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Introduction to Game Theory

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1 Foreword

This book has set itself the task of providing an overview of the field of game theory. The focus here is above all on imparting a fundamental understanding of the mechanisms and solution approaches of game theory to readers without prior knowledge in a short time. Because game theory is in the first place a mathematic discipline with very high formal demands, the book does not claim to be complete. Often, the solution concepts of game theory are mathematically very complex and impenetrable for outsiders. However, as long we remain on the surface, some principles can be explained plausibly with relatively simple means. For this reason the book is eminently suitable in particular as introductory reading, so that the interested reader can create a solid basis, which can then be intensified through advanced literature.

What are the advantages of reading this book? I believe that through the fundamental understanding of game theory concepts, the solution approaches that are introduced can enlighten in nearly all areas of life – after all, along with economics, it is not for nothing that game theory is applied in a huge number of disciplines, from sociology through politics and law to biology.

With this in mind I hope you have a lot of fun reading this book and thinking!

2 Introduction

2.1 Aim and task of game theory

Game theory is a mathematical branch of economic theory and analyses *decision situations* that have the character of games (e.g. auctions, chess, poker) and that go far beyond economics in their application. The significance of game theory can also be seen in the award of the Nobel prize in 1994 to the game theoreticians John Forbes Nash, John Harsanyi and Reinhard Selten.

Decision situations usually consist of several players who have to decide between various strategies, each of which influences their utility or the *payoffs* of the game. The primary aim here is not to defeat fellow players but to *maximise the player's own (expected) payoff*. Games are not necessarily modelled so that the gains of one player result from the losses of the opponent (or opponents). These types of games are simply a special case and are referred to as zero-sum games.

Game theory is therefore concerned with analysing all the framework conditions of a game (insofar as they are known) and, taking account of all possible strategies, with identifying those strategies that optimise one's own *utility* or one's own *payoff*. The decisive point in game theory is that it is not sufficient to consider your own strategies. A player must also anticipate which strategies are optimal for the opponent, because his choice has a direct effect on one's own payoff. There is therefore *reciprocal influencing* of the players. In the ideal case there are equilibriums in games, which, roughly speaking, means that the optimal strategies of players 'are in harmony with one another' and are 'stable' in their direct environment. This obviously does not apply to zero-sum games such as 'rock, paper, scissors', in which no constellation of strategies is optimal for all players.

In classical game theory it is assumed that all players act *rationally* and *egoistically*. According to this, each player wants to maximise his (expected) benefit. The final chapter shows that this does not always conform to reality.

2.2 Applications of game theory

There is a series of applications of game theory in different areas. Game theory is above all interesting where the framework conditions can be easily modelled as a game, that is, in which strategies and payoffs can be identified and there exists a clear dependency of the payoffs of the different players on the selected strategies.

In economics, for example, applications can be found in the fields of price and product policy and market entry, auctions, internal incentive systems, strategic alliances, or mergers, acquisition or takeovers of companies. In the legal sector, game theory is significant among others for the areas of contract design, patent protection and mediation and arbitration proceedings. Game theory is applied in politics (coalitions, power struggles, negotiations), in environmental protection (emission trading, resource economics), in sociology (for example in the distribution of a good), in warfare, or in biology in the field of evolutionary game theory. The latter models how successful modes of behaviour assert themselves in nature through selection mechanisms, and less successful ones disappear.

A classical example of game theory modelling (and unfortunately not applied) in economics is the auction of UMTS licences in Germany in 2000. The licences were distributed between six bidders for a total of DM 100 billion – a sum that dramatically exceeded expectations. The high price also signalled the great expectations regarding the economic importance of the UMTS standards, but could have turned out much less, because in the end the six bidders bid each other up to induce other bidders to drop out. However, because in the end no one dropped out, the high price had to be paid without an additional licence. The book by Stefan Niemeier *Die deutsche UMTS-Auktion. Eine spieltheoretische Analyse* published in 2002 shows, for example that from a game theory aspect the result is not always based on rational decisions, and that, given a suitable game theory analysis, some bidders could have saved money.

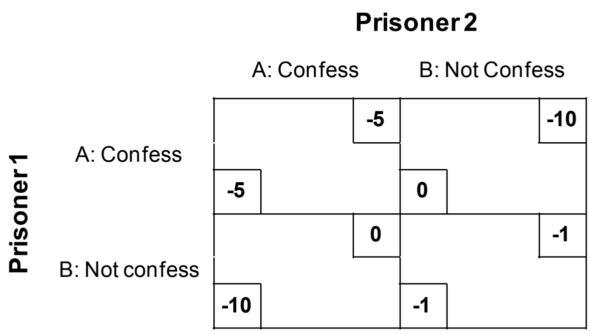
2.3 An example: the prisoner's dilemma

Probably the most famous game theory problem is the *prisoner's dilemma*, which will be introduced briefly here, and which provides an initial impression of how games can be modelled. Essential terms will also be introduced that are important for reading the following chapters.

Two criminals are arrested. They are suspected of having robbed a bank. Because there is very little evidence, the two can only be sentenced to a year's imprisonment on the basis of what evidence there is. For this reason, the two are questioned separately, with the aim of getting them to confess to the crime through incentives, and because of the uncertainty regarding what the other is saying. A deal is offered to each of them: if they confess, they will be freed – but only if the other prisoner does not confess; in this case he will go down for 10 years. If they both confess, they will each go to prison for five years.

The terms introduced up to now enable some statements to be made on the game theory modelling of this game. The two criminals are two players, each of whom has two strategies available: to confess or not to confess. Their payoff corresponds in this case to the years that they will have to spend in prison, whereby here, of course, the aim is not to maximise the payoff but to minimise it. The payoff depends not only on a prisoner's own strategy but also on the strategy of the other prisoner. It is also important that the two criminals make their decisions simultaneously and that each of them is unaware of the other's decision. In addition, this information is known to both players. Games like this are known in game theory as simultaneous games under complete information. Simultaneous games are also referred to as games in *normal form*, while sequential games – in other words, games in which 'play' takes place sequentially – are known as games in *extensive form*. Because two persons play the game, it is a *2-person game* or a *2-person normal game*.

With this information, the following model can be set up using game theory:



This 2x2 matrix is developed as follows: the strategies of the prisoner (prisoner 1) are on the left in the line legends, and the strategies of the second prisoner (prisoner 2) are at the top in the column legends. There are a total of four constellations, and a field in the matrix is reserved for each of these:

1. Both prisoners confess (top left field)
2. Prisoner 1 confesses, prisoner 2 does not confess (top right field)
3. Prisoner 1 does not confess, prisoner 2 confesses (bottom left field)
4. Neither prisoner confesses (bottom right field)

The two numbers in the four fields correspond to the payoffs of the two prisoners. The payoffs in the bottom left accrue to prisoner 1 in the respective constellations, while the payoffs in the top right are for prisoner 2.

So much for the notation. But what is it about this game that has enabled it to become so famous? The response is found in the paradoxical result that this game entails, namely that both confess and go to prison for five years, although if they had just said nothing, they would each have been sentenced to only one year's imprisonment.

We arrive at this result if we consider a prisoner's strategies more exactly from the aspect of the other prisoner. Let us assume that I am prisoner 1. I then consider my *best response* for each of the other prisoner's two strategies. If prisoner 2 confesses, I will confess as well, because in this case I will only have to go to prison for five years, instead of 10 years if I do not confess. In contrast, if I assume that prisoner 2 will not confess, I will confess myself, because I will then be released, which I naturally prefer to going to prison for one year, if I confess as well. This means I always choose the 'confess' strategy, completely regardless of which strategy the other prisoner chooses. Because the same case applies to the other prisoner, he will also confess, which leads to the paradoxical result described above.

This case can, of course, be regarded as a construction that is relevant only in theory. However, this can be countered by saying that life is full of prisoner's dilemma, namely whenever two (or more) parties do not move from their positions because they are afraid of being the only party to make concessions while the other parties do not move (for example, between management and union representatives).

2.4 Game theory terms

2.4.1 Preferences

Preference relations are extremely important in game theory. They state which alternatives a player prefers to other alternatives, and to which alternatives a player is indifferent. If a player prefers strategy (A) to strategy (B), we write $A > B$; if he is indifferent with regard to both strategies we write $A \sim B$.

Let us assume that a player has the choice of travelling by car (A), bus (B) or tram (C). The following is to apply with regard to his preferences:

1. The player prefers to travel by car rather than by bus: $A > B$
   ("The player *prefers* A to B")

2. It is all the same to him whether he travels by bus or tram: $B \sim C$
   ("The player is *indifferent* with regard to B and C")

Because of transitivity, $A > C$ then follows from (1) and (2).

2.4.2 Strategies

The *strategy* of a game is designated below as *S*. Let the strategy of player 1 be S_1 and the strategy of player 2 be S_2.

In the example of the prisoner's dilemma, the strategy selected by player 1 would be:

S_1 = confess,

or for player 2:

S_2 = confess

and for the whole game

$S = (S_1, S_2)$ = (confess, confess).

S is described in this case as a *strategy pair* as well.

S_1 (S_2) may also stand for a set of strategies of player 1 (player 2) to choose from, for example:

S_1 = (confess, do not confess)

and for player 2:

S_2 = (confess, do not confess)

and for the whole game

$S = (S_1, S_2)$ = ((confess, do not confess), (confess, do not confess)).

If the actions of the players in a game consist of the decision for one of the available strategies, we speak of a *pure strategy*. The strategies of the game itself are also referred to as pure strategies. In contrast, if several pure strategies of a player are each played with a certain probability, we speak of a *mixed strategy*. A classical example of a game in which the player pursues a mixed strategy is game 'rock, scissors, paper'.

2.4.3 Payoffs

Payoff A is used below to designate what is 'paid out' to a player on a given constellation of strategies. Because the payoff depends not only on a player's own strategy, but also on the strategies of all other players, payoff A is a function over strategy of all players.

For the prisoner's dilemma the payoff for player 1 would then be:

$A_1(S) = A_1(S_1, S_2) = A_1$(confess, confess) = -5

This term shows the payoff for player 1 (A_1) in the event that player 1 confesses (S_1 = confess) and player 2 confesses (S_2 = confess).

For player 2, the corresponding payoff over the same strategy pair is:

$A_2(S) = A_2(S_1, S_2) = A_2$(confess, confess) = -5

The payoff is therefore dependent not only on a player's own strategy, but also on the strategy of all players.

3 Simultaneous games

3.1 Foundations

As was shown in the example of the prisoner's dilemma, in a *simultaneous game* all players make their decisions at the same time, without knowing what the other players decide. The information about fellow players and their strategies and payoffs is, in contrast (in the cases dealt with here), general information and known to all players (complete information).

The following sections provide an overview of the different types of *strategies* and *equilibriums* in a *two-person simultaneous game*. Although at first only simultaneous games between two persons will be discussed – because they can be represented in a two-dimensional matrix – multi-person games (n-person simultaneous games) for which the same principles and mechanisms apply are also possible, as will be shown in conclusion in this chapter by means of a three-person simultaneous game.

3.2 Strategies

3.2.1 The maximin strategy

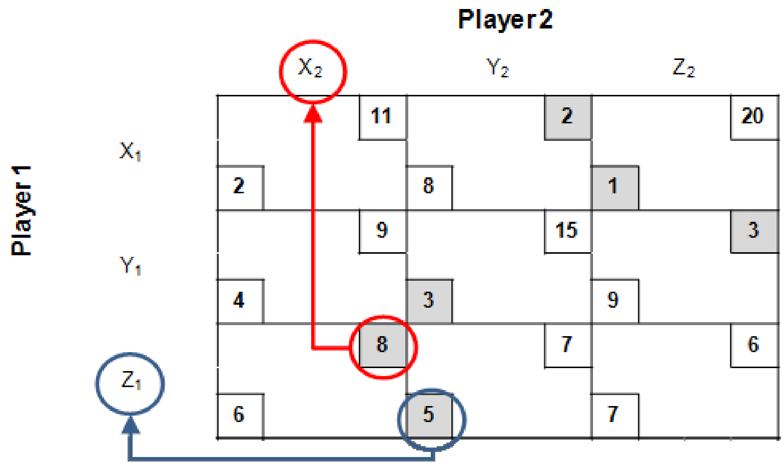
The *maximin strategy* corresponds to that strategy of a player with which he still achieves the best payoff in the most unfavourable case. Its objective is therefore damage limitation.

