Strings, Branes and Extra Dimensions

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Abstract

This review is devoted to strings and branes. Firstly, perturbative string theory is introduced. The appearance of various types of branes is discussed. These include orbifold fixed planes, D-branes and orientifold planes. The connection to BPS vacua of supergravity is presented afterwards. As applications, we outline the role of branes in string dualities, field theory dualities, the AdS/CFT correspondence and scenarios where the string scale is at a TeV. Some issues of warped compactifications are also addressed. These comprise corrections to gravitational interactions as well as the cosmological constant problem.

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Chapter 1

Introduction

One of the most outstanding problems of theoretical physics is to unify our picture of electroweak and strong interactions with gravitational interactions. We would like to view the attraction of masses as appearing due to the exchange of particles (gravitons) between the masses. In conventional perturbative quantum field theory this is not possible because the theory of gravity is not renormalizable. A promising candidate providing a unified picture is string theory. In string theory, gravitons appear together with the other particles as excitations of a string.

On the other hand, also from an observational point of view gravitational interactions show some essential differences to the other interactions . Masses always attract each other, and the strength of the gravitational interaction is much weaker than the electroweak and strong interactions. A way how this difference could enter a theory is provided by the concept of "branes". The expression "brane" is derived from membrane and stands for extended objects on which interactions are localized. Assuming that gravity is the only interaction which is not localized on a brane, the special features of gravity can be attributed to properties of the extra dimensions where only gravity can propagate. (This can be either the size of the extra dimension or some curvature.)

The brane picture is embedded in a natural way in string theory. Therefore, string theory has the prospect to unify gravity with the strong and electroweak interactions while, at the same time, explaining the difference between gravitational and the other interactions.

This set of notes is organized as follows. In chapter 2, we briefly introduce the concept of strings and show that quantized closed strings yield the graviton as a string excitation. We argue that the quantized string lives in a ten dimensional target space. It is shown that an effective field theory description of strings is given by (higher dimensional supersymmetric extensions of) the Einstein Hilbert theory. The concept of compactifying extra dimensions is introduced and special stringy features are emphasized. Thereafter, we introduce the orbifold fixed planes as higher dimensional extended objects where closed string twisted sector excitations are localized. The quantization of the open string will lead us to the concept of D-branes, branes on which open string excitations live. We compute the tensions and charges of D-branes and derive an effective field theory on the world volume of the D-brane. Finally, perturbative string theory contains orientifold planes as extended objects. These are branes on which excitations of unoriented closed strings can live. Compactifications containing orientifold planes and D-branes are candidates for phenomenologically interesting models. We demonstrate the techniques of orientifold compactifications at a simple example.

In chapter 3, we identify some of the extended objects of chapter 2 as stable solutions of the effective field theory descriptions of string theory. These will be the fundamental string and the D-branes. In addition we will find another extended object, the NS five brane, which cannot be described in perturbative string theory.

Chapter 4 discusses some applications of the properties of branes derived in the previous chapters. One of the problems of perturbative string theory is that the string concept does not lead to a unique theory. However, it has been conjectured that all the consistent string theories are perturbative descriptions of one underlying theory called M-theory. We discuss how branes fit into this picture. We also present branes as tools for illustrating duality relations among field theories. Another application, we are discussing is based on the twofold description of three dimensional D-branes. The perturbative description leads to an effective conformal field theory (CFT) whereas the corresponding stable solution to supergravity contains an AdS space geometry. This observation results in the AdS/CFT correspondence. We present in some detail, how the AdS/CFT correspondence can be employed to compute Wilson loops in strongly coupled gauge theories. An application which is of phenomenological interest is the fact that D-branes allow to construct models in which the string scale is of the order of a TeV. If such models are realized in nature, they should be discovered experimentally in the near future.

Chapter 5 is somewhat disconnected from the rest of these notes since it considers brane models which are not directly constructed from strings. Postulating the existence of branes on which certain interactions are localized, we present the construction of models in which the space transverse to the brane is curved. We discuss how an observer on a brane experiences gravitational interactions. We also make contact to the AdS/CFT conjecture for a certain model. Also other questions of phenomenological relevance are addressed. These are the hierarchy problem and the problem of the cosmological constant. We show how these problems are modified in models containing branes.

Chapter 6 gives hints for further reading and provides the sources for the current text.

Our intention is that this review should be self contained and be readable by people who know some quantum field theory and general relativity. We hope that some people will enjoy reading one or the other section.

Chapter 2

Perturbative description of branes

2.1 The Fundamental String

2.1.1 Worldsheet Actions

2.1.1.1 The closed bosonic string

Let us start with the simplest string – the bosonic string. The string moves along a surface through space and time. This surface is called the worldsheet (in analogy to a worldline of a point particle). For space and time in which the motion takes place we will often use the term target space. Let d be the number of target space dimesnions. The coordinates of the target space are X^μ, and the worldsheet is a surface $X^\mu(\tau, \sigma)$, where τ and σ are the time and space like variables parameterizing the worldsheet. String theory is defined by the requirement that the classical motion of the string should be such that its worldsheet has minimal area. Hence, we choose the action of the string proportional to the worldsheet. The resulting action is called Nambu Goto action. It reads

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-g}. \tag{2.1.1.1}$$

The integral is taken over the parameter space of σ and τ. (We will also use the notation $\tau = \sigma^0$, and $\sigma = \sigma^1$.) The determinant of the induced metric is called g. The induced metric depends on the shape of the worldsheet and the shape of the target space,

$$g_{\alpha\beta} = G_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{2.1.1.2}$$

where μ, ν label target space coordinates, whereas α, β label worldsheet parameters. Finally, we have introduced a constant α'. It is the inverse of the string tension and has the mass dimension -2. The choice of this constant sets the string scale. By construction, the action (2.1.1.1) is invariant under reparametrizations of the worldsheet.

