ v_x^+ &= \frac{v_x}{U} \\ v_y^+ &= \frac{v_y}{V} \\ p^+ &= \frac{p}{\Pi} \end{aligned}$$

where V is a characteristic velocity in the y-direction, and Π is a characteristic pressure, both of which are presently unknown. Each of the dimensionless variables that we have defined are all of the same order of magnitude: they are all of order one. In addition, any derivatives of these dimensionless variables with other dimensionless variables are also expected to be of order one.

If we substitute the dimensionless variables into Eq. (1.11), we find

$$\begin{aligned} \frac{U}{L}\frac{\partial v_x^+}{\partial x^+} + \frac{V}{\delta}\frac{\partial v_y^+}{\partial y^+} &= 0 \\ \frac{\partial v_x^+}{\partial x^+} + \frac{V}{U}\frac{L}{\delta}\frac{\partial v_y^+}{\partial y^+} &= 0 \end{aligned}$$

Because we know that the dimensionless variables are all of order one, the only way we can satisfy this equation is if we have

$$V \sim U\frac{\delta}{L}$$

In other words, the vertical component of the velocity of the fluid inside the boundary layer is extremely small (of order δ/L) compared with the horizontal velocity of the fluid. This sets the characteristic velocity in the y-direction.

Now we express Eq. (12.2) in terms of dimensionless variables:

$$\begin{aligned} \frac{\rho U^2}{L} v_x^+ \frac{\partial v_x^+}{\partial x^+} + \frac{\rho U V}{\delta} v_y^+ \frac{\partial v_x^+}{\partial y^+} &= \frac{\mu U}{\delta^2}\left(\frac{\delta^2}{L^2}\frac{\partial^2 v_x^+}{\partial x^{+2}} + \frac{\partial^2 v_x^+}{\partial y^{+2}}\right) - \frac{\Pi}{L}\frac{\partial p^+}{\partial x^+} \\ v_x^+ \frac{\partial v_x^+}{\partial x^+} + \frac{V}{U}\frac{L}{\delta} v_y^+ \frac{\partial v_x^+}{\partial y^+} &= \frac{\mu}{\rho U \delta}\frac{L}{\delta}\left(\frac{\delta^2}{L^2}\frac{\partial^2 v_x^+}{\partial x^{+2}} + \frac{\partial^2 v_x^+}{\partial y^{+2}}\right) - \frac{\Pi}{\rho U^2}\frac{\partial p^+}{\partial x^+} \\ v_x^+ \frac{\partial v_x^+}{\partial x^+} + v_y^+ \frac{\partial v_x^+}{\partial y^+} &= \frac{\mu}{\rho U \delta}\frac{L}{\delta}\left(\frac{\delta^2}{L^2}\frac{\partial^2 v_x^+}{\partial x^{+2}} + \frac{\partial^2 v_x^+}{\partial y^{+2}}\right) - \frac{\Pi}{\rho U^2}\frac{\partial p^+}{\partial x^+} \end{aligned}$$

The first two terms on the left side of the equation represent convective momentum transport. Far from the surface, they dominate the flow (along with the pressure term). The first term on the right side of the equation represents the viscous contributions to the momentum transport. Far from the surface, these are negligible; however, in the boundary layer, these should contribute significantly to the momentum transport. Consequently, we expect:

$$\frac{\mu}{\rho U \delta}\frac{L}{\delta} \sim 1$$

$$\delta \sim L\left(\frac{\rho U L}{\mu}\right)^{-1/2}$$

This gives us an estimate of the boundary layer thickness. Here we see that δ varies inversely with the square root of the Reynolds number. From the assumption that $\delta/L \ll 1$, we find that the $\partial^2 v_x^+/\partial x^{+2}$ term is negligible compared to the $\partial^2 v_y^+/\partial y^{+2}$ term.

We expect that the contribution of the pressure term to be the same order of magnitude as the convective and viscous terms; therefore, we have

$$\frac{\Pi}{\rho U^2} \sim 1$$

$$\Pi \sim \rho U^2$$

From this order of magnitude analysis, we find that, inside the boundary layer, Eq. (12.2) can be approximated as

$$\rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} \approx \mu \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial p}{\partial x} \tag{12.5}$$

Now, we proceed to Eq. (12.3)

$$\begin{aligned}
\frac{\rho U V}{L} v_x^+ \frac{\partial v_y^+}{\partial x^+} + \frac{\rho V^2}{\delta} v_y^+ \frac{\partial v_y^+}{\partial y^+} &= \frac{\mu V}{\delta^2} \left(\frac{\delta^2}{L^2} \frac{\partial^2 v_y^+}{\partial x^{+2}} + \frac{\partial^2 v_y^+}{\partial y^{+2}} \right) - \frac{\Pi}{\delta} \frac{\partial p^+}{\partial y^+} \\
v_x^+ \frac{\partial v_y^+}{\partial x^+} + \frac{V}{U} \frac{L}{\delta} v_y^+ \frac{\partial v_y^+}{\partial y^+} &= \frac{\mu}{\rho U \delta} \frac{L}{\delta} \left(\frac{\delta^2}{L^2} \frac{\partial^2 v_y^+}{\partial x^{+2}} + \frac{\partial^2 v_y^+}{\partial y^{+2}} \right) - \frac{\Pi}{\rho U^2} \frac{L}{\delta} \frac{\partial p^+}{\partial y^+} \\
v_x^+ \frac{\partial v_y^+}{\partial x^+} + v_y^+ \frac{\partial v_y^+}{\partial y^+} &= \left(\frac{\delta^2}{L^2} \frac{\partial^2 v_y^+}{\partial x^{+2}} + \frac{\partial^2 v_y^+}{\partial y^{+2}} \right) - \frac{L}{\delta} \frac{\partial p^+}{\partial y^+}
\end{aligned}$$

From this equation, we note that the pressure term is much larger than any of the other terms. Therefore, we can approximate Eq. (12.3) as

$$\frac{\partial p}{\partial y} \approx 0 \tag{12.6}$$

In other words, the pressure is approximately constant in a vertical direction through the boundary layer. This means that the pressure inside the boundary layer can be obtained directly from the velocity U of the outer flow from Eq. (12.4).

To summarize the results of this subsection, the boundary layer equations for flow past a submerged object are

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{12.7}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} \approx \nu \frac{\partial^2 v_x}{\partial y^2} + U \frac{\partial U}{\partial x} \tag{12.8}$$

$$\frac{\partial p}{\partial y} \approx 0 \tag{12.9}$$

12.3 von Kármán momentum integral analysis

Although the boundary layer equations (see Eqs. (12.7)–(12.9)) are significantly simpler than the full equations of motion of the fluid (see Eqs. (12.1)–(12.3)), they are still quite difficult to solve. In many cases, however, we are not interested in the full, detailed solution of these equations (e.g., the local velocity of the fluid at every point in the system), but only in a few "global" properties (e.g., the drag of the fluid on the object). von Kármán developed an elegant method for using the boundary layer equations to estimate the drag force on an object submerged in a flowing fluid. This method is known as the "von Kármán momentum integral analysis".