The maximin strategy can be determined in a matrix with any number of strategies n for player 1 and m for player 2 in two steps:

1. First off all, the smallest possible own payoff (min A) is selected for each own strategy taking account of all possible strategies of the opponent. If this occurs more than once, these are to be selected accordingly.
2. Following this, the largest (max min A) of these smallest payoffs is selected. The corresponding strategy is called the maximin strategy of the corresponding player. Several maximin strategies can exist for one player.

The maximin strategy of player 1 is designated MS_1, the corresponding maximin strategy of player 2 as MS_2.

The following example with $S_1 = (X_1, Y_1, Z_1)$ and $S_2 = (X_2, Y_2, Z_2)$ is by way of illustration:



Player 1 is looking for the smallest possible payoff (1, 3 and 5) and among these the largest (5), for strategies X_1, Y_1 and Z_1. The following applies for player 1 (S_2 stands for the set of strategies of player 2):

$$\left.\begin{array}{l} \min A_1(X_1, S_2) = 1 \\ \min A_1(Y_1, S_2) = 3 \\ \min A_1(Z_1, S_2) = 5 \end{array}\right\} \quad \max(1, 5, 3) = 5 \rightarrow \underline{MS_1 = Z_1}$$

The maximin strategy for player 1 is therefore strategy Z_1. The following applies for the maximin strategy of player 2:

$$\left.\begin{array}{l} \min A_2(S_1, X_2) = 8 \\ \min A_2(S_1, Y_2) = 2 \\ \min A_2(S_1, Z_2) = 3 \end{array}\right\} \quad \max(8, 2, 3) = 8 \rightarrow \underline{MS_2 = X_2}$$

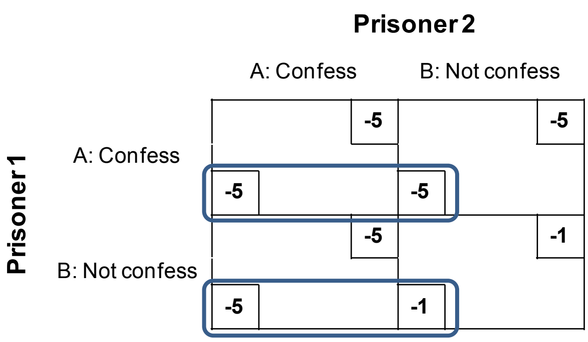
The maximin strategy for player 2 is therefore strategy X_2. The maximin strategy is the strategy with the least risk of a small payoff, without making assumptions about the preferences of the opponent (or opponents).

3.2.2 Dominant strategy

A strategy is designated as a *dominant strategy* if it holds for every other strategy that the latter do not put the player in a better position, and put him in a worse position in at least one case. A dominant strategy is thus ‘resistant’ to any possible change of strategy by the opponent, and is selected in each instance.

We can find an example of a dominant strategy in the example of the prisoner’s dilemma shown above. In this game, the dominant strategy for the prisoner is to confess, because in each instance this strategy puts him in a better position than the alternative strategy of not confessing.

A modification of the prisoner’s dilemma also provides a good illustration of the principle of the dominant strategy. In this modified version, both prisoners will definitely go to prison for 1 year. As soon as one of the two confesses to the crime, both must go to prison for 5 years. This situation can be mapped in the following matrix:



S_1 = (Do not confess) is the dominant strategy for prisoner 1, because on a change of the strategy to S_1 = (Confess) he is by no means in a worse position – it is of no concern if prisoner 2 chooses strategy S_2 = (Confess) – and in at least one case – with S_2 = (Do not confess) – he is better off.

The following applies therefore

S_1 = (Do not confess) is the dominant strategy for player 1

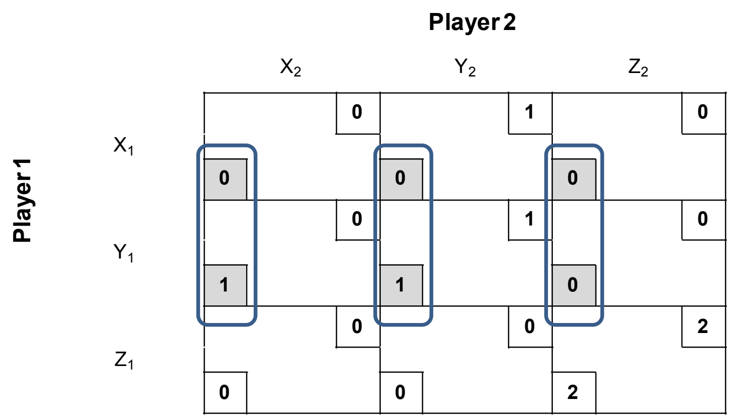
S_2 = (Do not confess) is the dominant strategy for player 2

If a dominant strategy for a player exists in a game, this player will always select the dominant strategy.

3.2.3 Dominated strategy

A *dominated strategy* has the characteristic for a player in a game that there is another strategy in this game that in each instance – that is, with every possible strategy of the opponent – is not worse, and is really better in at least one case. A dominated strategy can be removed from the matrix for further analysis, because in no case does it bring an advantage for the player in comparison with the strategy that dominates it.

The following example is intended to illustrate this situation:



It is easy to understand that player 1 prefers strategy Y_1 to strategy X_1, because it either places him at an advantage (if player 2 chooses X_2 or Y_2) or not in a worse position (if player 2 chooses Z_2). From the point of view of player 1 therefore:

$A_1(X_1, X_2) < A_1(Y_1, X_2)$, because $0 < 1$
$A_1(X_1, Y_2) < A_1(Y_1, Y_2)$, because $0 < 1$
$A_1(X_1, Z_2) = A_1(Y_1, Z_2)$, because $0 = 0$

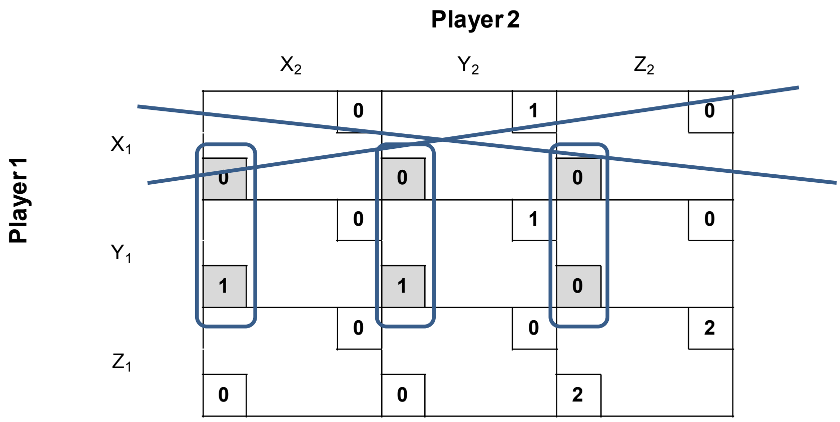
Therefore,

$Y_1 > X_1$.

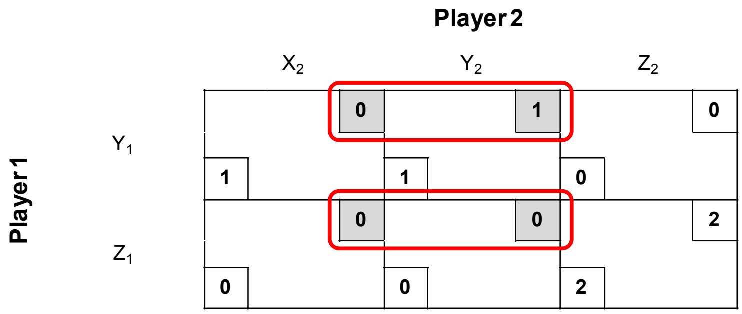
applies correspondingly for the preferences of player 1 with regard to his strategies.

Player 1 will therefore select Y_1 in preference to X_1. We can also say *strategy X_1 is dominated by strategy Y_1*. A statement cannot be made regarding Y_1 and Z_1. Which strategy player 1 prefers here depends on the strategy that player 2 selects.

As player 1 will never play X_1 because he prefers Y_1, strategy X_1 can be deleted from the game:



This gives us the following modified game:



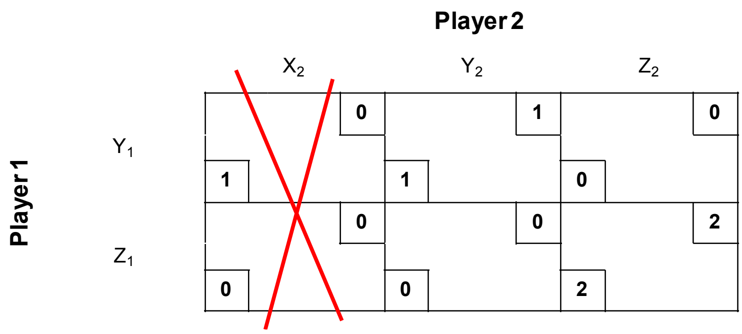
The result is that in this modified game for player 2 strategy X_2 is dominated by strategy Y_2 because:

$$A_2(Y_1, X_2) < A_2(Y_1, Y_2), \text{ because } 0 < 1$$
$$A_2(Z_1, X_2) = A_2(Z_1, Y_2), \text{ because } 0 = 0$$

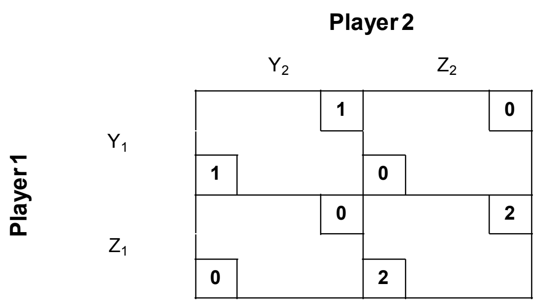
and thus

$$X_2 < Y_2.$$

This means that in this modified game strategy X_2 can also be deleted:



We then receive the following as a new game:



3.3 Equilibriums in pure strategies

3.3.1 Best responses

The concept of the *best response* is important for determining and understanding equilibriums. A player's best response refers to the respective strategies of the opponent, i.e. for each strategy of player 2 there exists one (or more) best response(s) for player 1.

Let us take another look at the prisoner's dilemma:

		Prisoner 2	
		A: Confess	B: Not Confess
Prisoner 1	A: Confess	-5 -5	-10 0
	B: Not confess	0 -10	-1 -1

If prisoner 2 confesses, the best response for prisoner 1 is to confess as well. If prisoner 2 does not confess, again, the best response to this for prisoner 1 is to confess.

In this case, we write:

BR_1(Confess) = (Confess)
BR_1(Do not confess) = (Confess)
BR_2(Confess) = (Confess)
BR_2(Do not confess) = (Confess)

If we start from a strategy and then determine the best responses – that is, the best response to a strategy, the best response to this best response, etc. – this process can be continued until the best response for a strategy pair is in each case the best response to itself as well. Because then an equilibrium has been found in which it is not worthwhile for any player to deviate unilaterally from this equilibrium. We will come across this situation again soon with the Nash equilibrium.