Alternatively, we could have introduced an independent metric $\gamma_{\alpha\beta}$ on the worldsheet. This enables us to write the action (2.1.1.1) in an equivalent form,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu. \tag{2.1.1.3}$$

For the target space metric we will mostly use the Minkowski metric $\eta_{\mu\nu}$ in the present chapter. Varying (2.1.1.3) with respect to $\gamma_{\alpha\beta}$ yields the energy momentum tensor,

$$T_{\alpha\beta} = -\frac{4\pi\alpha'}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\delta\gamma}\partial_\delta X^\mu\partial_\gamma X_\mu, \tag{2.1.1.4}$$

where the target space index μ is raised and lowered with $G_{\mu\nu} = \eta_{\mu\nu}$. Thus, the $\gamma_{\alpha\beta}$ equation of motion, $T_{\alpha\beta} = 0$, equates $\gamma_{\alpha\beta}$ with the induced metric (2.1.1.2), and the actions (2.1.1.1) and (2.1.1.3) are at least classically equivalent. If we had just used covariance as a guiding principle we would have written down a more general expression for (2.1.1.3). We will do so later. At the moment, (2.1.1.3) with $G_{\mu\nu} = \eta_{\mu\nu}$ describes a string propagating in the trivial background. Upon quantization of this theory we will see that the string produces a spectrum of target space fields. Switching on non trivial vacua for those target space fields will modify (2.1.1.3). But before quantizing the theory, we would like to discuss the symmetries and introduce supersymmetric versions of (2.1.1.3).

First of all, (2.1.1.3) respects the target space symmetries encoded in $G_{\mu\nu}$. In our case $G_{\mu\nu} = \eta_{\mu\nu}$ this is nothing but d dimensional Poincaré invariance. From the two dimensional point of view, this symmetry corresponds to field redefinitions in (2.1.1.3). The action is also invariant under two dimensional coordinate changes (reparametrizations). Further, it is Weyl invariant, i.e. it does not change under

$$\gamma_{\alpha\beta} \to e^{\varphi(\tau,\sigma)}\gamma_{\alpha\beta}. \tag{2.1.1.5}$$

It is this property which makes one dimensional objects special. The two dimensional coordinate transformations together with the Weyl transformations are sufficient to transform the worldsheet metric locally to the Minkowski metric,

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta}. \tag{2.1.1.6}$$

It will prove useful to use instead of σ^0, σ^1 the light cone coordinates,

$$\sigma^- = \tau - \sigma \ \text{, and } \ \sigma^+ = \tau + \sigma. \tag{2.1.1.7}$$

So, the gauged fixed version[1] of (2.1.1.3) is

$$S = \frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^- \partial_- X^\mu \partial_+ X_\mu. \tag{2.1.1.8}$$

However, the reparametrization invariance is not completely fixed. There is a residual invariance under the conformal coordinate transformations,

$$\sigma^+ \to \tilde{\sigma}^+ \left(\sigma^+\right) \quad , \quad \sigma^- \to \tilde{\sigma}^- \left(\sigma^-\right). \tag{2.1.1.9}$$

This invariance is connected to the fact that the trace of the energy momentum tensor (2.1.1.4) vanishes identically, $T_{+-} = 0$[2] . However, the other $\gamma_{\alpha\beta}$ equations are not identically satisfied and provide constraints, supplementing (2.1.1.8),

$$T_{++} = T_{--} = 0. \tag{2.1.1.10}$$

The equations of motion corresponding to (2.1.1.8) are[3]

$$\partial_+ \partial_- X^\mu = 0 \tag{2.1.1.11}$$

Employing conformal invariance (2.1.1.9) we can choose τ to be an arbitrary solution to the equation $\partial_+\partial_-\tau = 0$. (The combination of (2.1.1.9) and (2.1.1.7) gives

$$\tau \to \frac{1}{2}\left(\tilde{\sigma}^+\left(\sigma^+\right) + \tilde{\sigma}^-\left(\sigma^-\right)\right), \tag{2.1.1.12}$$

which is the general solution to (2.1.1.11)). Hence, without loss of generality we can fix

$$X^+ = \frac{1}{\sqrt{2}}\left(X^0 + X^1\right) = x^+ + p^+\tau, \tag{2.1.1.13}$$

where x^+ and p^+ denote the center of mass position and momentum of the string in the + direction, respectively. The constraint equations (2.1.1.10) can now be used to fix

$$X^- = \frac{1}{\sqrt{2}}\left(X^0 - X^1\right) \tag{2.1.1.14}$$

as a function of X^i $(i = 2, \ldots, d-1)$ uniquely up to an integration constant corresponding to the center of mass position in the minus direction. Thus we are left with

[1] Gauge fixing means imposing (2.1.1.6).

[2] The corresponding symmetry is called conformal symmetry. It means that the action is invariant under conformal coordinate transformations while keeping the worldsheet metric fixed. In two dimensions this is equivalent to Weyl invariance.

[3] For the time being we will focus on closed strings. That means that we impose periodic boundary conditions and hence there are no boundary terms when varying the action. We will discuss open strings when turning to the perturbative description of D-branes in section 2.3.

$d-2$ physical degrees of freedom X^i. Their equations of motion are (2.1.1.11) without any further constraints. By employing the symmetries of (2.1.1.3) we managed to reduce the system to $d-2$ free fields (satisfying (2.1.1.11)). Since these symmetries may suffer from quantum anomalies we will have to be careful when quantizing the theory in section 2.1.2.

2.1.1.2 Worldsheet supersymmetry

In this section we are going to modify the previously discussed bosonic string by enhancing its two dimensional symmetries. We will start from the gauge fixed action (2.1.1.8) which had as residual symmetries two dimensional Poincaré invariance and conformal coordinate transformations (2.1.1.9).[4] A natural extension of Poincaré invariance is supersymmetry. Therefore, we will study theories which are supersymmetric from the two dimensional point of view. In order to construct a supersymmetric extension of (2.1.1.8) one should first specify the symmetry group and then use Noether's method to build an invariant action. We will be brief and just present the result,

$$S=\frac{1}{2\pi\alpha'}\int d\sigma^+ d\sigma^-\left(\partial_- X^\mu\partial_+ X_\mu+\frac{i}{2}\psi_+^\mu\partial_-\psi_{+\mu}+\frac{i}{2}\psi_-^\mu\partial_+\psi_{-\mu}\right), \qquad (2.1.1.15)$$

where $\psi_\pm$ are two dimensional Majorana-Weyl spinors. To see this, we first note that

$$i\psi_+\partial_-\psi_+ + i\psi_-\partial_+\psi_- = -\frac{1}{2}\left(\psi_+,-\psi_-\right)\left(\rho^+\partial_+ + \rho^-\partial_-\right)\begin{pmatrix}\psi_-\\ \psi_+\end{pmatrix}, \qquad (2.1.1.16)$$

where

$$\rho^\pm=\rho^0\pm\rho^1, \qquad (2.1.1.17)$$

with

$$\rho^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ \text{and}\ \rho^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \qquad (2.1.1.18)$$