We start by using the continuity equation, Eq. (12.7), to express the y-component of the velocity in terms of the x-component of the velocity:

$$\begin{aligned}
\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}&=0\\
\frac{\partial v_y}{\partial y}&=-\frac{\partial v_x}{\partial x}\\
v_y&=-\int_0^y dy'\frac{\partial v_x}{\partial x}
\end{aligned}\tag{12.10}$$

where we have used the fact that $v_y=0$ at $y = 0$. Therefore, if we have an expression for $v_x(y)$,, we can use the above equation to determine $v_y(y)$.

Now, we integrate the x-momentum equation across the boundary layer. That is, we integrate Eq. (12.8) with respect to y from 0 to $\delta(x)$. This yields:

$$\begin{aligned}
\int_0^\delta dy\left[v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right]&\approx\int_0^\delta dy\left[\nu\frac{\partial^2 v_x}{\partial y^2}+U\frac{\partial U}{\partial x}\right]\\
\int_0^\delta dy\left[v_x\frac{\partial v_x}{\partial x}-\left(\int_0^y dy'\frac{\partial v_x}{\partial x}\right)\frac{\partial v_x}{\partial y}\right]&\approx\int_0^\delta dy\left[\nu\frac{\partial^2 v_x}{\partial y^2}+U\frac{\partial U}{\partial x}\right]\\
-\nu\int_0^\delta dy\frac{\partial^2 v_x}{\partial y^2}&\approx-\int_0^\delta dy v_x\frac{\partial v_x}{\partial x}+\int_0^\delta dy\left(\int_0^y dy'\frac{\partial v_x}{\partial x}\right)\frac{\partial v_x}{\partial y}\\
&\quad+\int_0^\delta dy U\frac{\partial U}{\partial x}
\end{aligned}\tag{12.11}$$

The left side of Eq. (12.11) can be expressed in terms of the shear stress at the wall:

$$
\begin{aligned}
-\nu \int_0^\delta dy \frac{\partial^2 v_x}{\partial y^2} &= -\frac{1}{\rho}\int_0^\delta dy \frac{\partial}{\partial y}\left(\mu \frac{\partial v_x}{\partial y}\right) \\
&= -\frac{1}{\rho}\int_0^\delta dy \frac{\partial \tau_{yx}}{\partial y} \\
&= -\frac{1}{\rho}[\tau_{yx}(\delta) - \tau_{yx}(0)] \\
&= \frac{\tau_w}{\rho}
\end{aligned}
$$

where $\tau_w = \tau_{yx}(0)$. The second term on the right side of Eq. (12.11) can be simplified as follows:

$$
\begin{aligned}
\int_0^\delta dy \left(\int_0^y dy' \frac{\partial v_x}{\partial x}\right)\frac{\partial v_x}{\partial y} &= \left[\left(\int_0^y dy' \frac{\partial v_x}{\partial x}\right) v_x\right]_0^\delta - \int_0^\delta dy \frac{\partial v_x}{\partial x} v_x \\
&= v_x(\delta)\int_0^\delta dy' \frac{\partial v_x}{\partial x} - \int_0^\delta dy \frac{\partial v_x}{\partial x} v_x \\
&= U \int_0^\delta dy \frac{\partial v_x}{\partial x} - \int_0^\delta dy \frac{\partial v_x}{\partial x} v_x \\
&= \int_0^\delta dy \frac{\partial v_x}{\partial x}(U - v_x)
\end{aligned}
$$

where we have used the facts that $v_x(\delta) \approx U$ and $v_x(0) = 0$ (i.e., no-slip boundary condition).

Combining all these relations, we find that Eq. (12.11) can be written as

$$\begin{aligned}
\frac{\tau_w}{\rho} &\approx -\int_0^\delta dy v_x \frac{\partial v_x}{\partial x} + \int_0^\delta dy \frac{\partial v_x}{\partial x}(U - v_x) + \int_0^\delta dy U \frac{\partial U}{\partial x} \\
&\approx \int_0^\delta dy v_x \frac{\partial}{\partial x}(U - v_x) + \int_0^\delta dy \frac{\partial v_x}{\partial x}(U - v_x) + \int_0^\delta dy (U - v_x) \frac{\partial U}{\partial x} \\
&\approx \int_0^\delta dy \frac{\partial}{\partial x}[v_x(U - v_x)] + \frac{\partial U}{\partial x} U \int_0^\delta dy \left(1 - \frac{v_x}{U}\right) \\
&\approx \frac{\partial}{\partial x}\int_0^\delta dy [v_x(U - v_x)] + [v_x(\delta)(U - v_x(\delta))]\frac{\partial \delta}{\partial x} + \frac{\partial U}{\partial x} U \int_0^\delta dy \left(1 - \frac{v_x}{U}\right) \\
&\approx \frac{\partial}{\partial x}\left[U^2 \int_0^\delta dy \frac{v_x}{U}\left(1 - \frac{v_x}{U}\right)\right] + \frac{\partial U}{\partial x} U \int_0^\delta dy \left(1 - \frac{v_x}{U}\right)
\end{aligned} \tag{12.12}$$

where we have used the fact that $v_x(\delta) \approx U$.

To simplify the notation, we define the displacement thickness δ_1 and momentum thickness δ_2:

$$\begin{aligned}
\delta_1(x) &= \int_0^{\delta(x)} dy \left(1 - \frac{v_x}{U}\right) \\
\delta_2(x) &= \int_0^{\delta(x)} dy \frac{v_x}{U}\left(1 - \frac{v_x}{U}\right)
\end{aligned}$$

So finally we have

$$\frac{\tau_w(x)}{\rho} \approx \frac{\partial}{\partial x}[U^2(x)\delta_2(x)] + \delta_1(x) U(x) \frac{\partial U(x)}{\partial x} \tag{12.13}$$

If we knew the expression for the velocity profile as a function of y (i.e., $v_x(y)$), then we could substitute it into Eq. (12.13) to determine the shear stress on the wall. von Kármán suggested that instead of using the exact velocity profile, which is unknown before we solve the full boundary layer equations, we could use a guess for the velocity profile. The more accurate the guess for the velocity profile, the more accurate the estimate for the drag on the wall.

The boundary conditions of the flow can be used to get an idea of the general shape of the velocity profile:

$$\begin{aligned}
v_x &= 0 && \text{at} \quad y = 0 \\
v_x &= U && \text{at} \quad y = \delta \\
\frac{\partial v_x}{\partial y} &= 0 && \text{at} \quad y = \delta \\
\mu \frac{\partial^2 v_x}{\partial y^2} &= \frac{\partial p}{\partial x} && \text{at} \quad y = 0
\end{aligned}$$

The first boundary condition is just the no-slip condition at the surface of the object. The second and third boundary conditions arise from the fact that the velocity profile inside the boundary layer must smoothly approach that of the outer flow as y approaches δ, the boundary layer thickness. The final boundary condition follows from Eq. (12.8) and the facts $v_x = v_y = 0$ at the surface of the object (i.e., at $y = 0$).

The four boundary conditions give us four sets of constraints that can be applied to any guess we make for our velocity profile. This means that if we choose a velocity profile with several free parameters, the boundary conditions can be used to specify four of these parameters. For example, if we choose the following cubic polynomial to represent the velocity profile:

$$\frac{v_x}{U} = c_0 + c_1\frac{y}{\delta} + c_2\left(\frac{y}{\delta}\right)^2 + c_3\left(\frac{y}{\delta}\right)^3 \tag{12.14}$$

we can use the boundary conditions to set the vaules of the parameters c_0, c_1, c_2, and c_3.