The fact that this process does not always end in a state of equilibrium is shown by the following example in which, starting from a specific strategy pair S = (Y_1, X_2), a 'best response circle' follows that does not lead to a stable equilibrium:

Player 2

Player 1	X_2	Y_2	Z_2
X_1	2, 11	8, 2	1, 20
Y_1	4, 9	3, 15	9, 3
Z_1	5, 8	6, 7	7, 6

This example assumes strategy Y_1 of player 1; player 2 reacts to this strategy accordingly with his best response $BR_2(Y_1) = Y_2$. If this thread is spun further, the result is the following best responses:

$BR_2(Y_1) = Y_2 \rightarrow S = (Y_1, Y_2)$
$BR_1(Y_2) = X_1 \rightarrow S = (X_1, Y_2)$
$BR_2(X_1) = Z_2 \rightarrow S = (X_1, Z_2)$
$BR_1(Z_2) = Y_1 \rightarrow S = (Y_1, Z_2)$
[…]

3.3.2 Nash equilibrium

The *Nash equilibrium* is one of the central solution concepts of game theory. It was developed in 1950 by John Nash and is relevant far beyond simultaneous games in nearly all forms of games. A Nash equilibrium is found when with a given strategy, it is not worthwhile for a player to be the only one to change his strategy. Therefore, in the Nash equilibrium no player has an incentive to depart from this equilibrium *unilaterally*. The expression “unilateral deviation is not worthwhile” can be used as a rule of thumb for the Nash equilibrium.

A Nash equilibrium does not make any statement as to whether the players can position themselves better if at least two players deviate from their strategy simultaneously.

Our prisoner’s dilemma serves once again as an example:

		Prisoner 2	
		A: Confess	B: Not Confess
Prisoner 1	A: Confess	-5 / -5	-10 / 0
	B: Not confess	0 / -10	-1 / -1

The Nash equilibrium S* = (Confess, confess) is a Nash equilibrium because it is not worthwhile for either player to be the only one to deviate from this equilibrium strategy (and to go to prison for twice as long). Only if both were to change their strategy would an incentive to change arise.

For Nash equilibriums we use the notation as shown in this example

S* = (Confess, confess)
S_1* = Confess
S_2* = Confess

S* designates the Nash equilibrium, S_1* the Nash equilibrium strategy of player 1 and S_2* the Nash equilibrium strategy of player 2.

But how can the Nash equilibrium be determined practically and methodically in a simultaneous game of two persons with any number of strategies? The easiest way is first of all to go through all possible strategies of player 2 for player 1, and in each case to mark the best payoff for player 1 for the various strategies (where there are several equally large best payoffs these are to be marked accordingly). This process is then applied vice versa for player 2. In this way, the own best responses (or rather, the payoffs belonging to them) are marked for all possible strategies of the opponent. The Nash equilibriums are then all those fields in which both payoff sizes are marked.

This method will be shown by means of the following example:

		Player 2		
		X_2	Y_2	Z_2
Player 1	X_1	0 0	1 0	0 0
	Y_1	0 1	1 1	0 0
	Z_1	0 0	0 0	2 2

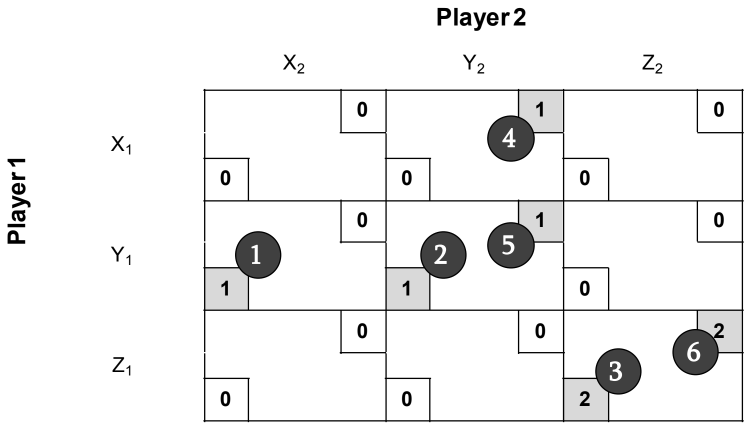
The method is now to be applied in 6 steps – one step per player (2) per strategy (3). We start with player 1 and strategy X_2:

1. If player 2 selects strategy X_2, the best response of player 1 is strategy Y_1 with the payoff $A_1(Y_1, X_2) = 1$, which is marked.
2. If player 2 selects strategy Y_2, the best response of player 1 is strategy Y_1 with the payoff $A_1(Y_1, Y_2) = 1$, which is marked.
3. If player 2 selects strategy Z_2, the best response of player 1 is strategy Z_1 with the payoff $A_1(Z_1, Z_2) = 2$, which is marked.
4. If player 1 selects strategy X_1, the best response of player 2 is strategy Y_2 with the payoff $A_2(X_1, Y_2) = 1$, which is marked.
5. If player 1 selects strategy Y_1, the best response of player 2 is strategy Y_2 with the payoff $A_2(Y_1, Y_2) = 1$, which is marked.
6. If player 1 selects strategy Z_1, the best response of player 2 is strategy Z_2 with the payoff $A_2(Z_1, Z_2) = 1$, which is marked.

In this way, 6 values are now marked, whereby the two payoffs are marked in the fields (Y_1, Y_2) and (Z_1, Z_2). These fields therefore each mark a Nash equilibrium, so that the game contains the following two Nash equilibriums:

$$S^* = (Y_1, Y_2),\ S_1^* = Y_1,\ S_2^* = Y_2$$
$$S^{**} = (Z_1, Z_2),\ S_1^{**} = Z_1,\ S_2^{**} = Z_2$$

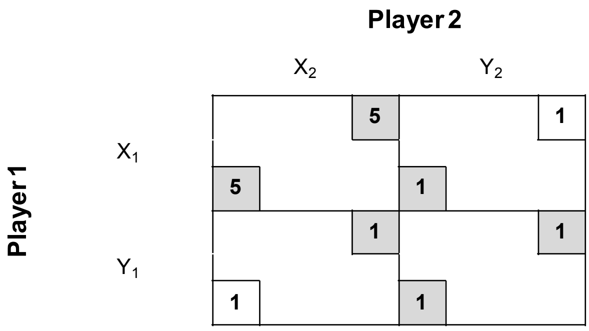
The six steps shown here for determining the Nash equilibriums are shown once again in the following illustration:



3.3.3 Strict equilibrium

If the strategies of all players in the Nash equilibrium are at the same time dominant strategies as well, this is referred to as a *strict Nash equilibrium*. The criterion of strictness requires that an equilibrium is strictly better than its environment. The consequence of a strict equilibrium is that each player has only a single best response to the strategies of his opponent. This means that there cannot be several strict equilibriums in a game.

The following shows the existence of a strict Nash equilibrium:



The strategies $S^* = (X_1, X_2)$ and $S^{**} = (Y_1, Y_2)$ are initially the game's Nash equilibriums of the game. Because it also holds that X_1 is the dominant strategy for player 1, and X_2 is the dominant strategy for player 2, S^* is also a strict Nash equilibrium.

In general, $S^* = (X_1, X_2)$ is held to be a strict Nash equilibrium if not only X_1 but also X_2 is a dominant strategy. Strict equilibriums are always Nash equilibriums as well.

3.3.4 Nash equilibriums and best responses

Nash equilibriums and best responses are directly connected. The following example serves to make this clear:

Player 2

Player 1	X_2	Y_2	Z_2
X_1	2, 11	8, 2	1, 20
Y_1	4, 9	3, 15	9, 3
Z_1	5, 8	6, 7	7, 6

The only Nash equilibrium in this game is $S^* = (Z_1, X_2)$. For the best response to the Nash equilibrium strategy of the opponent the following applies for player 1 and player 2 respectively:

$$BR_1(X_2) = Z_1$$
$$BR_2(Z_1) = X_2$$

The best responses to the Nash equilibrium result again in the same Nash equilibrium. This is why it is said that a Nash equilibrium is the "best response to itself."

3.4 Equilibriums in mixed strategies

3.4.1 Mixed strategies and expected payoffs

In contrast to pure strategies, in mixed strategies a player does not decide on one (pure) strategy, but plays several pure strategies, each of which has specific probability.

A classical example of a game in mixed strategies is tossing a coin in which there is a 50% chance of heads or tails being on top. Assume that a player tosses a coin. If it turns up tails he receives $2, if heads, he receives nothing. The coin receives nothing in this game. The payoff matrix then looks like this:

Coin

Player	X_2: Heads	Y_2: Tails
X_1: Toss the coin	0, 0	2, 0

Because there is a probability of ½ that the coin will show head or tail, we write the following for the mixed strategy of the coin:

$S_{Coin} = (½, ½)$

With this notation the strategies are no longer given but instead the probability with which the strategies are played. Anyone who is unable to imagine that a coin can pursue a strategy can imagine instead a player who decides on the strategies X_2 and Y_2 by tossing a coin in each case.

How high is the player's *expected payoff* now? Because the player has a 50% chance of receiving $0 (heads) and a 50% of receiving $2 (tails), his expected payoff is:

$$½ \cdot \$0 + ½ \cdot \$2 = \$1$$

For the expected payoff we write:

$$E(A_{player}(\text{toss coin}, (½, ½))) = ½ \cdot 0 + ½ \cdot 2 = \mathbf{1}$$

("*The expected payoff of the player when he tosses the coin is 1*")

A slightly more complex example should make this situation clearer:

Player 1 \ Player 2	X_2	Y_2
X_1	2, 2	3, 1
Y_1	1, 3	4, 4

Let player 2 now play strategy X_2 to 25% and strategy Y_2 to 75%. His strategy is therefore

$$S_2 = (¼, ¾).$$

It now has to be calculated how high the expected payoff of player 1 is if he plays X_1:

$$E(A_1(X_1, (¼, ¾))) = ¼ \cdot 2 + ¾ \cdot 3 = ½ + 9/4 = 11/4 = \mathbf{2{,}75}$$

("*The expected payoff of player 1 if he plays X1 and player 2 plays S2 = (¼, ¾) is 11/4*")

Let player 1 now decide to play a mixed strategy $S_1 = (½, ½)$ as well. Each strategy combination then occurs with a specific probability:

$S = (X_1, X_2)$	to 12.5%	(= ½ · ¼)
$S = (X_1, Y_2)$	to 37.5%	(= ½ · ¾)
$S = (Y_1, X_2)$	to 12.5%	(= ½ · ¼)
$S = (Y_1, Y_2)$	to 37.5%	(= ½ · ¾)
	= 100%	= 1

The expected payoff of player 1 is therefore calculated as follows:

$$\begin{aligned} E(A_1((½, ½), (¼, ¾))) &= ½ \cdot ¼ \cdot 2 + ½ \cdot ¾ \cdot 3 + ½ \cdot ¼ \cdot 1 + ½ \cdot ¾ \cdot 4 \\ &= 2/8 + 9/8 + 1/8 + 12/8 \\ &= 24/8 \\ &= \mathbf{3} \end{aligned}$$

3.4.2 Mixed Nash equilibriums

Mixed Nash equilibriums are Nash equilibriums that consist of mixed strategies. This can be illustrated very well with the 'rock, scissors, paper' game, in which rock beats scissors, scissors beats paper and paper beats rock. The winner receives a payoff of $1, which the loser has to pay. If they both chose the same strategy, no one gets anything. The payoff matrix then looks like this:

		Player 2		
		X_2: Rock	Y_2: Scissor	Z_2: Paper
Player 1	X_1: Rock	0, 0	1, -1	-1, 1
	Y_1: Scissor	-1, 1	0, 0	1, -1
	Z_1: Paper	1, -1	-1, 1	0, 0

This game does not have a Nash equilibrium in pure strategies. But which strategies will the players now pursue? If the two players play this game a few times successively, they will usually both try to play inscrutably. This means over the long term not to play any strategy more often than any other where possible, because the other player could gain a benefit from this and play the corresponding counterstrategy more frequently. Only if each player plays each of his strategies with the same probability is there no incentive for any player to change his strategy.