It is easy to check that the above matrices form a two dimensional Clifford algebra,

$$\left\{\rho^\alpha,\rho^\beta\right\}=-2\eta^{\alpha\beta}. \qquad (2.1.1.19)$$

Also, note that $i\left(\psi_+,-\psi_-\right)$ is the Dirac conjugate of $\begin{pmatrix}\psi_-\\ \psi_+\end{pmatrix}$ for real $\psi_\pm$, i.e. of the Majorana spinor $\begin{pmatrix}\psi_-\\ \psi_+\end{pmatrix}$. In addition to two dimensional Poincaré invariance and

[4] Alternatively, we could start from the action (2.1.1.3). This we would modify such that it becomes locally supersymmetric. Finally, we would fix symmetries in the locally supersymmetric action.

invariance under conformal coordinate transformations (2.1.1.9)[5] the action (2.1.1.15) is invariant under worldsheet supersymmetry,

$$\delta X^\mu = \bar{\epsilon}\psi^\mu = i\epsilon_+\psi_-^\mu - i\epsilon_-\psi_+^\mu, \tag{2.1.1.20}$$

$$\delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon. \tag{2.1.1.21}$$

In components (2.1.1.21) gives rise to the two equations

$$\delta\psi_-^\mu = -2\epsilon_+\partial_- X^\mu, \tag{2.1.1.22}$$

$$\delta\psi_+^\mu = 2\epsilon_-\partial_+ X^\mu. \tag{2.1.1.23}$$

When checking the invariance of (2.1.1.15) under (2.1.1.20), (2.1.1.22), (2.1.1.23) one should take into account that spinor components are anticommuting, e.g. $\epsilon_+\psi_- = -\psi_-\epsilon_+$. Since the supersymmetry parameters $\epsilon_\pm$ form a non chiral Majorana spinor, the above symmetry is called $(1,1)$ supersymmetry. (In the end of this section we will also discuss the chiral $(1,0)$ supersymmetry.) To summarize, the action (2.1.1.15) has the following two dimensional global symmetries: Poincaré invariance and supersymmetry. The corresponding Noether currents are the energy momentum tensor,

$$T_{++} = \partial_+ X^\mu\partial_+ X_\mu + \frac{i}{2}\psi_+^\mu\partial_+\psi_{+\mu}, \tag{2.1.1.24}$$

$$T_{--} = \partial_- X^\mu\partial_- X_\mu + \frac{i}{2}\psi_-^\mu\partial_-\psi_{-\mu}, \tag{2.1.1.25}$$

and the supercurrent

$$J_+ = \psi_+^\mu\partial_+ X_\mu, \tag{2.1.1.26}$$

$$J_- = \psi_-^\mu\partial_- X_\mu. \tag{2.1.1.27}$$

The vanishing of the trace of the energy momentum tensor $T_{+-} \equiv 0$ is again a consequence of the invariance under the (local) conformal coordinate transformations (2.1.1.9). The supercurrent is a spin–$\frac{3}{2}$ object and naively one would expect to get four independent components. That there are only two non-vanishing components is a consequence of the fact that the supersymmetries (2.1.1.20), (2.1.1.22), (2.1.1.23) leave the action invariant also when we allow instead of constant $\epsilon_\pm$ for

$$\epsilon_- = \epsilon_-\left(\sigma^+\right) \text{ and } \epsilon_+ = \epsilon_+\left(\sigma^-\right), \tag{2.1.1.28}$$

i.e. they are "partially" local symmetries. Once again, the vanishing of the energy momentum tensor is an additional constraint on the system. We did not derive this

[5] Under the transformation (2.1.1.9) the spinor components transform as $\psi_\pm \to \left(\tilde{\sigma}^{\pm\prime}\right)^{-\frac{1}{2}}\psi_\pm$.

explicitly here. But it can be easily inferred as follows. In two dimensions the Einstein tensor vanishes identically. Thus, if we were to couple to two dimensional (Einstein) gravity, the constraint $T_{\alpha\beta} = 0$ would correspond to the Einstein equation. Similarly, the supercurrents (2.1.1.26), (2.1.1.27) are constrained to vanish. (If the theory was coupled to two dimensional supergravity, this would correspond to the gravitino equations of motion.)

As in the bosonic case we can employ the symmetry (2.1.1.9) to fix

$$X^+ = x^+ + p^+\tau. \tag{2.1.1.29}$$

The local supersymmetry transformation (2.1.1.21) with ϵ given by (2.1.1.28) can be used to gauge

$$\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}^{\mu=+} = 0. \tag{2.1.1.30}$$

(We have written here the target space (light cone) index as $\mu = +$ in order to avoid confusion with the worldsheet spinor indices.) Note, that the gauge fixing condition (2.1.1.30) is compatible with (2.1.1.29) and the supersymmetry transformations (2.1.1.20), (2.1.1.21), as (2.1.1.30) implies the supersymmetry transformation

$$\delta X^+ = 0. \tag{2.1.1.31}$$

The constraints (2.1.1.24), (2.1.1.25), (2.1.1.26), (2.1.1.27) can be solved for X^-, and $\psi_\alpha^{\mu=-}$ (here, α denotes the worldsheet spinor index). Therefore, after fixing the local symmetries completely we are left with $d-2$ free bosons and $d-2$ free fermions (from a two dimensional point of view).

We should note that in the closed string case (periodic boundary conditions in bosonic directions) we have two choices for boundary conditions on the worldsheet fermions. Boundary terms appearing in the variation of the action vanish for either periodic or anti periodic boundary conditions on worldsheet fermions. (Later, we will call the solutions with antiperiodic fermions Neveu Schwarz (NS) sector and the ones with periodic boundary conditions Ramond (R) sector.