12.3.1 Flow past a flat plate

In this subsection, we will apply the von Kármán integral analysis to uniform flow past a flat plate. In this case, the outer flow is U = const, and, therefore, the pressure gradient is zero (i.e., $\partial p/\partial x = 0$). If we choose to use the velocity profile given in Eq. (12.14), the coefficients can be determined by using the four boundary conditions to yield:

$$\frac{v_x}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{12.15}$$

Given that we know the velocity profile, we can determine the shear stress at the wall from Newton's law of viscosity:

$$\begin{aligned}\tau_w &= \tau_{yx}(0)\\ &= \mu\left.\frac{\partial v_x}{\partial y}\right|_{y=0}\\ &= \frac{3\mu}{2}\frac{U}{\delta}\end{aligned} \tag{12.16}$$

In addition, the momentum thickness δ_2 can be determined:

$$\begin{aligned}\delta_2(x) &= \int_0^{\delta(x)} dy\frac{v_x}{U}\left(1-\frac{v_x}{U}\right)\\ &= \int_0^{\delta(x)} dy\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1-\frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\\ &= \delta\int_0^1 d\xi\left(\frac{3}{2}\xi - \frac{1}{2}\xi^3\right)\left(1-\frac{3}{2}\xi+\frac{1}{2}\xi^3\right)\\ &= \frac{39}{280}\delta(x)\end{aligned} \tag{12.17}$$

where $\xi = y/\delta$.

Substituting both these relations into the von Kármán integral relation, Eq. (12.13), we find a differential equation for the boundary layer thickness $\delta(x)$

$$\begin{aligned}
\frac{\tau_w}{\rho} &\approx \frac{\partial}{\partial x}[U^2\delta_2] + \frac{\partial U}{\partial x}U\delta_1 \\
\frac{3\mu}{2\rho}\frac{U}{\delta(x)} &\approx U^2\frac{\partial \delta_2(x)}{\partial x} = \frac{39}{280}U^2\frac{\partial \delta(x)}{\partial x} \\
\delta(x)\frac{\partial \delta(x)}{\partial x} &\approx \frac{140\nu}{13U} \\
\frac{\partial \delta^2(x)}{\partial x} &\approx \frac{280\nu}{13U}
\end{aligned} \tag{12.18}$$

This equation can be integrated to give

$$\delta(x) \approx \sqrt{\frac{280}{13}}\left(\frac{U}{\nu x}\right)^{-1/2} \tag{12.19}$$

where we have used the fact that $\delta = 0$ at $x = 0$. This should be compared with the exact result:

$$\delta(x) \approx 5.0\left(\frac{U}{\nu x}\right)^{-1/2} \tag{12.20}$$

Now that we know how the boundary layer thickness varies with the position on the plate, we can determine the local shear stress on the wall

$$\begin{aligned}
\tau_w(x) &= \frac{3\mu}{2}\frac{U}{\delta(x)} \\
&= \frac{3\mu U}{2}\sqrt{\frac{13}{280}}\left(\frac{U}{\nu x}\right)^{1/2} \\
&= \sqrt{\frac{117}{1120}}\mu U\left(\frac{U}{\nu x}\right)^{1/2}
\end{aligned} \tag{12.21}$$

The total force F on the wall is

$$\begin{aligned}
F &= W\int_0^L dx\tau_w(x) \\
&= W\int_0^L dx\sqrt{\frac{117}{1120}}\mu U\left(\frac{U}{\nu x}\right)^{1/2} \\
&= W\sqrt{\frac{117}{1120}}\mu U\left(\frac{U}{\nu}\right)^{1/2}2L^{1/2} \\
&= W\sqrt{\frac{117}{280}}\mu U\left(\frac{UL}{\nu}\right)^{1/2}
\end{aligned} \tag{12.22}$$

If we define a drag coefficient C_D and a Reynolds number as

$$C_D = \frac{F/(LW)}{\rho U^2/2}$$
$$\mathrm{Re} = \frac{UL}{\nu}$$

then we find the relation

$$C_D = \sqrt{\frac{117}{70}}\mathrm{Re}^{-1/2} \tag{12.23}$$

The exact solution to the boundary layer equations gives a prefactor of 1.328, which differs from our approximate result by about 3%. So our simple analysis gives a fairly accurate prediction of the drag on the plate.

13 Boundary layers in energy and mass transport

13.1 Transport of energy

The boundary layer method that was developed in the previous section to determine the flow across an object submerged in a fluid can be extended to study heat transfer. If we assume that the temperature at the surface of the object T_w differs from the temperature of the bulk fluid $T_\infty,$ then there will be heat flow between the object and the fluid. The energy balance equation for an incompressible fluid in a steady state, two-dimensional flow is given by

$$\rho C_v v_j \frac{\partial T}{\partial x_j} = k \frac{\partial}{\partial x_i} \frac{\partial T}{\partial x_i} + \tau_{ji} \frac{\partial v_i}{\partial x_j} \tag{13.1}$$

The final term on the left side of the equation represents the conversion of kinetic energy to thermal energy. This term is typically much smaller than the other terms in the equation, and so we neglect it. Therefore, the equations that we deal with for a two-dimensional flow are given by

$$\begin{aligned} v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} &= \frac{k}{\rho C_v} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \\ v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} &= \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \end{aligned} \tag{13.2}$$

where $\alpha = k/(\rho C_v)$ is the thermal diffusivity.

We can perform the same order of magnitude analysis for the energy equation as we did for the momentum equation. In this case, however, we need to define a dimensionless temperature T^+ as

$$T^+ = \frac{T - T_w}{T_\infty - T_w}$$

Expressing the energy equation in dimensionless form leads to

$$\frac{U}{L} v_x^+ \frac{\partial T^+}{\partial x^+} + \frac{V}{\delta} v_y^+ \frac{\partial T^+}{\partial y^+} = \frac{\alpha}{\delta^2} \left(\frac{\delta^2}{L^2} \frac{\partial^2 T^+}{\partial x^{+2}} + \frac{\partial^2 T^+}{\partial y^{+2}} \right) \tag{13.3}$$

$$v_x^+ \frac{\partial T^+}{\partial x^+} + \frac{V}{U} \frac{L}{\delta} v_y^+ \frac{\partial T^+}{\partial y^+} = \frac{\alpha}{U\delta} \frac{L}{\delta} \left(\frac{\delta^2}{L^2} \frac{\partial^2 T^+}{\partial x^{+2}} + \frac{\partial^2 T^+}{\partial y^{+2}} \right) \tag{13.4}$$

$$v_x^+ \frac{\partial T^+}{\partial x^+} + v_y^+ \frac{\partial T^+}{\partial y^+} = \frac{\alpha}{U\delta} \frac{L}{\delta} \left(\frac{\delta^2}{L^2} \frac{\partial^2 T^+}{\partial x^{+2}} + \frac{\partial^2 T^+}{\partial y^{+2}} \right) \tag{13.5}$$

We find that the first term on the right side of the equation is negligible compared to the second term; therefore, we find that the boundary layer equation for the balance of energy is

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \approx \alpha \frac{\partial^2 T}{\partial y^2} \tag{13.6}$$

13.1.1 Approximate integral analysis

Equation (13.6) is a partial differential equation for the temperature T in the fluid, and, in general, it can not be solved analytically. So, we will extend the von Kármán integral analysis of the previous subsection to apply to the transport of energy. This will allow us to obtain an estimate of the heat transfer coefficient from the object.