The strategy

$$S^* = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$$

is accordingly a Nash equilibrium that consists of mixed strategies and in which no player (as shown) has an incentive to deviate from this strategy.

In principle, every game has a Nash equilibrium. If there is no Nash equilibrium in pure strategies, there is at least one Nash equilibrium in mixed strategies. However, a game can also have Nash equilibriums in pure and mixed strategies.

But how is a mixed Nash equilibrium determined? For this purpose, the following simple game is to be considered in which player 1 plays strategy X_1 with the probability p and strategy Y_1 with the corresponding probability (1-p), and player 2 plays the strategy X_2 with the probability q and strategy Y_2 with the corresponding probability (1-q):

Player 2

Player 1	X_2(q)	Y_2(1-q)
X_1 (p)	1, 2	2, 0
Y_1 (1-p)	2, 0	0, 2

This game has no Nash equilibrium in pure strategies. However, because every game has at least one Nash equilibrium, there exists at least one mixed Nash equilibrium.

To determine a mixed Nash equilibrium considerations must be made regarding the probabilities with which a player has to play his strategies so that his opponent is indifferent with regard to these strategies. Looking back at the 'rock, scissors, paper' game, this means that I have to select my strategy in such a way that my opponent does not prefer a specific strategy with which he can outsmart me – which would then give me an incentive to change the strategy.

Coming back to the example: let player 1 now consider which strategies he has to play so that player 2 is indifferent with regard to his strategies, i.e. the expected payoffs for player 2 must be equal for X_2 and Y_2. This results in the following equation:

$$E(A_2((p, (1\text{-}p)), X_2)) = E(A_2((p, (1\text{-}p)), X_2))\ !$$

This equation can be determined in detail with the concrete probabilities and payoffs:

$$p \cdot 2 + (1\text{-}p) \cdot 0 = p \cdot 0 + (1\text{-}p) \cdot 2$$
$$\rightarrow 2p = 2 - 2p$$
$$\rightarrow 4p = 2$$
$$\rightarrow \mathbf{p = 1/2}$$

Let player 2 consider the same:

$$E(A_1(X_1, (q, (1\text{-}q)))) = E(A_1(Y_1, (q, (1\text{-}q))))\ !$$
$$\rightarrow q \cdot 1 + (1\text{-}q) \cdot 2 = q \cdot 2$$
$$\rightarrow q + 2 - 2q = 2q$$
$$\rightarrow 2 = 3q$$
$$\rightarrow \mathbf{q = 2/3}$$

The probabilities of the mixed Nash equilibrium were now determined. The game has the mixed Nash equilibrium

$$S^* = ((1/2, 1/2), (2/3, 1/3))$$

But how can we calculate a mixed Nash equilibrium if there are more than two strategies per player? For this purpose we take another look at the 'rock, scissors, paper' game, by working out the familiar solution again arithmetically.

First of all, we assume that player 1 plays rock with probability p, scissors with probability q and paper with probability (1-p-q):

Player 2

Player 1	X_2: Rock	Y_2: Scissor	Z_2: Paper
X_1: Rock (p)	0, 0	1, -1	-1, 1
Y_1: Scissor (q)	-1, 1	0, 0	1, -1
Z_1: Paper (1-p-q)	1, -1	-1, 1	0, 0

Let player 1 now consider which strategy he has to play so that player 2 is indifferent with regard to his strategies. The following must apply:

$E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Rock}))$
$= E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Scissor}))$
$= E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Paper}))$

or

(1) $E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Rock})) = E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Scissor}))$
(2) $E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Scissor})) = E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Paper}))$
(3) $E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Rpck})) = E(A_2((p, q, (1\text{-}p\text{-}q)), \text{Paper}))$

These three equations have to be fulfilled in the mixed Nash equilibrium. We first solve equation **(1)**:

(1): $p \cdot 0 + q \cdot 1 + (1\text{-}p\text{-}q) \cdot (-1) = p \cdot (-1) + q \cdot 0 + (1\text{-}p\text{-}q) \cdot 1$
$q - 1 + p + q = -p + 1 - p - q$
$2q + p = -2p + 2 - q$
$3q = -3p + 2$
$\rightarrow q = 2/3 - p$ **(4)**

(2): $p \cdot (-1) + q \cdot 0 + (1\text{-}p\text{-}q) \cdot 1 = p \cdot 1 + q \cdot (-1) + (1\text{-}p\text{-}q) \cdot 0$
$-p + 1 - p - q = p - q$
$1 - 2p - q = p - q$
$3p = 1$
$\rightarrow p = 1/3$ **(5)**

(5) in (4): $q = 2/3 - 1/3 = 1/3$

$\rightarrow p = 1/3,\ q = 1/3,\ 1\text{-}p\text{-}q = 1/3$ **(6)**

Now all we have to do is to check the correctness of equation **(3)**:

(6) in (3): $p \cdot 0 + q \cdot 1 + (1\text{-}p\text{-}q) \cdot (-1) = p \cdot 1 + q \cdot (-1) + (1\text{-}p\text{-}q) \cdot 0$

$1/3 - 1 + 1/3 + 1/3 = 1/3 - 1/3$

$0 = 0$

There is thus no contradiction in equation 3, and therefore the strategy applies for player 1 in the Nash equilibrium

$$S_1^* = (1/3, 1/3, 1/3).$$

The same results for player 2, so that, in absolute terms, the mixed Nash equilibrium

$$S^* = ((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$$

results.

However, there are also games that have a Nash equilibrium in both pure and in mixed strategies, as the following example shows:

		Player 2	
		X_2 (q)	Y_2 (1-q)
Player 1	X_1 (p)	4, 2	0, 0
	Y_1 (1-p)	0, 0	2, 4

This game contains the two Nash equilibriums in pure strategies

$S^* = (X_1, X_2)$
$S^{**} = (Y_1, Y_2)$

However, it also contains the mixed Nash equilibrium

$S^{***} = ((2/3, 1/3), (1/3, 2/3))$

3.5 Special forms of games

3.5.1 Fair games

A game is called *fair* when both players are in exactly the same position, i.e. when both players have the same payoffs with the same strategy combinations. We can also recognise fair games in graphs because they reflect the payoff matrix on a diagonal from top left to bottom right.

The following applies in fair games, for example

$A_1(X_1, X_2) = A_2(X_1, X_2)$
$A_1(X_1, Y_2) = A_2(Y_1, X_2)$
$E(A_1((1/2, 1/2), (1/3, 2/3))) = E(A_2((1/3, 2/3), (1/2, 1/2)))$

Examples of fair games are the 'rock, scissors, paper' game and the prisoner's dilemma.

3.5.2 Zero-sum games

In a *zero-sum game* the sum of the payoffs in each field of the payoff matrix always equals zero – i.e. what one player wins, another loses. An example of a zero-sum game is the 'rock, scissors, paper' game, which is also a fair game. However, a zero-sum game does not have to be fair. For example, in one game only one player can ever win, and his opponent has to 'pay' this profit. The game is then a zero-sum game, but is not fair.

3.6 Simultaneous games in economics

3.6.1 Market entry game

A familiar example of simultaneous games is the so-called market entry game, in which the competitors Airbus and Boeing think about entering the market. Because development costs are very high on the one hand, and the market is limited on the other, neither Airbus nor Boeing gains anything if both enter the market and share it, because the development costs are then higher than the expected benefit of entering the market. In contrast, if one of them leaves the market to the competitor, they will not have any development costs, but the competitor will acquire huge profits because they have the market to themselves. If neither of the two companies enters the market, neither costs nor profits accrue.

Assuming that the development costs are $50 per company and the market has a volume of $80, and under the premise that the market is divided equally, the following payoff matrix results:

		Boeing	
		X_2: Enter the market	Y_2: Keep out of the market
Airbus	X_1: Enter the market	-10 -10	0 30
	Y_1: Keep out of the market	30 0	0 0

The two strategies that one company enters the market and one keeps out of the market result as Nash equilibriums:

Airbus \ Boeing	X_2: Enter the market	Y_2: Keep out of the market
X_1: Enter the market	Airbus: -10, Boeing: -10	Airbus: 30, Boeing: 0
Y_1: Keep out of the market	Airbus: 0, Boeing: 30	Airbus: 0, Boeing: 0

It follows from the Nash equilibriums S* = (Y_1, X_2) and S** = (X_1, Y_2) that in these points neither of the two companies can improve its payoff as long as the competitor keeps to its strategy.

3.6.2 Market agreements (the OPEC game)

In this game, two OPEC states have agreed to offer only a limited volume of oil on the global market together in order to keep the oil price high. With this agreement each state achieves annual sales of $15bn. If one of the two states breaks the agreement it can expect increased sales of $15bn, while the other state receives nothing. If both states break the agreement, because of the lower oil price resulting from the glut of oil both states only have sales of $5bn.

Let X_1, X_2 each be the strategy to keep to the agreement and Y_1, Y_2 the strategy to break it. The payoff matrix of the game then looks like this:

		Player 2	
		X_2	Y_2
Player 1	X_1	10, 10	0, 15
	Y_1	15, 0	5, 5

The game possesses a Nash equilibrium in pure strategies:

$$S^* = (Y_1, Y_2)$$

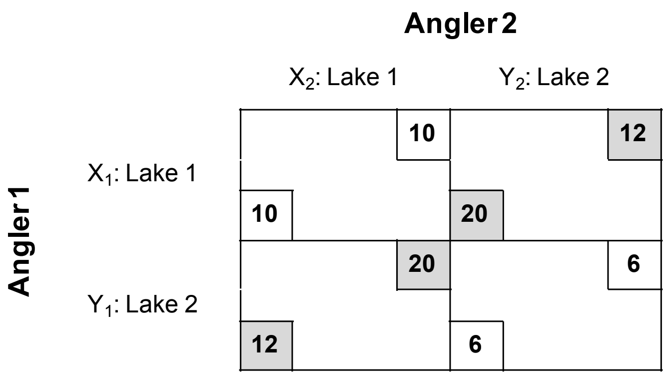
This means that both states have an incentive to break the agreement although they are put into a worse position in this Nash equilibrium than if both were to keep to the agreement. However, this situation is unstable, because each state has an incentive to increase the output.

The situation of the OPEC game is very similar to the prisoner's dilemma.

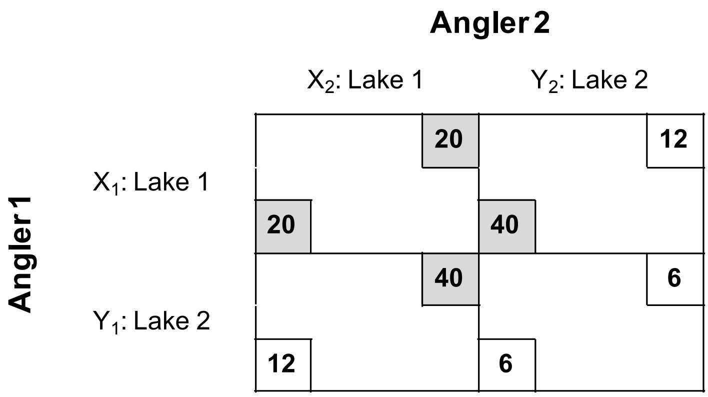
3.6.3 Resource economics

In this game, two fishermen each has the possibility of deciding between one of two lakes (lake 1 and lake 2) for fishing. Lake 1 contains 20 fish, lake 2 contains 12 fish. If both fish the same lake, they share the number of fish between them; otherwise one fisherman receives the complete number of fish in one lake.