Going back to (2.1.1.15), we note that alternatively we could have written down a $(1,0)$ supersymmetric action by setting the left handed fermions $\psi_+^\mu = 0$. The supersymmetries are now given by (2.1.1.20) and (2.1.1.22), only. The parameter ϵ_- does not occur anymore, and hence we have reduced the number of supersymmetries by one half. More generally one can add left handed fermions λ_+^A which do not transform under supersymmetries. Therefore, they do not need to be in the same representation of the target space Lorentz group as the X^μ (therefore the index A instead of μ).

Summarizing we obtain the following $(1,0)$ supersymmetric action

$$S = \frac{1}{2\pi\alpha'}\int d\sigma^+ d\sigma^- \left(\partial_- X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_-^\mu \partial_+ \psi_{-\mu} + \frac{i}{2}\sum_{A=1}^{N} \lambda_+^A \partial_- \lambda_{+A}\right). \quad (2.1.1.32)$$

this will turn out to be the worldsheet action of the heterotic string. The energy momentum tensor is as given in (2.1.1.24), (2.1.1.25) with λ_+^A replacing ψ_+^μ in (2.1.1.25). There is only one conserved supercurrent (2.1.1.26).

Finally, we should remark that there are also extended versions of two dimensional supersymmetry (see for example [456]). We will not be dealing with those in this review.

2.1.1.3 Space-time supersymmetric string

In the above we have extended the bosonic string (2.1.1.3) to a superstring from the two dimensional perspective. We called this worldsheet supersymmetry. Another direction would be to extend (2.1.1.3) such that the target space Poincaré invariance is enhanced to target space supersymmetry. This concept leads to the Green Schwarz string. Space time supersymmetry means that the bosonic coordinates X^μ get fermionic partners θ^A (where A labels the number of supersymmetries N) such that the targetspace becomes a superspace. In addition to Lorentz symmetry, the supersymmetric extension mixes fermionic and bosonic coordinates,

$$\begin{aligned} \delta\theta^A &= \epsilon^A, & (2.1.1.33)\\ \delta\bar{\theta} &= \bar{\epsilon}^A, & (2.1.1.34)\\ \delta X^\mu &= i\bar{\epsilon}\Gamma^\mu\theta^A, & (2.1.1.35) \end{aligned}$$

where the global transformation parameter ϵ^A is a target space spinor and Γ^μ denotes a target space Dirac matrix. In order to construct a string action respecting the symmetries (2.1.1.33) – (2.1.1.35) one tries to replace $\partial_\alpha X^\mu$ by the supersymmetric combination

$$\Pi_\alpha^\mu = \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A. \quad (2.1.1.36)$$

This leads to the following proposal for a space time supersymmetric string action

$$S_1 = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\Pi_\alpha^\mu\Pi_{\beta\mu}. \quad (2.1.1.37)$$

Note that in contrast to the previously discussed worldsheet supersymmetric string, (2.1.1.37) consists only of bosons when looked at from a two dimensional point of view. The action (2.1.1.37) is invariant under global target space supersymmetry, i.e. Lorentz transformations plus the supersymmetry transformations (2.1.1.33) – (2.1.1.35). From the worldsheet perspective we have reparametrization invariance and Weyl invariance (2.1.1.5). This is again enough to fix the worldsheet metric $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ (cf (2.1.1.6)). The resulting action will exhibit conformal coordinate transformations (2.1.1.9) as residual symmetries. The energy momentum tensor ((2.1.1.4) with $\partial_\alpha X^\mu$ replaced by Π_α^μ (2.1.1.36)) is again traceless. Like in section 2.1.1.1, the vanishing of the energy momentum tensor gives two constraints. We have seen that in the non-supersymmetric case fixing conformal coordinate transformations and solving the constraints leaves effectively $d-2$ (transversal) bosonic directions.[6] In order for the target space supersymmetry not to be spoiled in this process, we would like to reduce the number of fermionic directions θ^A by a factor of

$$\frac{2^{\frac{[d-2]}{2}}}{2^{\frac{[d]}{2}}} = \frac{1}{2}$$

simultaneously. So, we need an additional local symmetry whose gauge fixing will remove half of the fermions θ^A. The symmetry we are looking for is known as κ symmetry. It exists only in special circumstances. First of all, the number of supersymmetries should not exceed $N = 2$ (i.e. $A = 1, 2$). Then, adding a further term

$$\begin{aligned} S_2 \quad = \quad & \frac{1}{2\pi\alpha'} \int d^2\sigma \left\{ -i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \left(\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 \right) \right. \\ & \left. + \epsilon^{\alpha\beta} \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 \right\} \end{aligned} \qquad (2.1.1.38)$$

to (2.1.1.37) results in a κ symmetric action. (We will give the explicit transformations below.) In (2.1.1.38) $\epsilon^{\alpha\beta}$ denotes the two dimensional Levi Civita symbol. If one is interested in less than $N = 2$ one can just put the corresponding θ^A to zero. The requirement that adding S_2 to the action does not spoil supersymmetry (2.1.1.33) – (2.1.1.35), leads to further constraints,

(*i*) $d = 3$ and θ is Majorana
(*ii*) $d = 4$ and θ is Majorana or Weyl
(*iii*) $d = 6$ and θ is Weyl
(*iv*) $d = 10$ and θ is Majorana-Weyl.

It remains to give the above mentioned κ symmetry transformations explicitly. By adding S_1 and S_2 one observes that the kinetic terms for the θ's (terms with one derivative acting on a fermion) contain the following projection operators

$$P_\pm^{\alpha\beta} = \frac{1}{2}\left(\gamma^{\alpha\beta} \pm \frac{\epsilon^{\alpha\beta}}{\sqrt{-\gamma}} \right). \qquad (2.1.1.39)$$

[6] Since the field equations are different for (2.1.1.37) the details of the discussion in the bosonic case will change. The above frame just gives a rough motivation for a modification of (2.1.1.37) carried out below.