The first step is to integrate Eq. (13.6) across the *thermal* boundary layer of the system. The thickness of this boundary layer $\delta_t(x)$ is defined as the distance from the surface of the object at which $T^+ = 0.99$. This thickness is not necessarily the same as the momentum boundary layer thickness (i.e., $\delta_t(x) \neq \delta(x)$). Performing this integration, we find

$$\begin{aligned}
\int_0^{\delta_t} dy \left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} \right] &\approx \int_0^{\delta_t} dy \alpha \frac{\partial^2 T}{\partial y^2} \\
\int_0^{\delta_t} dy v_x \frac{\partial T}{\partial x} - \int_0^{\delta_t} dy \left(\int_0^{y} dy' \frac{\partial v_x}{\partial x} \right) \frac{\partial T}{\partial y} &\approx \alpha \left[\frac{\partial T(\delta_t)}{\partial y} - \frac{\partial T(0)}{\partial y} \right] \\
\int_0^{\delta_t} dy v_x \frac{\partial T}{\partial x} - \left(\int_0^{\delta_t} dy' \frac{\partial v_x}{\partial x} \right) T(\delta_t) + \int_0^{\delta_t} dy \frac{\partial v_x}{\partial x} T &\approx -\frac{k}{\rho C_v} \frac{\partial T(0)}{\partial y} \\
\int_0^{\delta_t} dy v_x \frac{\partial T}{\partial x} + \int_0^{\delta_t} dy \frac{\partial v_x}{\partial x} [T - T_\infty] &\approx -\frac{k}{\rho C_v} \frac{\partial T(0)}{\partial y} \\
\int_0^{\delta_t} dy \frac{\partial}{\partial x} [v_x (T - T_\infty)] &\approx \frac{q_w}{\rho C_v}
\end{aligned}$$

where we have used the fact that $T \approx T_\infty$ and $\partial T / \partial y \approx 0$ at $y = \delta_t$. This equation can be rearranged to yield

$$\begin{aligned}
\frac{q_w}{\rho C_v} &\approx \frac{\partial}{\partial x} \left[U (T_\infty - T_w) \int_0^{\delta_t} dy \frac{v_x}{U} \left(\frac{T - T_\infty}{T_\infty - T_w} \right) \right] \\
-\frac{q_w / (T_\infty - T_w)}{\rho C_v} &\approx \frac{\partial}{\partial x} \left[U \int_0^{\delta_t} dy \frac{v_x}{U} \left(1 - \frac{T - T_w}{T_\infty - T_w} \right) \right]
\end{aligned} \tag{13.7}$$

If we knew the temperature profile in the system, we could use Eq. (13.7) to determine the rate of heat transfer between the object and the surrounding fluid. We can guess a form for the temperature profile, as we did for the velocity profile. The better the guess, the better the estimate for the rate of heat transfer. To aid in developing a good guess, we note the following boundary conditions for the temperature profile

$$\begin{aligned}
T &= T_w && \text{at} \quad y = 0 \\
T &= T_\infty && \text{at} \quad y = \delta \\
\frac{\partial T}{\partial y} &= 0 && \text{at} \quad y = \delta \\
\frac{\partial^2 T}{\partial y^2} &= 0 && \text{at} \quad y = 0
\end{aligned}$$

The last boundary conditions follows from Eq. (13.6) and the fact that $v_x = v_y = 0$ when $y = 0$.

13.1.2 Flow across a flat plate

In this subsection, we determine the rate of heat transfer between a fluid at temperature T_∞, flowing at velocity U across a flat plate at temperature T_w. A simple estimate for the temperature profile that statisfies all four of the boundary conditions is:

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad (13.8)$$

With this estimate for the temperature, we can determine the heat flux q_w from the surface of the plate:

$$\begin{aligned} q_w &= -k \left.\frac{\partial T}{\partial y}\right|_{y=0} \\ &= -\frac{3k}{2\delta_t}(T_\infty - T_w) \\ \frac{q_w}{T_\infty - T_w} &= -\frac{3k}{2\delta_t} \end{aligned} \tag{13.9}$$

If we substitute Eq. (13.8) into Eq. (13.7), we find

$$\begin{aligned} -\frac{q_w/(T_\infty - T_w)}{\rho C_v} &\approx \frac{\partial}{\partial x}\left[U\int_0^{\delta_t} dy \frac{v_x}{U}\left(1 - \frac{T - T_w}{T_\infty - T_w}\right)\right] \\ \frac{3k}{2\delta_t \rho C_v} &\approx \frac{\partial}{\partial x}\left\{U\int_0^{\delta_t} dy\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\left(\frac{y}{\delta_t}\right) + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\right\} \\ \frac{3\alpha}{2\delta_t} &\approx \frac{\partial}{\partial x}\left\{U\delta_t\int_0^1 d\xi\left[\frac{3}{2}\left(\frac{\delta_t}{\delta}\right)\xi - \frac{1}{2}\left(\frac{\delta_t}{\delta}\right)^3\xi^3\right]\left[1 - \frac{3}{2}\xi + \frac{1}{2}\xi^3\right]\right\} \\ &\approx \frac{\partial}{\partial x}\left\{U\delta_t\left[\frac{3}{20}\left(\frac{\delta_t}{\delta}\right) - \frac{3}{280}\left(\frac{\delta_t}{\delta}\right)^3\right]\right\} \end{aligned} \tag{13.10}$$

The thermal boundary layer thickness is proportional to the momentum boundary layer thickness; therefore, we can write

$$\delta_t(x) = K\delta(x)$$

where *K* is a constant that we will determine later. If we substitute this relation into Eq. (13.10), we find

$$\begin{aligned} \frac{3\alpha}{2\delta_t} &\approx \frac{3}{20}K\left(1 - \frac{K^2}{14}\right)U\frac{\partial\delta_t(x)}{\partial x} \\ \frac{\alpha}{K\delta(x)} &\approx \frac{K^2}{10}\left(1 - \frac{K^2}{14}\right)U\frac{\partial\delta(x)}{\partial x} \\ \frac{\alpha}{K}\sqrt{\frac{13}{280}}\left(\frac{U}{\nu x}\right)^{1/2} &\approx \frac{K^2}{10}\left(1 - \frac{K^2}{14}\right)U\sqrt{\frac{280}{13}}\left(\frac{\nu}{U}\right)^{1/2}\frac{x^{-1/2}}{2} \\ K^3\left(1 - \frac{K^2}{14}\right) &\approx \frac{13}{14}\frac{\alpha}{\nu} = \frac{13}{14}\frac{1}{\mathrm{Pr}} \end{aligned} \tag{13.11}$$

where $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number, and we have substituted the expression for $\delta(x)$ given in Eq. (12.19). For a given Prandtl number, Eq. (13.11) can be solved to determine *K*, the ratio of the thermal boundary layer thickness to the momentum boundary layer thickness.