The payoff matrix looks like this:



In this game, each of the two Nash equilibriums consists of the strategy of fishing in two different lakes. However, a distribution of this type is not self-evident and also depends on the difference between the stocks of fish in the lakes not being too great. For example, if the fish yield in lake 1 is increased from 20 to 40 fish, the result is the following game:



Because of the changes to the framework conditions, the only Nash equilibrium consists of sharing lake 1.

3.7 3-person games

A game does not necessarily have to consist of two players. In principle, any number of players is possible, e.g. if any number of fishermen dispute the two lakes in the example of resource economics.

In the following, it will be shown briefly for three players at least how a Nash equilibrium can be determined in this case. In this case, a third dimension would have to be inserted into the payoff matrix for the third player. In order to remain with a two-dimensional representation, this third dimension is represented instead by a separate payoff matrix per strategy of the third player (this can be imagined three-dimensionally as well as successively). The respective payoff matrices still consist in the lines of player 1 and in the columns of player 2. However, a third payoff for player 3 is added in the individual payoff fields.

The procedure will be illustrated by means of the resource economics game. Now, three fishermen are in dispute about the two lakes. Let lake 1 contain 18 fish and lake 2 contain 12 fish. This results in the following two payoff matrices:

Angler 3: (X_3: Lake 1)

Angler 1 \ Angler 2	X_2: Lake 1	Y_2: Lake 2
X_1: Lake 1	6, 6, 6	9, 9, 12
Y_1: Lake 2	12, 9, 9	6, 18, 6

Angler 3: (Y_3: Lake 2)

Angler 1 \ Angler 2	X_2: Lake 1	Y_2: Lake 2
X_1: Lake 1	9, 12, 9	18, 6, 6
Y_1: Lake 2	6, 6, 18	4, 4, 4

The Nash equilibriums now result with the help of the definition "it is not worthwhile for any player to be the only one to deviate from his strategy". For example, if player 1 goes to lake 1 and player 2 to lake 1, player 3 then has the choice between lake 1 (6 fish for player 3) and lake 2 (12 fish). Because 12 fish are naturally better than 6 fish, the value 12 is marked accordingly. This procedure is now implemented in the usual manner for all combinations. In the end, the following marked payoff matrices are obtained:

Angler 3: (X_3: Lake 1)

Angler 1 \ Angler 2	X_2: Lake 1	Y_2: Lake 2
X_1: Lake 1	6, 6, 6	**9**, **9**, **12**
Y_1: Lake 2	**12**, **9**, **9**	6, **18**, 6

Angler 3: (Y_3: Lake 2)

Angler 1 \ Angler 2	X_2: Lake 1	Y_2: Lake 2
X_1: Lake 1	**9**, **12**, **9**	**18**, 6, 6
Y_1: Lake 2	6, 6, **18**	4, 4, 4

The game therefore has three Nash equilibriums in pure strategies:

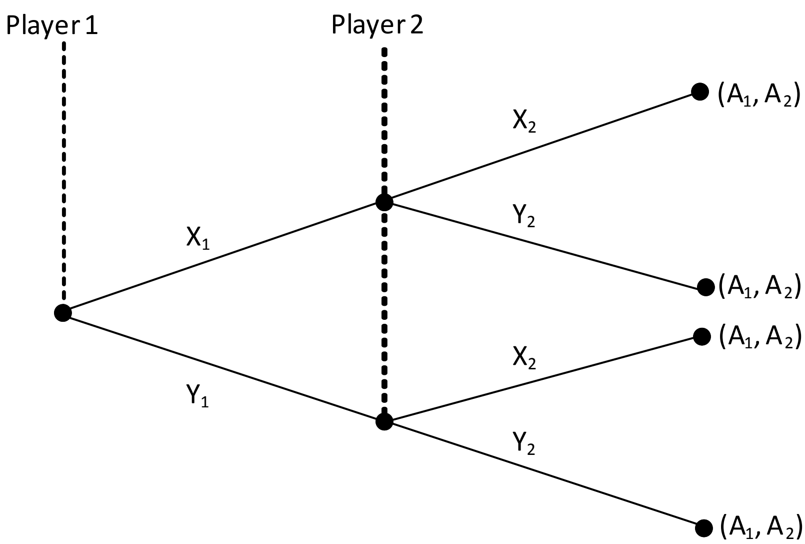
S* = (Lake 1, Lake 2, Lake 1)
S** = (Lake 2, Lake 1, Lake 1)
S*** = (Lake 1, Lake 1, Lake 2)

4 Sequential games

4.1 Foundations

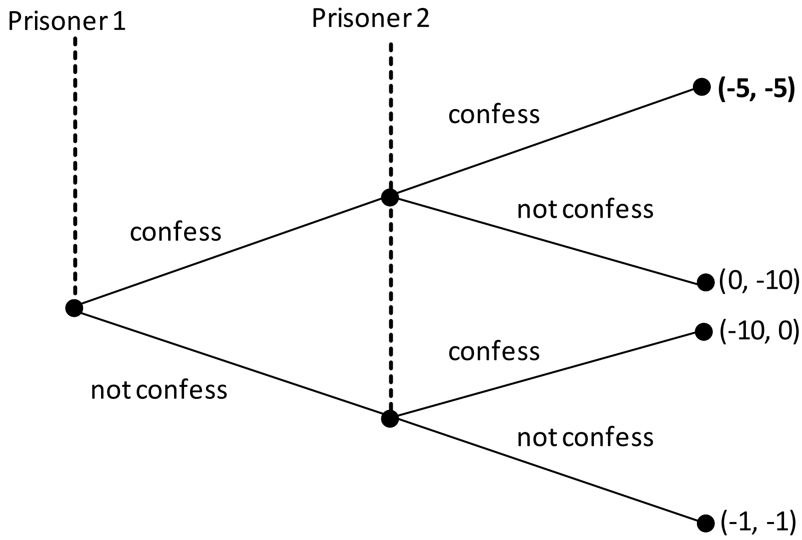
In contrast to simultaneous games *sequential games* are characterised by the fact that the choice of the strategies is not made simultaneously but successively – in other words sequentially. When the player who makes the second move chooses his strategy, he knows the first move. He can thus make his decision in dependence on the first move. Sequential games are also known as games in *extensive form*.

While simultaneous games are illustrated by means of matrices, sequential games are shown by means of a decision tree:



In this game, player 1 makes a decision first; he has the choice between X_1 and Y_1. Player 2 then makes his choice between X_2 and Y_2. Depending on what the choice of player 1 is, player 2 makes his decision in another node. Because the game's information is known to the players, player 2 can already consider from the start how he will decide in which case. In this way, a compete decision tree is created at the end of which stand the payoffs for player 1 and 2.

The prisoner's dilemma is taken again as an example – this time in extensive form, whereby prisoner 1 makes his decision first:



If prisoner 1 confesses, prisoner 2 will also confess, because he will then have to go to prison for 5 years instead of 10. If prisoner 1 does not confess, prisoner 2 will also confess, because he will then go free. Because prisoner 1 knows how prisoner 2 will react, he has the choice of confessing and then going to prison for 5 years (because prisoner 2 will confess in this case as well) or not confessing. In the latter case he will go to prison for 10 years, because prisoner 2 confesses with this choice. In other words, both confess – as in the simultaneous game as well.

4.2 Terms

After this basic overview of the structure of a sequential game, a few essential terms will now be introduced.

A *decision node* is any point in a game at which a player can make a decision. The prisoner's dilemma, for example, has 3 decision nodes.

Path P designates the path from the first decision node to any payoff. One possible path in the prisoner's dilemma would be, for example:

P = (do not confess, do not confess)

The payoff for this path would be

A(P) = A(do not confess, do not confess) = (-1, -1)

A player must pursue a strategy for each of his decision nodes. In the prisoner's dilemma, prisoner 1 has the choice between 'confess' and 'do not confess' at node 1. His strategy could therefore be

S_1 = confess

Prisoner 2 has two decision nodes. For example, if he decides at node 2 on 'confess' and at node 3 on 'do not confess', he has the strategy

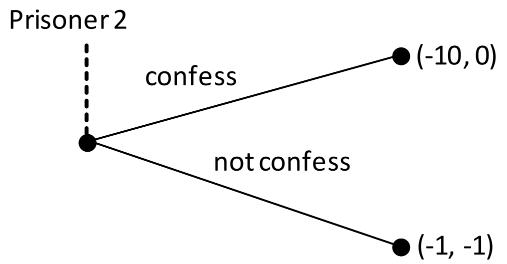
$$S_2 = (\text{confess, do not confess})$$

The strategy of the whole game is then

$$S = (\text{confess, (confess, do not confess)})$$

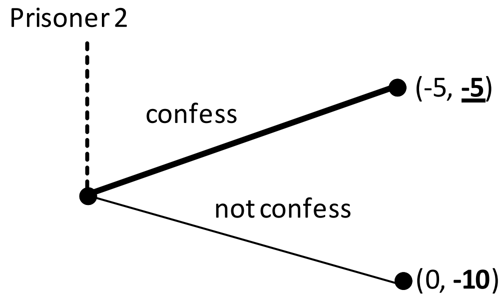
4.3 Subgames and subgame perfect equilibriums

Subgames are all games that start in any decision node, i.e. in each game there are exactly as many subgames as decision nodes. The whole game itself is a subgame as well. A subgame of the prisoner's dilemma would be, e.g.



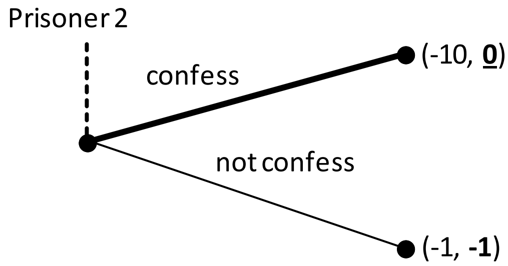
To find out which strategy is played in the equilibrium, starting with the smallest subgames the path is marked at each decision node that is selected for this subgame. The decision tree is, as it were, 'rolled up from behind'.

In the example of the prisoner's dilemma the first subgame is:

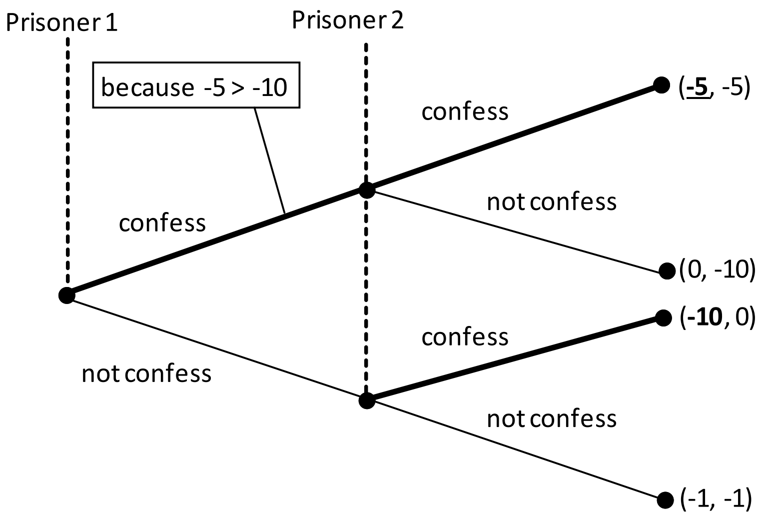


Because the equilibrium strategy for prisoner 2 in this subgame is 'confess', the appropriate path is marked.