The transformation parameter for the additional local symmetry is called κ_α^A. It is a spinor from the target space perspective and in addition a worldsheet vector subject to the following constraints

$$\kappa^{1\alpha} = P_-^{\alpha\beta}\kappa_\beta^1, \tag{2.1.1.40}$$

$$\kappa^{2\alpha} = P_+^{\alpha\beta}\kappa_\beta^2, \tag{2.1.1.41}$$

$$\tag{2.1.1.42}$$

where the worldsheet indices α, β are raised and lowered with respect to the worldsheet metric $\gamma_{\alpha\beta}$. Now, we are ready to write down the κ transformations,

$$\delta\theta^A = 2i\Gamma^\mu\Pi_{\alpha\mu}\kappa^{A\alpha}, \tag{2.1.1.43}$$

$$\delta X^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A, \tag{2.1.1.44}$$

$$\delta\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\right) = -16\sqrt{-\gamma}\left(P_-^{\alpha\gamma}\bar{\kappa}^{1\beta}\partial_\gamma\theta^1 + P_+^{\alpha\gamma}\bar{\kappa}^{2\beta}\partial_\gamma\theta^2\right). \tag{2.1.1.45}$$

For a proof that these transformations leave $S_1 + S_2$ indeed invariant we refer to[222] for example.

Once we have established that the number of local symmetries is correct, we can now proceed to employ those symmetries and reduce the number of degrees of freedom by gauge fixing. We will go to the light cone gauge in the following. Here, we will discuss only the most interesting case of $d = 10$. As usual we use reparametrization and Weyl invariance to fix $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. We can fix κ symmetry (2.1.1.43)–(2.1.1.45) by the choice

$$\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0, \tag{2.1.1.46}$$

where

$$\Gamma^\pm = \frac{1}{\sqrt{2}}\left(\Gamma^0 \pm \Gamma^9\right). \tag{2.1.1.47}$$

This sets half of the components of θ to zero. With the κ fixing condition (2.1.1.46) the equations of motion for X^+ and X^i ($i = 2, \ldots, d-1$) turn out to be free field equations (cf (2.1.1.11)). The reason for this can be easily seen as follows. After imposing (2.1.1.46), out of the fermionic terms only those containing $\bar{\theta^A}\Gamma^-\theta^A$ remain in the action $S_1 + S_2$. Especially, the terms fourth order in θ^A have gone. The above mentioned terms with Γ^- couple to $\partial_\alpha X^+$, and hence they will only have influence on the X^- equation (obtained by taking the variation of the action with respect to X^+). Thus we can again fix the conformal coordinate transformations by the choice (2.1.1.13). The X^- direction is then fixed (up to a constant) by imposing the constraint of vanishing energy momentum tensor. Since the coupling of bosons and fermions is reduced to a coupling to $\partial_\alpha X^+$, there is just a constant p^+ in front of the free kinetic terms of the fermions.

In the light-cone gauge described above the target space symmetry has been fixed up to the subgroup $SO(8)$, where the X^i and the θ^A transform in eight dimensional representations.[7] For $SO(8)$ there are three inequivalent eight dimensional representations, called $\mathbf{8_v}$, $\mathbf{8_s}$, and $\mathbf{8_c}$. The group indices are chosen as i, j, k for the $\mathbf{8_v}$, a, b, c for the $\mathbf{8_s}$, and $\dot{a}, \dot{b}, \dot{c}$ for the $\mathbf{8_c}$. In particular, X^i transforms in the vector representation $\mathbf{8_v}$. For the target space spinors we can choose either $\mathbf{8_s}$ or $\mathbf{8_c}$. Absorbing also the constant in front of the kinetic terms in a field redefinition we specify this choice by the following notation

$$\sqrt{p^+}\theta^1 \quad \to S^{1a} \qquad \text{or} \quad S^{1\dot{a}} \tag{2.1.1.48}$$
$$\sqrt{p^+}\theta^2 \quad \to S^{2a} \qquad \text{or} \quad S^{2\dot{a}}. \tag{2.1.1.49}$$

Essentially, we have here two different cases: we take the same $SO(8)$ representation for both θ's or we take them mutually different. The first option results in type IIB theory whereas the second one leads to type IIA.

So, the gauge fixing procedure simplifies the theory substantially. The equations of motion for the remaining degrees of freedom are just free field equations. For example for the type IIB theory they read,

$$\partial_+\partial_- X^i \;=\; 0, \tag{2.1.1.50}$$
$$\partial_+ S^{1a} \;=\; 0, \tag{2.1.1.51}$$
$$\partial_- S^{2a} \;=\; 0. \tag{2.1.1.52}$$

They look almost equivalent to the equations of motion one obtains from the worldsheet supersymmetric action (2.1.1.15) after eliminating the $\pm$ directions by the light cone gauge. Especially, (2.1.1.51) and (2.1.1.52) have the form of two dimensional Dirac equations where S^1 and S^2 appear as 2d Majorana-Weyl spinors. An important difference is however, that in (2.1.1.15) all worldsheet fields transform in the vector representation of the target space subgroup $SO(d-2)$.

In the rest of this chapter we will focus only on the worldsheet supersymmetric formulation. There, target space fermions will appear in the Hilbert space when quantizing the theory. We will come back to the Green Schwarz string only when discussing type IIB strings living in a non-trivial target space ($AdS_5 \times S^5$) in section 4.3.

2.1.2 Quantization of the fundamental string

[7] A Majorana-Weyl spinor in ten dimensions has 16 real components. Imposing (2.1.1.46) leaves eight.

2.1.2.1 The closed bosonic string

Our starting point is equation (2.1.1.11).

$$\partial_+\partial_- X^i = 0. \tag{2.1.2.1}$$

Imposing periodicity under shifts of σ^1 by π we obtain the following general solutions[8]

$$X^\mu = X_R^\mu\left(\sigma^-\right) + X_L^\mu\left(\sigma^+\right), \tag{2.1.2.2}$$

with

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}, \tag{2.1.2.3}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}. \tag{2.1.2.4}$$

Here, all σ^α dependence is written out explicitly, i.e. x^μ, p^μ, α_n^μ, and $\tilde{\alpha}_n^\mu$ are σ^α independent operators. Classically, one can associate x^μ with the center of mass position, p^μ with the center of mass momentum and α_n^μ $(\tilde{\alpha}_n^\mu)$ with the amplitude of the n'th right moving (left moving) vibration mode of the string in x^μ direction. Reality of X^μ imposes the relations

$$\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu \text{ and } \tilde{\alpha}_n^{\mu\dagger} = \tilde{\alpha}_{-n}^\mu. \tag{2.1.2.5}$$