For most fluids, the Prandtl number ranges from about order 1 to up to order 103. In this range, $K^2/14 \ll 1$, and Eq. (13.11) can be approximately solved explicitly for *K*:

$$K \approx \left(\frac{13}{14}\right)^{1/3}\mathrm{Pr}^{-1/3} + \cdots$$

The thickness of the thermal boundary layer is given by

$$\begin{aligned}\delta_t(x) &= K\delta(x)\\ &= K\sqrt{\frac{280}{13}}\left(\frac{U}{\nu x}\right)^{-1/2}\\ &\approx \sqrt{\frac{280}{13}}\left(\frac{13}{14}\right)^{1/3}\mathrm{Pr}^{-1/3}\left(\frac{U}{\nu x}\right)^{-1/2}\end{aligned} \tag{13.12}$$

The local heat flux q_w from the plate can be determined from $\delta t(x)$ by using Eq. (13.9)

$$\begin{aligned}-\frac{q_w}{T_\infty - T_w} &= \frac{3k}{2\delta_t(x)}\\ \frac{q_w/(T_w - T_\infty)}{k} &= \frac{3}{2K}\sqrt{\frac{13}{280}}\left(\frac{U}{\nu x}\right)^{1/2}\\ &= \frac{3}{2}\sqrt{\frac{13}{280}}\left(\frac{14}{13}\right)^{1/3}\mathrm{Pr}^{1/3}\left(\frac{U}{\nu x}\right)^{1/2}\end{aligned} \tag{13.13}$$

The total rate of heat lost Q over the surface of the entire plate can be determined by integrating the heat flux

$$\begin{aligned}Q &= W\int_0^L dx q_w(x)\\ &= Wk(T_w - T_\infty)\int_0^L dx\frac{3k}{2}\sqrt{\frac{13}{280}}\left(\frac{14}{13}\right)^{1/3}\mathrm{Pr}^{1/3}\left(\frac{U}{\nu x}\right)^{1/2}\\ \frac{Q/(T_w - T_\infty)}{kW} &= 3\sqrt{\frac{13}{280}}\left(\frac{14}{13}\right)^{1/3}\mathrm{Pr}^{1/3}\left(\frac{UL}{\nu}\right)^{1/2}\\ &= \sqrt{\frac{117}{280}}\left(\frac{14}{13}\right)^{1/3}\mathrm{Pr}^{1/3}\mathrm{Re}^{1/2}\end{aligned} \tag{13.14}$$

The average heat transfer coefficient h is

$$\frac{Q}{WL} = h(T_w - T_\infty)$$

If we define the Nusselt number Nu as

$$\mathrm{Nu} = \frac{hL}{k}$$

then we find the relation

$$\mathrm{Nu} = \sqrt{\frac{117}{280}}\left(\frac{14}{13}\right)^{1/3}\mathrm{Pr}^{1/3}\mathrm{Re}^{1/2} \tag{13.15}$$

13.2 Transport of mass

Let's consider the problem where we have a fluid with species A dissolved at concentration $c_{A,\infty}$ flowing at velocity U past an object. At the surface of the object, the concentration of A is $c_{A,w}$. In this situation, there will be transfer of species A between the surface of the object and the bulk fluid.

In direct analogy with the boundary layer equation for energy, we can develop a boundary layer equation for the transport of A:

$$v_x \frac{\partial c_A}{\partial x} + v_y \frac{\partial c_A}{\partial y} \approx \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial y^2} \tag{13.16}$$

To determine the rate of transfer of species *A*, we can extend the von Kármán integral analysis to Eq. (13.16). In this case, we integrate the mass transfer equation across the *concentration* boundary layer of the system. The thickness δ_c of this boundary layer is different from both the momentum and thermal boundary layers (i.e., $\delta_c(x) \neq \delta(x)$ and $\delta_c(x) \neq \delta_t(x)$). Integrating Eq. (13.16), we find

$$\begin{aligned}
\int_0^{\delta_c} dy \left[v_x \frac{\partial c_A}{\partial x} + v_y \frac{\partial c_A}{\partial y} \right] &\approx \int_0^{\delta_c} dy \mathcal{D}_{AB} \frac{\partial^2 c_A}{\partial y^2} \\
\int_0^{\delta_c} dy v_x \frac{\partial c_A}{\partial x} - \int_0^{\delta_c} dy \left(\int_0^{y} dy' \frac{\partial v_x}{\partial x} \right) \frac{\partial c_A}{\partial y} &\approx \mathcal{D}_{AB} \left[\frac{\partial c_A(\delta_c)}{\partial y} - \frac{\partial c_A(0)}{\partial y} \right] \\
\int_0^{\delta_c} dy v_x \frac{\partial c_A}{\partial x} - \left(\int_0^{\delta_c} dy' \frac{\partial v_x}{\partial x} \right) c_A(\delta_c) + \int_0^{\delta_c} dy \frac{\partial v_x}{\partial x} c_A &\approx -\mathcal{D}_{AB} \frac{\partial c_A(0)}{\partial y} \\
\int_0^{\delta_c} dy v_x \frac{\partial c_A}{\partial x} + \int_0^{\delta_c} dy \frac{\partial v_x}{\partial x} [c_A - c_{A,\infty}] &\approx -\mathcal{D}_{AB} \frac{\partial c_A(0)}{\partial y} \\
\int_0^{\delta_c} dy \frac{\partial}{\partial x} [v_x (c_A - c_{A,\infty})] &\approx N_{A,w}
\end{aligned}$$

where we have used the fact that $c_A(\delta_c) = c_{A,\infty}$ and $\partial c_A / \partial y = 0$ at $y = \delta_c$. $N_{A,w}$ is the flux of *A* at the wall; we have assumed that Fick's law is applicable. The above equation can be rearranged to yield

$$\begin{aligned}
N_{A,w} &\approx \frac{\partial}{\partial x} \left[U (c_{A,\infty} - c_{A,w}) \int_0^{\delta_t} dy \frac{v_x}{U} \left(\frac{c_A - c_{A,\infty}}{c_{A,\infty} - c_{A,w}} \right) \right] \\
-\frac{N_{A,w} / (c_{A,\infty} - c_{A,w})}{\mathcal{D}_{AB}} &\approx \frac{\partial}{\partial x} \left[U \int_0^{\delta_t} dy \frac{v_x}{U} \left(1 - \frac{c_A - c_{A,w}}{c_{A,\infty} - c_{A,w}} \right) \right]
\end{aligned}$$

This is the von Kármán integral for mass transport.

The associated boundary conditions for the concentration profile are

$$\begin{aligned}
c_A &= c_{A,w} & &\text{at} \quad y = 0 \\
c_A &= c_{A,\infty} & &\text{at} \quad y = \delta \\
\frac{\partial c_A}{\partial y} &= 0 & &\text{at} \quad y = \delta \\
\frac{\partial^2 c_A}{\partial y^2} &= 0 & &\text{at} \quad y = 0
\end{aligned}$$

The last boundary conditions follows from Eq. (13.16) and the fact that $v_x = v_y = 0$ when $y = 0$.

We can determine the dependence of the mass transfer coefficient on the bulk flow rate of the fluid, using an analysis similar to the one used to determine the heat transfer coefficient.