This is repeated for the second subgame, in which the equilibrium path is marked again ('confess'):



The third subgame then corresponds to the complete decision tree, whereby the markings of the first two subgames are retained:



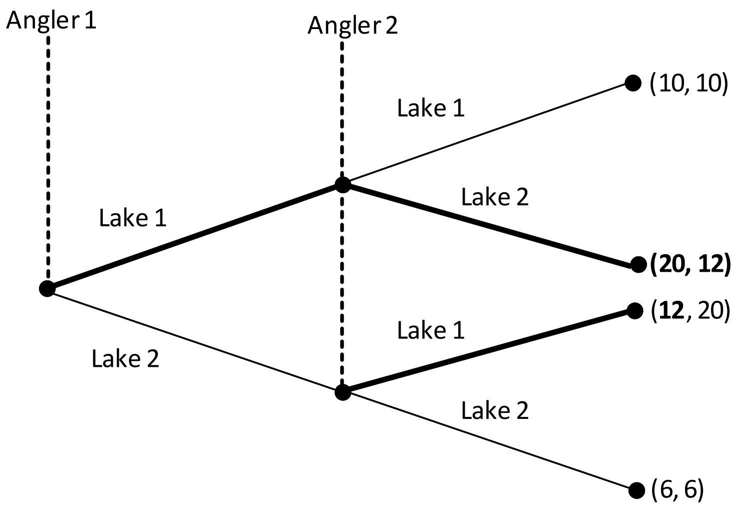
If prisoner 2 confesses in nodes 2 and 3, prisoner 1 has the choice in node 1 between 5 years (S_1 = 'confess') and 10 years (S_1 = 'do not confess'). Prisoner 1 therefore decides in the equilibrium for S_1 = confess.

The strategy that is in equilibrium for each subgame is called the *subgame perfect equilibrium*. Subgame perfect equilibriums are also Nash equilibriums. The subgame perfect equilibrium in the prisoner's dilemma is:

S^* = (confess, (confess, confess))

4.4 Sequential games played simultaneously and Nash equilibriums

Sequential games can always be modelled as simultaneous games as well. This will be illustrated with the example of the two anglers who have to share two lakes. Angler 1 is to decide first:



The game has the subgame perfect equilibrium S^* = (lake 1, (lake 1, lake 2)). In this equilibrium, angler 1 will decide on lake 1, whereupon angler 2 goes to the other lake.

Angler 2 has two selection options: lake 1 and lake 2. In contrast, angler 2 has a total of four selection options, because he has to make a choice for two decision nodes. His selection options are

- (lake1, lake1)
- (lake1, lake2)
- (lake2, lake1)
- (lake2, lake2)

The simultaneous game can already be modelled with this information:

Player 2

Player 1	Lake 1; Lake 1	Lake 1; Lake 2	Lake 2; Lake 1	Lake 2; Lake 2
Lake 1	10, 10	10, 10	20, 12	20, 12
Lake 2	12, 20	6, 6	12, 20	6, 6

The game has a total of three 3 Nash equilibriums:

S* = (Lake 1, (Lake 2, Lake 1))
S** = (Lake 1, (Lake 2, Lake 2))
S*** = (Lake 2, (Lake 1, Lake 1))

The Nash equilibrium S* is also a subgame perfect Nash equilibrium. As shown here, not every Nash equilibrium is also subgame perfect (however, each subgame perfect equilibrium is also a Nash equilibrium).

How can a Nash equilibrium in a sequential game be imagined? We recall that in a Nash equilibrium no player has an incentive to depart from the equilibrium strategy unilaterally. We check this criterion at S***, because this Nash equilibrium obviously takes another course as the subgame perfect equilibrium. In S*** player 1 goes to lake 2, player 2 then goes to lake 1. Player 1 receives 12 fish, player 2 receives 20 fish.

What would happen if one of the two deviates unilaterally from this strategy? Let us assume that player 1 would go to lake 1 instead of to lake 2. Player 2 – who does not change his strategy according to assumption – would then also go to lake 1 and player 1 then only receives 10 fish. Therefore, player 1 does not have an incentive to depart from the equilibrium strategy unilaterally.

Now let it be assumed that player 2 changes his strategy. He has two possibilities for this: he can change to lake 2 in the second node and in the third node. Let us start with the third node: if player 2 changes to lake 2 in this node, he then has a payoff of just 6 fish instead of 20 in equilibrium S***. If player 2 changes to lake 2 in the second node, at first nothing happens, because player 1 in S*** decides on lake 2 in the first node and thus the decision is made in the third node, and not in the second. However, because player 1 would no longer go to lake 2, the new strategy is unstable (and not "best response to itself"). For this reason, S*** is a Nash equilibrium.

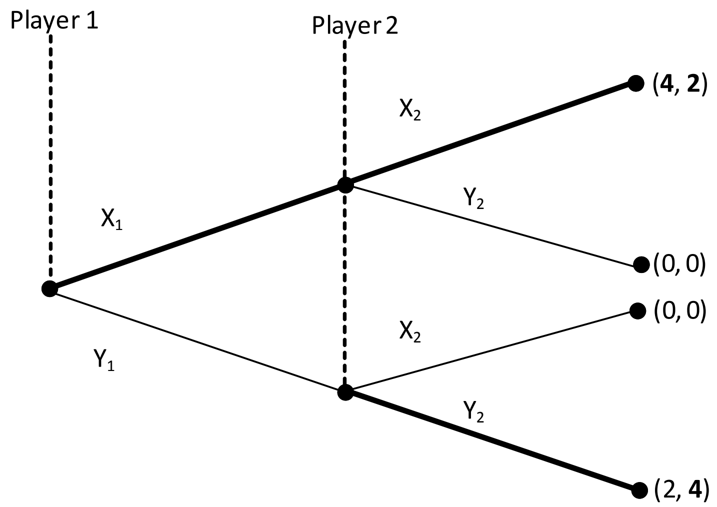
4.5 The First Mover's Advantage (FMA)

It can be seen easily that in some games it is an advantage to be the first to make a decision to which the other player then has to react. The player making the first decision then has the *First Mover's Advantage* (FMA).

The prisoner's dilemma does not have an FMA. Whatever is decided, both go to prison for five years, whether the game is played simultaneously or sequentially. We will now check whether there is an FMA for the following game:

		Player 2	
		X_2 (q)	Y_2 (1-q)
Player 1	X_1 (p)	4, 2	0, 0
	Y_1 (1-p)	0, 0	2, 4

This game is to be carried out as a sequential game, whereby player 1 begins:



In the subgame perfect equilibrium player 1 receives payoff 4 and player 2 receives payoff 2. If player 2 were to be the first to decide, he would receive payoff 4, and player 1 only payoff 2. The player who decided first therefore has an advantage, so that this game has an FMA.

4.6 An example: the Cuba crisis

Nuclear war almost broke out in 1964. Russia under Khrushchev wanted to station atom bombs in Cuba. The USA, under Kennedy, then threatened to attack Russia. At the last minute, Russia decided against the atom bombs. This situation will now be modelled in a game that has both sequential and simultaneous components.

Russia provokes the USA. The USA then has two possibilities

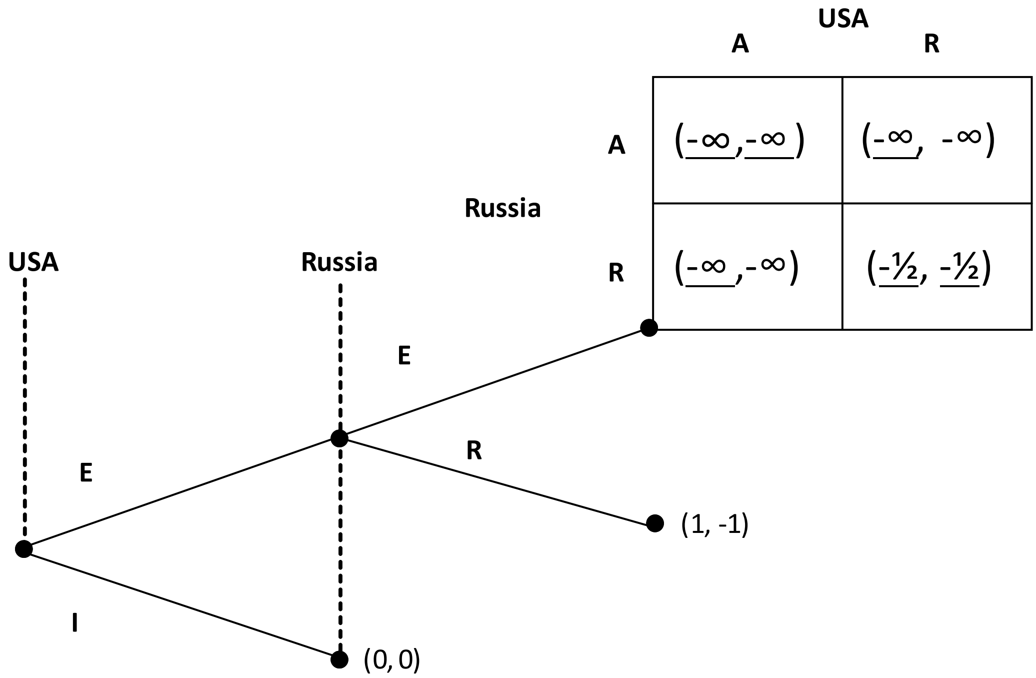
- It can ignore the provocation **(I)** (both receive 0) or
- let the situation escalate **(E)**.

If the USA lets the situation escalate, Russia can either

- retreat **(R)** (the USA wins the conflict and receives 1, Russia loses and receives -1) or
- let the tension escalate further **(E)**.

If both countries let the situation escalate, they have to decide simultaneously whether they want to start a nuclear war **(A)** or to retreat **(R)**. As soon as a country decides on a nuclear war, the payoff for both countries is ($-\infty$). If both countries retreat, they admit their own defeat and each receives ($-½$). Because a joint defeat is less painful than a sole defeat, this payoff of ($-½$) is less negative than in the case of a unilateral defeat (-1).

These situations will be modelled in the following game:

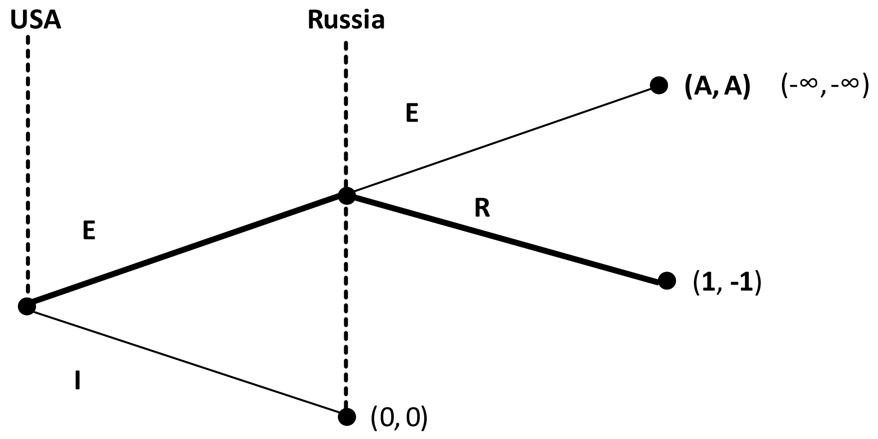


In game theory, we start with the consideration of the game with the last decision of the game. There are two Nash equilibriums in the simultaneous part of the game:

S* = (A, A) and
S** = (R, R).