We also define a zeroth vibration coefficient via

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}p^\mu. \tag{2.1.2.6}$$

Since the canonical momentum is obtained by varying the action (2.1.1.8) with respect to $\dot{X}^\mu$ (where the dot means derivative with respect to τ) we obtain the following canonical quantization prescription. The equal time commutators are given by

$$\left[X^\mu\left(\sigma\right), X^\nu\left(\sigma'\right)\right] = \left[\dot{X}^\mu\left(\sigma\right), \dot{X}^\nu\left(\sigma'\right)\right] = 0, \tag{2.1.2.7}$$

and

$$\left[\dot{X}^\mu\left(\sigma\right), X^\nu\left(\sigma'\right)\right] = -i\pi\delta\left(\sigma-\sigma'\right)\eta^{\mu\nu} \tag{2.1.2.8}$$

where the delta function is a distribution on periodic functions. Formally it can be assigned a Fourier series

$$\delta\left(\sigma\right) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{2ik\sigma}. \tag{2.1.2.9}$$

[8]Frequently, we will put $\alpha' = \frac{1}{2}$. Since it is the only dimensionfull parameter (in the system with $\hbar = c = 1$), it is easy to reinstall it when needed.

With this we can translate the canonical commutators (2.1.2.7) and (2.1.2.8) into commutators of the Fourier coefficients appearing in (2.1.2.3) and (2.1.2.4),

$$
\begin{aligned}
[p^\mu, x^\nu] &= -i\eta^{\mu\nu}, && (2.1.2.10)\\
[\alpha_n^\mu, \alpha_k^\nu] &= n\delta_{n+k}\eta^{\mu\nu}, && (2.1.2.11)\\
[\tilde{\alpha}_n^\mu, \tilde{\alpha}_k^\nu] &= n\delta_{n+k}\eta^{\mu\nu}, && (2.1.2.12)
\end{aligned}
$$

where δ_{n+k} is shorthand for $\delta_{n+k,0}$. So far, we did not take into account the constraints of vanishing energy momentum tensor (2.1.1.10). To do so we go again to the light cone gauge (2.1.1.13), i.e. set

$$\alpha_n^+ = \tilde{\alpha}_n^+ = 0 \ \text{ for } \ n \neq 0. \tag{2.1.2.13}$$

Now the constraint (2.1.1.10) can be used to eliminate X^- (up to x^-), or alternatively the α_n^- and $\tilde{\alpha}_n^-$,

$$
\begin{aligned}
p^+\alpha_n^- &= \sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: -2a\delta_n, && (2.1.2.14)\\
p^+\tilde{\alpha}_n^- &= \sum_{m=-\infty}^{\infty} :\tilde{\alpha}_{n-m}^i\tilde{\alpha}_m^i: -2a\delta_n && (2.1.2.15)
\end{aligned}
$$

where a sum over repeated indices i from 2 to $d-1$ is understood. The colon denotes normal ordering to be specified below. We have parameterized the ordering ambiguity by a constant a. (In principle one could have introduced two constants a, $\tilde{a}$. But this would lead to inconsistencies which we will not discuss here.) Equations (2.1.2.14) and (2.1.2.15) are not to be read as operator identities but rather as conditions on physical states which we will construct now. We choose the vacuum as an eigenstate of the p^μ

$$p^\mu |k\rangle = k^\mu |k\rangle\,, \tag{2.1.2.16}$$

with k^μ being an ordinary number. Further, we impose that the vacuum is annihilated by half of the vibration modes,

$$\alpha_n^i |k\rangle = \tilde{\alpha}_n^i |k\rangle = 0 \ \text{ for } \ n > 0. \tag{2.1.2.17}$$

The rest of the states can now be constructed by acting with a certain number of α_{-n}^i and $\tilde{\alpha}_{-n}^i$ ($n > 0$) on the vacuum. But we still need to impose the constraint (2.1.1.10). Coming back to (2.1.2.14) and (2.1.2.15) we can now specify what is meant by the normal ordering. The α_k^i ($\tilde{\alpha}_k^i$) with the greater Fourier index k is written to the right[9]. For $n \neq 0$ (2.1.2.14) and (2.1.2.15) just tell us how any α_n^- or $\tilde{\alpha}_n^-$ can be

[9] E.g. for $k > 0$ this implies that $:\alpha_k^i\alpha_{-k}^i := \alpha_{-k}^i\alpha_k^i$, i.e. the annihilation operator acts first on a state.

expressed in terms of the α_k^i and $\tilde{\alpha}_l^i$. The nontrivial information is contained in the $n=0$ case. It is convenient to rewrite (2.1.2.14) and (2.1.2.15) for $n=0$,

$$2p^+p^- - p^ip^i = 8(N-a) = 8(\tilde{N}-a), \tag{2.1.2.18}$$

where (doing the normal ordering explicitly)

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i, &(2.1.2.19)\\ \tilde{N} &= \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i. &(2.1.2.20)\end{aligned}$$

The N ($\tilde{N}$) are number operators in the sense that they count the number of creation operators α_{-n}^i ($\tilde{\alpha}_{-n}^i$) acting on the vacuum. To be precise, the N ($\tilde{N}$) eigenvalue of a state is this number multiplied by the index n and summed over all different kinds of creation operators acting on the vacuum (for left and right movers separately). Interpreting the p^μ eigenvalue k^μ as the momentum of a particle (2.1.2.18) looks like a mass shell condition with the mass squared M^2 given by

$$M^2 = 8(N-a) = 8(\tilde{N}-a). \tag{2.1.2.21}$$

The second equality in the above equation relates the allowed right moving creation operators acting on the vacuum to the left moving ones. It is known as the level matching condition.