14 Analogies in Momentum, Heat, and Mass Transfer

14.1 Introduction

In the previous sections, we have seen the equations that govern the transport of momentum, energy, and mass. If we look more closely at these equations, we can see strong analogies between the three types of transport.

14.2 Comparison of diffusive transport

To begin with, we will compare diffusive transport. Let's first look at the change in the velocity profile around a plate immersed in an initially stationary liquid. At time $t = 0$, the plate is instantaneously accelerated to a velocity V. We are interested in how the velocity profile changes as a function of time. For this problem, the momentum balance equation reduces to:

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{14.1}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, which has units of length squared over time (e.g., $\mathrm{m^2\,s^{-1}}$). The boundary conditions are that $v_x = V$ at $y = 0$ and $v_x = 0$ as $y \to \infty$. The initial condition is that $v_x = 0$ when $t = 0$. The solution of this equation is given by

$$\begin{aligned} v_x(y,t) &= V\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] \\ \frac{v_x(y,t)}{V} &= 1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) \end{aligned} \tag{14.2}$$

where erf is the error function, which is defined as:

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}$$

Now, let's look at the change in the temperature of a plate that is suddenly immersed in a pool of liquid initially at a uniform temperature T_0. The temperature of the plate is kept constant at T_1. If there is no convection in the fluid (i.e., all heat transport is due to conduction), then the governing equation is

$$\rho C_v \frac{\partial T}{\partial t} = k \frac{\partial^2 c_A}{\partial y^2} \tag{14.3}$$

where $\alpha = k/(\rho C_v)$ is the thermal diffusivity, which also has units of length squared over time. The boundary conditions are $T = T_1$ at $y = 0$, and $T = T_0$ as $y \to \infty$. The initial condition is $T = T_0$ when $t = 0$. The solution of this equation is given by

$$\begin{aligned} T(y,t) &= T_0 + (T_1 - T_0)\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right)\right] \\ \frac{T(y,t) - T_0}{T_1 - T_0} &= 1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\alpha t}}\right) \end{aligned} \tag{14.4}$$

Finally, let's look at the change in the concentration profile of a dye *A* about a plate (which is saturated with the dye) that is suddenly immersed in a pool of pure liquid *B*. The concentration of *A* on the plate remains constant at c_{A0}. The governing equation for this problem is

$$\frac{\partial c_A}{\partial t} = \mathcal{D}_{AB}\frac{\partial^2 c_A}{\partial y^2} \tag{14.5}$$

where $\mathcal{D}_{AB}$ is the diffusion coefficient of species *A* in *B*, which again has units of length squared over time. The boundary conditions are $c_A = c_{A0}$ at $y = 0$, and $c_A = 0$ as $y \to \infty$. The initial condition is $c_A = 0$ when *t* = 0. The solution of this equation is given by

$$\begin{aligned} c_A(y,t) &= c_{A0}\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{D_{AB}t}}\right)\right] \\ \frac{c_A(y,t)}{c_{A0}} &= 1 - \operatorname{erf}\left(\frac{y}{2\sqrt{D_{AB}t}}\right) \end{aligned} \tag{14.6}$$

If we compare Eqs. (14.2), (14.4), and (14.6), we can see that they are essential identical, if we make the identification $\nu \leftrightarrow \alpha \leftrightarrow \mathcal{D}_{AB}$. In the first case, momentum diffuses from the plate into the bulk fluid; in the second case, thermal energy diffuses into the bulk fluid, while in the third case, species *A* into the bulk fluid.

As time progresses, the property (i.e., momentum, heat, or mass) slowly diffuses into the bulk of the system. The distance *δ* from the wall at which this property is significantly different from the bulk value varies with time as:

$$\begin{aligned} &\delta \sim \sqrt{\nu t} && \text{for momentum} \\ &\delta \sim \sqrt{\alpha t} && \text{for energy} \\ &\delta \sim \sqrt{\alpha t} && \text{for mass} \end{aligned}$$

The relative rate at which heat diffuses with respect to momentum is given by the Prandtl number

$$\mathrm{Pr} \equiv \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

The relative rate at which mass diffuses with respect to momentum is given by the Schmidt number

$$\mathrm{Sc} \equiv \frac{\nu}{\mathcal{D}_{AB}} = \frac{\mu}{\rho \mathcal{D}_{AB}}$$

14.3 Reynolds analogy

In the previous subsection, we compared the diffusive transport of momentum, heat, and mass. In this subsection, we will compare the transport of these properties under turbulent flow conditions. In the fully turbulent region, the turbulent stresses dominate the momentum transport and the eddy thermal conductivity dominates heat transport. In this situation, the momentum balance is approximately given by

$$\begin{aligned} \tau_{yx}^{(t)} &\approx \tau_w \\ \rho l^2 \left(\frac{\partial \langle v_x \rangle}{\partial y}\right)^2 &\approx \tau_w \end{aligned} \tag{14.7}$$

The energy balance is approximately given by

$$\begin{aligned} q^{(t)} &\approx q_w \\ -\rho C_v l^2 \left|\frac{\partial \langle v_x \rangle}{\partial y}\right| \left(\frac{\partial \langle T \rangle}{\partial y}\right) &\approx q_w \end{aligned} \tag{14.8}$$

Dividing the energy balance by the momentum balance yields

$$\begin{aligned} C_v \frac{\partial \langle T \rangle / \partial y}{\partial \langle v_x \rangle / \partial y} &= -\frac{q_w}{\tau_w} \\ \frac{\partial \langle T \rangle}{\partial \langle v_x \rangle} &= -\frac{q_w}{C_v \tau_w} \end{aligned} \tag{14.9}$$

If we assume that the entire fluid can be considered turbulent, even near the wall, then this equation can be integrated from the bulk (where the temperature is T_∞, and the velocity is *V*) to the wall (where the temperature is T_w, and the velocity is zero)

$$T_\infty - T_w = -\frac{q_w}{C_v \tau_w} V \tag{14.10}$$

The heat transfer coefficient *h* on the wall is defined by the relation

$$q_w \equiv h(T_w - T_\infty) \tag{14.11}$$

If we then introduce the Nusselt number

$$\mathrm{Nu} \equiv \frac{hL}{k}$$

where *L* is some characteristic length of the system, such as a pipe diameter, the Reynolds number

$$\mathrm{Re} \equiv \frac{L\rho V}{\mu},$$

and the friction factor:

$$f = \frac{2\tau_w}{\rho V^2}$$

then we can rearrange Eq. (14.10) to the form

$$\frac{\mathrm{Nu}}{\mathrm{PrRe}} = \frac{f}{2} \tag{14.12}$$

This relation is known as Reynolds analogy and has been found to agree well with experimental data for systems where $\mathrm{Pr} \approx 1$. It allows us to predict heat transfer coefficients from knowledge of the flow behavior of a system.

The Reynolds analogy provides a similar expression for mass transfer. If we consider a component A in a system that is diffusing from the bulk fluid, which is at concentration $c_{A,\infty}$, to a surface surface, which is at concentration $c_{A,w}$, then the flux of A is given by

$$N_A = k_c(c_{A,w} - c_{A,\infty}) \tag{14.13}$$

where k_c is the mass transfer coefficient. Then the Reynolds analogy for mass transfer is

$$\frac{\mathrm{Sh}}{\mathrm{ScRe}} = \frac{f}{2} \tag{14.14}$$

where Sh is the Sherwood number, which is defined as

$$\mathrm{Sh} = \frac{k_c L}{\mathcal{D}_{AB}} \tag{14.15}$$

Similarly to the case of heat transfer, this relationship is only accurate when $\mathrm{Sc} \approx 1$.