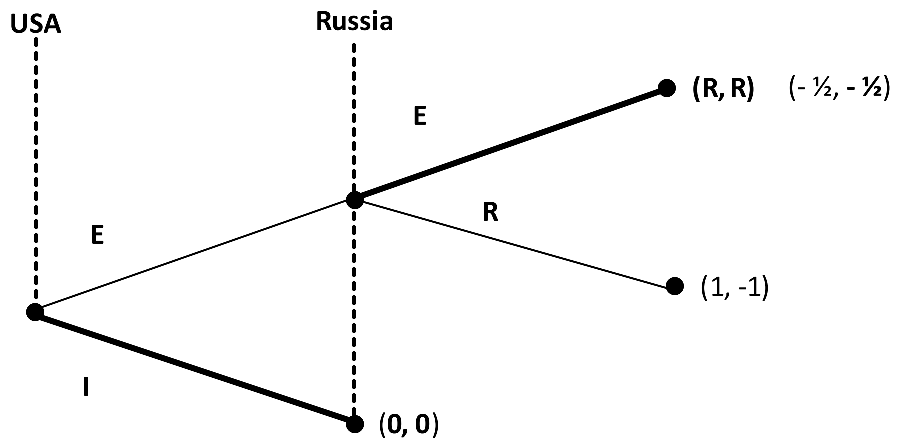
Because we do not know which Nash equilibrium will be played, we have to consider both possibilities.

Let us assume first of all that the first Nash equilibrium S* = (A, A) is played; the result is the following sequential game:



In this case, the USA will let the situation escalate and threaten with nuclear war; following this, Russia will retreat. There will therefore be no nuclear war.

Now let us assume that the second Nash equilibrium S** = (R, R) is played in the simultaneous game. This results in the sequential game:



In this case, the USA will ignore the provocation right at the start, and nothing happens. Both receive a zero payoff, which means keeping the peace without effort.

If we take a closer look at the simultaneous game, we find that there is a dominant strategy for each player, namely to retreat. For this reason we can assume that the second Nash equilibrium S** = (R, R) will be played, because this is a strict Nash equilibrium.

5 Negotiations (cooperative games)

5.1 Foundations

After presenting non-cooperative games in chapters 3 and 4, we will now take a brief look at cooperative games. In non-cooperative games each player attempts to maximise his own benefit, without considering the opponent's benefit, but in cooperative games (negotiations) players team up to maximise and distribute the joint benefit. In this way, opponents become fellow players.

The game theory *negotiation problem* is characterised by the fact that several players have a joint interest in an agreement with regard to the object of the negotiations, but pursue different objectives. At the common distribution of a cake, the players, for example, must agree on the one hand to a distribution, however, on the other hand, individual players will want to claim as large a piece as possible for themselves.

The objectives of negotiations are, on the one hand, that the cooperation of all participants is intended to bring an additional benefit in comparison with non-cooperative behaviour; on the other hand, this additional benefit is to be distributed among the players as equitably as possibly.

5.2 Coalitions

A coalition is a combination of several players. Let *coalition K* below designate the cooperation of a specific number of players from the player set *N*. A coalition may consist of the total set of players as well as of a subset of these players.

The following example should make this clear: there are five farms in a region, of which farms 1, 4 and 5 combine in order to increase their profits. For the coalition and the player set:

Player set N = {1, 2, 3, 4, 5}
Coalition K = {1, 4, 5}

A coalition whose player set consists of one element is referred to below as *ones coalition*; a coalition that consists of all players is known as an *all coalition*.

5.3 The characteristic function

The joint payoff of a coalition is designated the *characteristic function v(K)* of a coalition. It is of decisive importance in the analysis of a game, because the characteristic function can be used to determine which coalitions bring an advantage in comparison with a non-cooperative game method (one coalition) and which of these coalitions are those that bring the greatest advantage.

As an example, let there be a player set N = {1, 2, 3}. The value of the characteristic function is now to be given for each possible coalition:

Coalition K	v(K)
{ 1 }	$10
{ 2 }	$10
{ 3 }	$15
{ 1, 2 }	$25
{ 1, 3 }	$25
{ 2, 3 }	$30
{ 1, 2, 3 }	$40

If players 1 and 2 form a coalition, they receive jointly $5 more than in the ones coalition. This added value can be formed through the difference in the values of the characteristic function:

Added value {1, 2} = v({1, 2}) – v({1}) – v({2}) = $25 – $10 – $10 = **$5**

The same constellation applies to the coalition between players 2 and 3, who would also receive $5 more together ($30 instead of $25). In the all coalition as well there is a total of $5 more available ($40 instead of $35). Only a coalition between players 1 and 3 does not bring any added value; in both cases they receive a total of $25.

But which coalition will now be formed and be stable enough against 'attempted poaching' by players not included in the coalition?

Because players 1 and 3 do not receive more together, they are dependent on player 2. He can pick his 'partner'. A coalition made up of all players does not make sense for any player, because they would not receive more in comparison with a two players coalition with player 2, and must share this 'extra amount' among more players.

The result is therefore that player 2 will form a coalition with either player 1 or player 3. He will decide on the player from whom he can claim the most of the $5 gained for himself. For this reason, players 1 and 3 will try to 'undercut' each other and name a minimal share of the $5 as an own share, in order to be considered for a coalition, because even a minimal amount is still more than the amount in a ones coalition.

5.4 The cake game

In the cake game, several players share a cake of size 1. In the following example, three players attempt to divide the cake among them, whereby a coalition of two players forms a 'majority', which can decide on the division without the agreement of the third player. This example will be used to show that not every game enables a stable coalition.

To approach a solution we assume at first in step 1 that one of the players has the idea of dividing the cake 'equitably': one third each. A step is preferred to the previous one, if two of the three players can improve their position and form a coalition with majority appeal. In the following table there is a (+) or (-) between the steps in the shares columns, depending on whether a player's position improves or deteriorates:

	Shares		
	Player 1	Player 2	Player 3
(1) Player 3 has the idea of dividing the cake so that each player receives 1/3	33%	33%	33%
(2) Players 1 and 2 reject this and form a coalition in which each receives half. Player 3 receives nothing.	50% (+)	50% (+)	0% (-)
(3) Because player 3 wants part of the cake as well, he has the following idea: he offers player 1 75% of the cake and takes 25% himself. In this way, they both receive more than in (2). Player 2 receives nothing.	75% (+)	0% (-)	25% (+)
(4) Because player 2 now receives nothing, he offers player 3 50%. Player 2 would then receive 50% as well. Player 1 receives nothing.	0% (-)	50% (+)	50% (+)

As we can see, step 2 corresponds to the division in step 4 (two players each get half, the third player receives nothing), so that an infinite circle can be constructed here. For this reason, this game does not have a stable solution – for each solution there exists a better solution, that is, each solution is *dominated* by another solution.

5.5 Negotiations between two players

On negotiations between two players it has to be determined how a profit made jointly is to be apportioned between the two. An example is intended to show what such an apportionment solution can look like.

The two negotiation partners Rene (player 1) and Jenny (player 2) buy a jewel in Rostock for €6,000. Rene pays a share of €2,000 and Jenny pays €4,000. They sell the jewel in Munich's Maximilianstrasse for €10,000, i.e. they earn a joint profit of €4,000. How do they apportion the profit?

The following parameters and constraints apply for this game

- Let p = 10,000 be the proceeds of the sale in Munich
- Neither wants to receive less than they paid, i.e. Rene wants at least €2,000 and Jenny at least €4,000. We call these points threat points. The game contains the two *threat points*
  - $d_1 = 2000$
  - $d_2 = 4000$
- Let the sought after apportionments of the profits be u_1 for Rene and u_2 Jenny.
- The constraint $u_1 + u_2 = 10{,}000$ applies, that is, the complete sum must be apportioned.

To determine the solution we apply the so-called *Nash solution*, which maximises the term

$$(u_1 - d_1) \bullet (u_2 - d_2)$$

under the above-mentioned constraint.

We thus obtain the maximisation problem

$$\max\ (u_1 - d_1) \bullet (u_2 - d_2)$$
$$\text{under the constraint } p - u_1 - u_2 = 0$$

The constraint is (as already done here) to be reformed so that there is a zero on the right-hand side.

The result is the *LaGrange function L*:

$$L = \text{maximisation function} + \lambda \bullet \text{constraint}$$
$$L = (u_1 - d_1) \bullet (u_2 - d_2) + \lambda \bullet (p - u_1 - u_2)$$

The function L is now derived successively in accordance with u_1, u_2 and λ and set equal to zero.

Differentiate with respect to u_1:

$u_2 - d_2 - \lambda = 0$
With $d_2 = 4000$ then
$u_2 - 4000 - \lambda = 0$ **(1)**

Differentiate with respect to u_2:

$u_1 - d_1 - \lambda = 0$
With $d_1 = 2000$ then
$u_1 - 2000 - \lambda = 0$ **(2)**

Differentiate with respect to λ:

$p - u_1 - u_2 = 0$
With $p = 10000$ then
$10000 - u_1 - u_2 = 0$ **(3)**

Because three equations are now contrasted with three variables, a value can be calculated for each variable. The result for the sought-for apportionments of profits u_1 and u_2 is thus

$u_1^* = 4000$
$u_2^* = 6000$

Under the Nash solution, Rene receives €4,000 from the proceeds and the sale and Jenny receives €6,000. This result is equitable in that each receives the same profit of €2,000. However, it is inequitable, in that Rene receives a much greater return on the capital he invested.

5.6 Distinguishing cooperative and non-cooperative games

While non-cooperative games were considered in chapters 3 and 4, the present chapter has provided a brief overview of some aspects of cooperative games. These two types of games differ as well in the way their questions are worded.

In cooperative games the question is which coalitions bring which profits and how the joint profits are apportioned to the members of the coalition. In contrast, in non-cooperative games the interesting question is which strategies the players should pursue and which equilibriums the game has (even if equilibrium solutions of course can be constructed in cooperative games as well).

Strategies in cooperative games are therefore not to do one thing or the other, but with which ‘partners’ a player should form a coalition and what gain he can expect from this.

6 Decisions under uncertainty

6.1 Modelling uncertainty

In game theory, uncertainty exists if a player does not know what to expect. In the *theory of economic uncertainty* this uncertainty is modelled with lotteries, in which results or payoffs are modelled with probabilities from which *expected payoffs* result. A player usually has the choice between several lotteries. The profit, or the payoff, is designated x, the set of possible payoffs as $X = \{x_1, x_2, \dots , x_n\}$ with the probabilities $\{p_1, p_2, \dots , p_n\}$. The expected payoff over all possible payoffs is designated accordingly E(X).

Let the following lottery be given as an example:

A player receives 2 million euros with a probability of 50%, with a further 50% he receives nothing. His payoff therefore amounts to 2 million euros with p = 0.5 and zero euros with (1-p) = 0.5. The following therefore applies for the parameters of this game:

$x_1 = 2000000,\ p = 0{,}5$
$x_2 = 0,\ (1\text{-}p) = 0{,}5$

The expected payoff is calculated from this as follows:

$$E(X) = x_1 \bullet p + x_2 \bullet (1\text{-}p)$$
$$= 2000000 \bullet 0{,}5 + 0 \bullet 0{,}5$$
$$= 1000000$$

The player therefore has an expected payoff of 1 million euros.