For example, the first excited state is

$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j \left|k\right\rangle . \tag{2.1.2.22}$$

By symmetrizing or antisymmetrizing with respect to i, j and splitting the symmetric expression into a trace part and a traceless part one sees easily that the states (2.1.2.22) form three irreducible representations of $SO(d-2)$. Since we have given the states the interpretation of being particles living in the targetspace, these should correspond to irreducible representations of the little group. Only when the above states are massless the little group is $SO(d-2)$ (otherwise it is $SO(d-1)$). Therefore, for unbroken covariance with respect to the targetspace Lorentz transformation, the states (2.1.2.22) must be massless. Comparing with (2.1.2.21) we deduce that the normal ordering constant a must be one,

$$a \stackrel{!}{=} 1. \tag{2.1.2.23}$$

In the following we are going to compute the normal ordering constant a. Requiring agreement with (2.1.2.23) will give a condition on the dimension of the targetspace to be 26. The following calculation may look at some points a bit dodgy when it comes to computing the exact value of a. So, before starting we should note that the compelling result will be that a depends on the targetspace dimension. The exact numerics can be verified by other methods which we will not elaborate on here for the sake of briefness. We will consider only N since the calculation with $\tilde{N}$ is a very straightforward modification (just put tildes everywhere). The initial assumption is that naturally the ordering in quantum expressions would be symmetric, i.e.

$$N - a = \frac{1}{2} \sum_{n=-\infty, n\neq 0}^{\infty} \alpha_{-n}^{i} \alpha_{n}^{i}. \tag{2.1.2.24}$$

By comparison with the definition of N (2.1.2.19) and using the commutation relations (2.1.2.11) we find

$$a = -\frac{d-2}{2} \sum_{n=1}^{\infty} n. \tag{2.1.2.25}$$

This expression needs to be regularized. A familiar method of assigning a finite number to the rhs of (2.1.2.25) is known as 'zeta function regularization'. One possible representation of the zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{2.1.2.26}$$

The above representation is valid for the real part of s being greater than one. The zeta function, however, can be defined also for complex s with negative real part. This is done by analytic continuation. The way to make sense out of (2.1.2.25) is now to replace the infinite sum by the zeta function

$$a = -\frac{d-2}{2} \zeta(-1) = \frac{d-2}{24}. \tag{2.1.2.27}$$

Comparing with (2.1.2.23) we see that we need to take

$$d = 26 \tag{2.1.2.28}$$

in order to preserve Lorentz invariance. This result can also be verified in a more rigid way. Within the present approach one can check that $a = 1$ and $d = 26$ are needed for the target space Lorentz algebra to close. In other approaches, one sees that the Weyl symmetry becomes anomalous for $d \neq 26$.

Since N and $\tilde{N}$ are natural numbers we deduce from (2.1.2.21) that the mass spectrum is an infinite tower starting from $M^2 = -8 = -4/\alpha'$ and going up in steps of $8 = 4/\alpha'$. The presence of a tachyon (a state with negative mass square) is a problem.

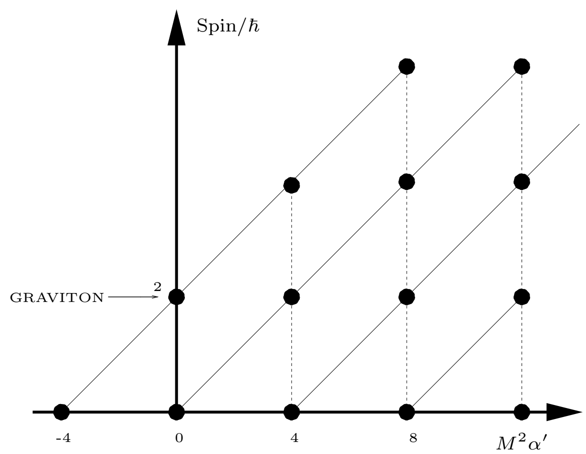

Figure 2.1: Mass spectrum of the closed bosonic string

It shows that we have looked at the theory in an unstable vacuum. One possibility that this is not complete nonsense could be that apart from the massterm the tachyon potential receives higher order corrections (like e.g. a power of four term) with the opposite sign. Then it would look rather like a Higgs field than a tachyon, and one would expect some phase transition (tachyon condensation) to occur such that the final theory is stable. For the moment, however, let us ignore this problem (it will not occur in the supersymmetric theories to be studied next).

The massless particles are described by (2.1.2.22). The part symmetric in i, j and traceless corresponds to a targetspace graviton. This is one of the most important results in string theory. There is a graviton in the spectrum and hence string theory can give meaning to the concept of quantum gravity. (Since Einstein gravity cannot be quantized in a straightforward fashion there is a graviton only classically. This corresponds to the gravitational wave solution of the Einstein equations. The particle aspect of the graviton is missing without string theory.) The trace-part of (2.1.2.22) is called dilaton whereas the piece antisymmetric in i, j is simply the antisymmetric tensor field (commonly denoted with B). A schematic summary of the particle spectrum of the closed bosonic string is drawn in figure 2.1.

As a consistency check one may observe that the massive excitations fit in $SO(25)$ representations, i.e. they form massive representations of the little group of the Lorentz group in 26 dimensions.

As we have already mentioned, this theory contains a graviton, which is good since it gives the prospect of quantizing gravity. On the other hand, there is the tachyon, at best telling us that we are in the wrong vacuum. (There could be no stable vacuum at all – for example if the tachyon had a run away potential.) Further, there are no target space fermions in the spectrum. So, we would like to keep the graviton but to get rid of the tachyon and add fermions. We will see that this goal can be achieved by quantizing the supersymmetric theories.

2.1.2.2 Type II strings

In this section we are going to quantize the (1,1) worldsheet supersymmetric string. We will follow the lines of the previous section but need to add some new ingredients. We start with the action (2.1.1.15). The equations of motion for the bosons X^μ are identical to the bosonic string. So, the mode expansion of the X^μ is not altered and given by (2.1.2.3) and (2.1.2.4). The equations of motion for the fermions are,

$$\partial_- \psi_+^\mu = 0, \tag{2.1.2.29}$$

$$\partial_+ \psi_-^\mu = 0. \tag{2.1.2.30}$$

Further, we need to discuss boundary conditions for the worldsheet fermions. Modulo the equations of motion (2.1.2.29) and (2.1.2.30) the variation of the action (2.1.1.15) with respect to the worldsheet fermions turns out to be[10]

$$\frac{i}{2\pi}\left(-\psi_{+\mu}\delta\psi_+^\mu + \psi_{-\mu}\delta\psi_-^\mu\right)\Big|_{\sigma=0}^{\pi}. \tag{2.1.2.31}$$

For the closed string we need to take the variation of ψ_+^μ independent from the one of ψ_-^μ at the boundary (because we do not want the boundary condition to break part of the supersymmetry (2.1.1.22) and (2.1.1.23)). Hence, the spinor components can be either periodic or anti-periodic under shifts of σ by π. The first option gives the Ramond (R) sector. In the R sector the general solution to (2.1.2.29) and (2.1.2.30) can be written in terms of the following mode expansion

$$\psi_-^\mu = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}, \tag{2.1.2.32}$$

$$\psi_+^\mu = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}. \tag{2.1.2.33}$$

The other option to solve the boundary condition is to take anti-periodic boundary conditions. This is called the Neveu Schwarz (NS) sector. In the NS sector the general solution to the equations of motion (2.1.2.29) and (2.1.2.30) reads[11]

$$\psi_-^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)}, \tag{2.1.2.34}$$

$$\psi_+^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}, \tag{2.1.2.35}$$

[10] Again we put $\alpha' = \frac{1}{2}$.