14.4 Prandtl analogy

In the development of the Reynolds analogy, we assumed that the turbulent core extended all the way down to the wall. However, at the wall, viscous forces dominate and turbulent eddies are almost nonexistent. Prandtl improved upon the previous derivation by accounting for the laminar sublayer.

In the laminar sublayer, the momentum transfer is governed primarily by viscous stress and can be written as

$$\mu \frac{\partial \langle v_x \rangle}{\partial y} \approx \tau_w \tag{14.16}$$

Heat transfer is dominated by conduction, and the energy balance is given by

$$-k \frac{\partial \langle T \rangle}{\partial y} = q_w \tag{14.17}$$

Dividing the energy balance equation by the momentum balance equation, we find

$$\begin{aligned} \frac{k}{\mu} \frac{\partial \langle T \rangle / \partial y}{\partial \langle v_x \rangle / \partial y} &= -\frac{q_w}{\tau_w} \\ \frac{\partial \langle T \rangle}{\partial \langle v_x \rangle} &= -\frac{q_w}{\tau_w} \frac{\mu}{k} \end{aligned} \tag{14.18}$$

Integrating this equation from the edge of the laminar sublayer $y = \xi$ to the wall $y = 0$, we find

$$\begin{aligned} \langle T(\xi) \rangle - \langle T(0) \rangle &= -\frac{q_w}{\tau_w} \frac{\mu}{k} \left(\langle v_x(\xi) \rangle - \langle v_x(0) \rangle \right) \\ &= -\frac{q_w}{\tau_w} \frac{\mu}{k} \langle v_x(\xi) \rangle \end{aligned} \tag{14.19}$$

In order to account for the turbulent core, Prandtl used Eq. (14.9). However, instead of integrating this equation from the bulk all the way to the wall, he integrated it from the bulk to the edge of the laminar sublayer:

$$T_\infty - \langle T(\xi) \rangle = -\frac{q_w}{C_v \tau_w} \left(U - \langle v_x(\xi) \rangle \right) \tag{14.20}$$

The temperature at the edge of the laminar sublayer $\langle T(\xi) \rangle$ is unknown; however, it can be eliminated by adding Eqs. (14.19) and (14.20) together:

$$\begin{aligned} T_\infty - T_w &= -\frac{q_w}{\tau_w} \frac{\mu}{k} \langle v_x(\xi) \rangle - \frac{q_w}{C_v \tau_w} (U - \langle v_x(\xi) \rangle) \\ &= -\frac{q_w}{C_v \tau_w} U \left[1 + \left(\frac{\mu C_v}{k} - 1 \right) \frac{\langle v_x(\xi) \rangle}{U} \right] \\ \frac{\text{Nu}}{\text{PrRe}} &= \frac{f/2}{1 + 5(\text{Pr} - 1)\sqrt{f/2}} \end{aligned} \tag{14.21}$$

where we have assume that the edge of the laminar sublayer is at $\xi = 5\mu/\sqrt{\tau_w \rho}$ (see Sec. 10.4 on the "universal velocity profile"). This relation is known as Prandtl's analogy.

By taking into account the buffer layer, von Karman developed the following relation:

$$\frac{\mathrm{Nu}}{\mathrm{PrRe}} = \frac{f/2}{1 + 5(\mathrm{Pr} - 1 + \ln(1 + (\mathrm{Pr} - 1)5/6))\sqrt{f/2}} \tag{14.22}$$

14.5 Chilton-Colburn analogy

An alternate approach to extending the range of applicability of the Reynolds analogy was taken by Chilton and Colburn. By comparing the boundary layer analysis for momentum transfer from a flat plate (see Eq. (12.23)) to that for heat transfer from a flat plate (see Eq. (13.15)), they find that

$$\frac{\mathrm{Nu}}{\mathrm{RePr}}\mathrm{Pr}^{2/3} \approx \frac{C_D}{2} \tag{14.23}$$

This relationship is quite similar to the Reynolds analogy, with the exception of the dependence on the Prandtl number. Even though it was developed for the flat plate geometry, empirically, it has been found to work well for a wide range of geometries and a wide range of Prandtl numbers $0.6 < \mathrm{Pr} < 100$.

In a similar manner, the Chilton-Colburn analogy for mass transfer can be developed as

$$\frac{\mathrm{Sh}}{\mathrm{ReSc}}\mathrm{Sc}^{2/3} \approx \frac{C_D}{2} \tag{14.24}$$

This relationship is found to work well for a wide range of geometries for $0.6 < \mathrm{Sc} < 2500$.

Note that these two equations can be combined to give a relationship between heat and mass transfer coefficients

$$\begin{aligned} \frac{\mathrm{Nu}}{\mathrm{Pr}}\mathrm{Pr}^{2/3} &\approx \frac{\mathrm{Sh}}{\mathrm{Sc}}\mathrm{Sc}^{2/3} \\ \frac{h}{\rho C_p k_c} &\approx \left(\frac{\mathrm{Sc}}{\mathrm{Pr}}\right)^{2/3} \end{aligned} \tag{14.25}$$

15 Interphase mass transfer

15.1 Introduction

Many separation processes involve the transfer of a species from one phase to another. For example, liquid-liquid extraction involves the transfer of a component between two immiscible liquid phases. Distillation involves the transfer of a more volatile component from the liquid phase to the vapor phase. In order to properly design and size a separation process, we need to know the rate at which mass transfer between the phases occurs.

15.2 Individual mass-transfer coefficients

Consider the adsorption of a species A from a gas stream to a liquid solution. The partial pressure of A in the bulk gas phase is p_{AG}, and the concentration of the A in the bulk liquid phase is c_{AL}. Typical concentration profiles are shown in Fig. 15.1. The driving force for mass transfer is the chemical potential difference of species A between the gas and liquid phases. Molecules have a tendency to diffuse from regions of high chemical potential to regions of low chemical potential.

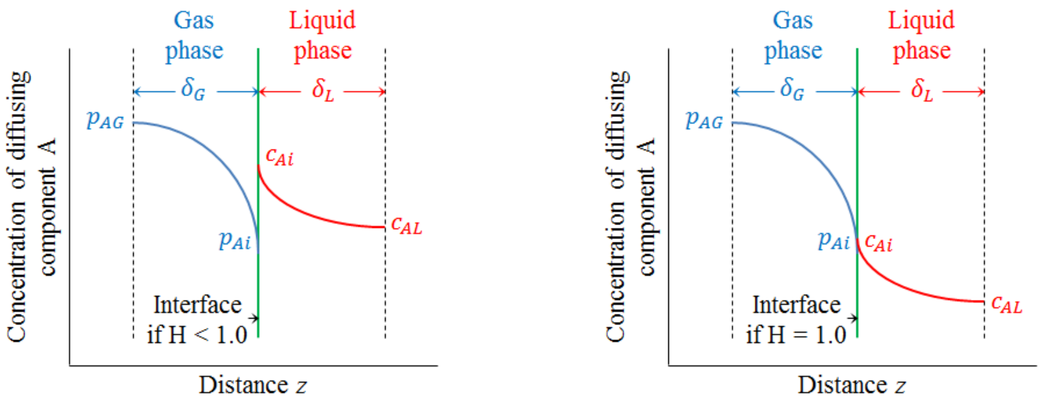


Figure 15.1: Concentration gradients between two contacting phases.