In contrast, let the following example serve as lottery 2:

The player receives 1 million euros as the certain payoff. Therefore:

$$x_1 = 1000000,\ p = 1$$

The expected payoff is

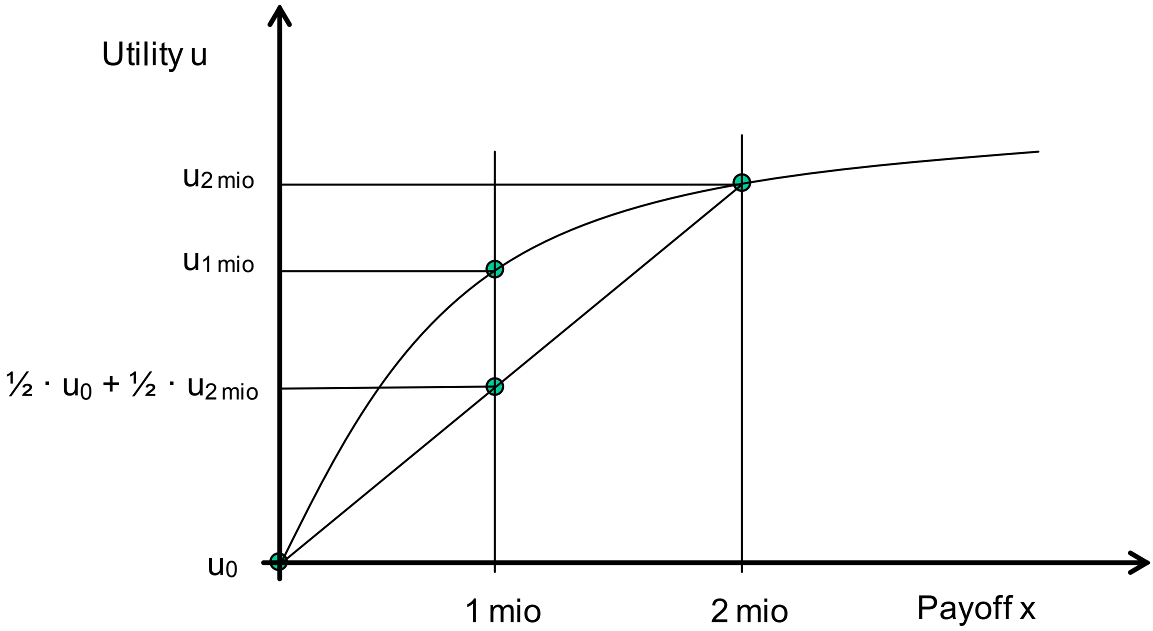
$$E(X) = x_1 \bullet p = 1000000 \bullet 1$$
$$= 1000000$$

The expected payoff in lottery 2 is thus 1 million euros as well. The player can now decide on one of the lotteries. Because both lotteries have the same expected payoff, he should be indifferent between lottery 1 and lottery 2. However, in fact, nearly all players prefer lottery 2, in which they receive 1 million euros with certainty. To explain this rationally, a utility function u(x) is introduced on the payoff x.

6.2 The utility function u(x)

As the above example has made clear, money does not always appear to be the same. The reason for this is the decreasing appreciation of constantly increasing amounts of money. A player who does not have any money himself will value the certain 1 million euros much higher than the uncertain 2 million euros, although both options have the same expectancy value. However, the second million euros has a lower value for the player than the first million euros – we can therefore say that 1 million euros is not always worth the same.

These circumstances can be shown in a graph by a concave utility function:



A utility function of this type can be modelled as follows, for example:

$$u(x) = \sqrt{x}$$

The utility of 1 million euros is then

$$u(10000000€) = \sqrt{10000000} = 1000$$

and the utility of 2 million euros is correspondingly

$$u(20000000€) = \sqrt{20000000} \approx 1414.$$

The second million therefore only has a utility of around 414, while the first million, with 1000, has even twice as much utility.

6.3 The expected utility

After the expected payoff and the utility function were introduced, the expected utility can also be calculated through the relationship u(x). The utility of a payoff is shown with 'u' and the utility of a lottery with 'U'.

To return to our example with the two lotteries, we know now that the utility of 2 million euros with the introduced utility function u(x) = $\sqrt{x}$ is less than twice as much as the utility of 1 million euros. As the graph makes clear, lottery 1 is preferred to lottery 2, because the following relationship applies:

$$u(1\text{ million}) > \tfrac{1}{2} \bullet u(0) + \tfrac{1}{2} \bullet u(2\text{ million})$$

We can therefore say as well that the *expected utility of* lottery 2 is greater than the expected utility for lottery 1.

The expected utility is calculated for lottery 1 as:

$$E(U_1) = \tfrac{1}{2} \bullet u(0) + \tfrac{1}{2} \bullet u(2\text{ Mio}) = \tfrac{1}{2} \bullet 0 + \tfrac{1}{2} \bullet 1414 = \mathbf{707}$$

The expected utility for lottery 2 results as follows:

$$E(U_2) = 1 \bullet u(1\text{ Mio}) = \mathbf{1000}$$

Therefore, $E(U_2) > E(U_1)$.

7 Anomalies in game theory

7.1 Foundations

Up to now, it was always assumed that all players always act rationally and egoistically and attempt to maximise their own (expected) utility. However, in reality there are some experiments whose implementation shows that this is not always the case and that participants do not always obey the rules of game theory. It can be seen that factors such as trust, fairness, punishment, or simply a false assessment of a situation, can also play a role.

In this context, a few experiments will be shown briefly below in which the participants (apparently) act irrationally.

7.2 Games under uncertainty: the Ellsberg paradox

In this game the player has the choice between 2 urns, each with 100 balls, from which a ball is to be removed. If the player forecasts the right colour, he wins a sum of money.

The following two options are available for the choice of the urns:

- Urn 1 contains 50 red and 50 black balls
- Urn 2 contains 100 balls, which are either all red or all black.

According to the rules of game theory, the player should be indifferent between the two urns, because the probability of forecasting the right colour is 50 percent in each case. However, in reality, the experimental subjects mainly choose urn 1.

7.3 Games without uncertainty

7.3.1 The ultimatum game

In the ultimatum game $100 are to be divided between two players. First of all, player 1 (proposer) makes a suggestion as to how the amount is to be divided. Player 2 (responder) can accept this proposal – and the amount is then split accordingly – or reject it. If he rejects it, neither player receives anything.

From the aspect of game theory, player 1 offers player 2 a minimal amount of $1, keeping the remaining $99 for himself. Player 2 accepts, because an amount of $1 is still preferable to nothing at all.

However, in reality, game theory is hardly considered at all. The proposer's most frequent offer here is 50/50, i.e. $50 for each player. In addition, responders are quite prepared to reject low offers, even though they then have a lower utility themselves. In this case, responders are obviously pursuing a *punishment strategy*.

7.3.2 The dictator game

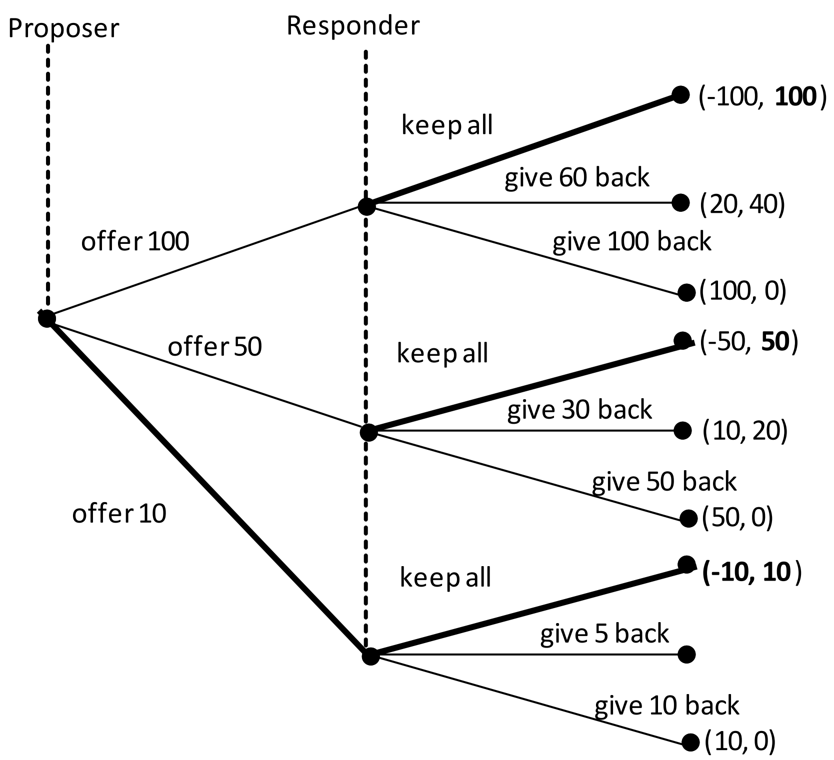
In the dictator game the rules are analogous to the ultimatum game, with the sole difference that the responder may not refuse. This means that a responder may not reject an offer that is disadvantageous for him in order to punish the proposer. According to game theory, the proposer would keep the complete amount for himself – because no one can stop him from pocketing the money.

However, in reality, although the share of 50/50 offers is clearly reduced in comparison with the ultimatum game, the proposer still usually makes the responder an offer of an amount between \$10 and \$50. This can be explained, for example, by a *fairness strategy* of the proposer.

7.3.3 The gift exchange game

In this game between a proposer and a responder each player has scope for apportioning. At first, the proposer gives the responder an amount between \$10 and \$100, which he pays himself. After this, the responder can either keep the money or give some of it back to the proposer. The amount that the proposer gets back is doubled for him.

If this game is simplified any restricted to just a few options, the following game can be modelled in an extensive form:



From the aspect of game theory, in a subgame perfect equilibrium the proposer will give the responder the minimal amount of $10, which the latter will then keep in full for himself.

In reality however, proposers tend to offer responders a large amount. For their part, responders are prepared to return a large part of this to the proposer. The more a responder receives, the more he is prepared to give back. This can also be interpreted as a *trust strategy* on the part of the proposer and as a *reward strategy* of the responder.

If it is assumed that the proposer gives the responder $100 and the latter gives $70 back to the proposer, the proposer receives $140 and the responder $30, which means that both are better placed than in the subgame perfect equilibrium. Although they both act irrationally from the aspect of game theory, they 'ensure' themselves more money than in game theory equilibrium.

7.3.4 Explanations: fairness, punishment, trust and reward

The results of the games without uncertainty suggest that

- players do not (always) maximise their payoff, and that
- players do not (always) act egoistically.

The question that arises here is therefore:

Do players not maximise (and therefore do not act rationally) or do players maximise in a different way to their monetary payoff (i.e. players act rationally, but there must be payoffs other than the monetary payoff)?

Attempts at explanations form *social norms* ("You don't treat people badly. If you treat someone badly, you feel bad yourself ") and *reciprocity* ("do good to those who do good to you, and harm those who harmed you ").

In a look back at the games introduced here, it will be examined for which games social norms are appropriate, and for which games reciprocity appears to be relevant:

- Ultimatum game: fairness norm and reciprocity
  - 50:50 rule is "fair" and provides the reason for the observed behaviour.
  - Unfair behaviour (infringement of the norm) is punished by rejection.
- Dictator game: fairness norm and egoism
  - The proposer thinks that it is fair to give up some of his money. However, half is rarely handed over, because he cannot be punished.
  - A mixture of egoism and fair behaviour is possible.
- Gift exchange game: trust and reward
  - The proposer trusts the responders.
  - The responders rewarded this trust and give a higher amount back. They appear to display reciprocal behaviour.

But when does a norm take effect, and which norm is this? Why does it apply? When do players act egoistically and when do they act fairly?

There have been some attempts to express these modes of behaviour in equations. One first attempt was Rabin's model, which includes the players' "friendliness" in the utility function:

$$u_i(a_i, b_j, c_i) = \pi(a_i, b_j) + f_i'(b_j, c_i) \bullet [1 + f_i(a_i, b_j)]$$

The parameters of this game are:

a_i	Decision by player i
b_j	Decision by j expected by i
c_i	Expectation of i as to the action by i that j expects
$\pi(a_i, b_j)$	Monetary payoff
$f_i'(b_j, c_i)$	Expectation of i regarding j's friendliness

$f_i(a_i, b_j)$ Expected friendliness of i towards j

This model offers an explanation for some experiments, but there are also experiments that negate this model. The fact is that game theory has recognised that a person's behaviour is not caused solely by maximisation of his utility. Many game theoreticians now believe, for example, in reciprocity. On the other hand, cognitive overload may lead to players not acting rationally in specific situations.

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