One method to model the adsorption of A from the gas phase is to consider it as a two step process. The first step is the transport of A from the bulk gas to the gas-liquid interface. The second step is the transport of A from the interface to the bulk liquid phase. These two processes offer the main resistances to mass transfer. In this model, it assumed that the interface itself offers no resistance to mass transfer, so the concentration of A in the gas and liquid phases are at equilibrium on the interface.

The flux of A through the gas phase is given by

$$N_A = k_G(p_{AG} - p_{Ai}) \tag{15.1}$$

where p_{Ai} is the partial pressure of A in the gas phase just at the interface, and k_G is the gas phase mass-transfer coefficient. The flux of A through the liquid phase is given by

$$N_A = k_L(c_{Ai} - c_{AL}) \tag{15.2}$$

where c_{Ai} is the concentration of A in the liquid phase just at the interface, and k_L is the liquid phase mass-transfer coefficients. The mass transfer coefficients k_G and k_L can be estimated using methods discussed in previous section (e.g., boundary layer theory).

At the interface, the system is at equilibrium, which implies that the concentration of A in the liquid phase is directly related to its partial pressure in the gas phase. If species A is only at dilute concentrations in the liquid phase, this can be written as Henry's law

$$p_{Ai} = \mathcal{H}_A c_{Ai}$$

15.3 Overall mass-transfer coefficients

In order to apply the model developed in the previous subsection, the concentration of the species at the interface needs to be known. This makes it a bit difficult to directly apply, since typically only the concentrations in the bulk phases are known (i.e., p_{AG} and c_{AL}).

Thus, it is convenient to define mass transfer coefficients that are based on the overall driving force in the system, so that the interfacial concentrations are not required. There are two manners in which to characterize the overall driving force for mass transfer. One is to base the driving force on the gas partial pressures. In this case, the flux of *A* is given by

$$N_A = K_G(p_{AG} - p_A^*) \tag{15.3}$$

where *KG* is the overall mass transfer coefficient based on the gas partial pressure, and $p_A^* = \mathcal{H}_A c_{AL}$. Note that p_A^* is the partial pressure of *A* that would be in equilibrium with the bulk liquid concentration of *A*. Note that, as expected, the flux of *A* vanishes when the bulk gas partial pressure is in equilibrium with the bulk liquid concentration (i.e., $p_{AG} = p_A^* = \mathcal{H}_A c_{AL}$).

The other choice is to base the driving force on the liquid concentration:

$$N_A = K_L(c_{AL} - c_A^*) \tag{15.4}$$

where k_L is the overall mass transfer coefficient based on the liquid concentrations, and $c_A^* = p_A/\mathcal{H}_A$ which is the concentration of *A* that would be in equilibrium with the gas partial pressure of *A*. Again, the flux of *A* vanishes when the two bulk phases are at equilibrium.

A sketch of the various driving forces associated with each of the definitions of the flux of *A* are given in Fig. 15.2.

Although we do not need the interfacial concentrations in this approach, the difficulty with using overall mass-transfer coefficients (i.e., k_G and k_L) is that there is no direct manner to estimate their values, where as there are methods to estimate the individual mass transfer coefficients (i.e., k_G and k_L). Therefore, we need to find a relationship between overall and individual mass transfer coefficients.

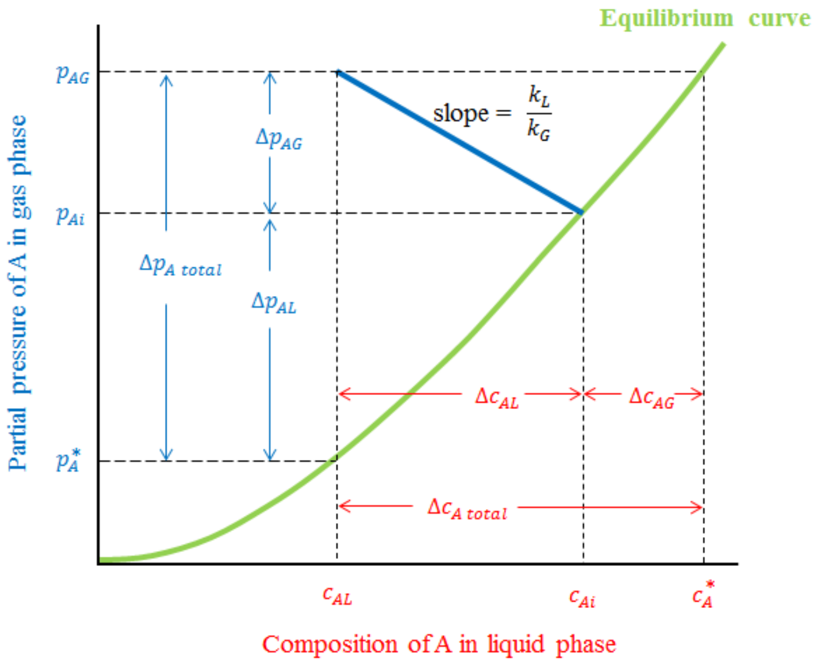


Figure 15.2: Interfacial compositions as predicted by two-resistance theory.

Let's obtain this relation for K_G. First we note that flux of A from the bulk gas phase to the gas-liquid interface (see Eq. (15.1)) can be rewritten as

$$\frac{N_A}{k_G} = p_{AG} - p_{Ai} \tag{15.5}$$

From Eq. 15.2, the flux of A from the interface to the bulk liquid phase can also be written as

$$\frac{N_A}{k_L} = c_{Ai} - c_{AL} \tag{15.6}$$

Multiplying this equation by the Henry's law constant $\mathcal{H}_A$ leads to

$$N_A \frac{\mathcal{H}_A}{k_L} = \mathcal{H}_A c_{Ai} - \mathcal{H}_A c_{AL} = p_{Ai} - p_A^* \tag{15.7}$$

By adding Eqs. (15.5) and (15.7), we find:

$$N_A \left(\frac{1}{k_G} + \frac{\mathcal{H}_A}{k_L} \right) = p_{AG} - p_A^* \tag{15.8}$$

where we have assumed that the system is at steady state so all the fluxes must be the same. Comparing this relation with Eq. (15.3), we can determine that

$$\frac{1}{K_G} = \frac{1}{k_G} + \frac{\mathcal{H}_A}{k_L} \tag{15.9}$$

This directly relates the overall mass-transfer coefficients to the two individual mass transfer coefficients. In a similar manner, we can determine the relationship for the overall mass-transfer coefficient K_L

$$\frac{1}{K_L} = \frac{1}{\mathcal{H}_A k_G} + \frac{1}{k_L} \tag{15.10}$$

The quantity $1/K_G$ can be interpreted as an overall resistance to mass transfer. Thus, by examining Eq. (15.9), we can see that the total resistance to mass transfer is composed of two contributions: the resistance of the gas phase $1/K_G$ and the resistance of the liquid phase $\mathcal{H}_A/k_L$.