[11] The reality (Majorana) condition on the worldsheet spinor components provides relations analogous to (2.1.2.5).

where now the sum is over half integer numbers $(\ldots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots)$.

For the bosons the canonical commutators are as given in (2.1.2.7), (2.1.2.8). Hence, the oscillator modes satisfy again the algebra (2.1.2.10) – (2.1.2.12). Worldsheet fermions commute with worldsheet bosons. The canonical (equal time) anti-commutators for the fermions are

$$\left\{\psi_+^\mu\left(\sigma\right), \psi_+^\nu\left(\sigma'\right)\right\} = \left\{\psi_-^\mu\left(\sigma\right), \psi_-^\nu\left(\sigma'\right)\right\} = \pi\eta^{\mu\nu}\delta\left(\sigma - \sigma'\right), \quad (2.1.2.36)$$

$$\left\{\psi_+^\mu\left(\sigma\right), \psi_-^\nu\left(\sigma'\right)\right\} = 0. \quad (2.1.2.37)$$

For the Fourier modes this implies

$$\left\{b_r^\mu, b_s^\nu\right\} = \left\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\right\} = \eta^{\mu\nu}\delta_{r+s} \quad (2.1.2.38)$$

in the NS sectors[12], and

$$\left\{d_m^\mu, d_n^\nu\right\} = \left\{\tilde{d}_m^\mu, \tilde{d}_n^\nu\right\} = \eta^{\mu\nu}\delta_{m+n} \quad (2.1.2.39)$$

in the R sectors. Like the bosonic Fourier modes these can be split into creation operators with negative Fourier index, and annihilation operators with positive Fourier index. What about zero Fourier index? For the NS sector fermions this does not occur. The vacuum is always taken to be an eigenstate of the bosonic zero modes where the eigenvalues are the target space momentum of the state. (This is exactly like in the bosonic string discussed in the previous section.) The Ramond sector zero modes form a target space Clifford algebra (cf (2.1.2.39)). This means that the Ramond sector states form a representation of the d dimensional Clifford algebra, i.e. they are target space spinors. We will come back to this later. Pairing left and right movers, there are altogether four different sectors to be discussed: NSNS, NSR, RNS, RR. In the NSNS sector for example the left and right moving worldsheet fermions have both anti-periodic boundary conditions. The vacuum in the NSNS sector is defined via (2.1.2.16), (2.1.2.17) and

$$b_r^\mu\left|k\right\rangle = \tilde{b}_r^\mu\left|k\right\rangle = 0 \;\text{ for }\; r > 0. \quad (2.1.2.40)$$

We can build states out of this by acting with bosonic left and right moving creation operators on it. Further, left and right moving fermionic creators from the NS sectors can act on (2.1.2.40). We should also impose the constraints (2.1.1.24) – (2.1.1.27) on those states. As before, we do so by going to the light cone gauge

$$\alpha_n^+ = \tilde{\alpha}_n^+ = b_r^+ = \tilde{b}_r^+ = 0. \quad (2.1.2.41)$$

[12] We say NS sectors and not NS sector because there are two of them: a left and a right moving one.

Then the constraints can be solved to eliminate the minus directions. The important information is again in the zero mode of the minus direction. This reads (2.1.2.18)

$$2p^+p^- - p^ip^i = 8(N_{NS} - a_{NS}) = 8(\tilde{N}_{NS} - a_{NS}). \tag{2.1.2.42}$$

The expressions for the number operators are modified due to the presence of (NS sector) worldsheet fermions

$$N_{NS} = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r, \tag{2.1.2.43}$$

and the analogous expression for $\tilde{N}_{NS}$. Its action on states is like in the bosonic case (see discussion below (2.1.2.20)) taking into account the appearance of fermionic creation operators. Again, we have put a so far undetermined normal ordering constant in (2.1.2.42) and taken normal ordered expressions for the number operators. Now, the first excited state is

$$b^i_{-\frac{1}{2}}\tilde{b}^j_{-\frac{1}{2}} \left|k\right\rangle . \tag{2.1.2.44}$$

Its target space tensor structure is identical to the one of (2.1.2.22). In particular it forms massless representations of the target space Lorentz symmetry. Thus, Lorentz covariance implies that

$$a_{NS} = \frac{1}{2} \tag{2.1.2.45}$$

should hold.

We compute now a_{NS} by first naturally assuming that a symmetrized expression appears on the rhs of (2.1.2.42). This gives (see also (2.1.2.25))

$$a_{NS} = -\frac{d-2}{2}\sum_{n=1}^{\infty} n + \frac{d-2}{2}\sum_{r=\frac{1}{2}}^{\infty} r. \tag{2.1.2.46}$$

We use again the zeta function regularization to make sense out of (2.1.2.46). For the second sum the following formula proves useful

$$\sum_{n=0}^{\infty}(n+c) = \zeta\left(-1, c\right) = -\frac{1}{12}\left(6c^2 - 6c + 1\right). \tag{2.1.2.47}$$

(Note, that splitting the lhs of (2.1.2.47) into $\zeta\left(-1\right) + c + c\zeta\left(0\right)$ gives a different (wrong) result. This is because we understand the infinite sum as an analytic continuation of a finite one: $\sum \left(n+a\right)^{-s}$ with real part of s greater than one. For generic s the above splitting is not possible.) Anyway, with the regularization prescription (2.1.2.47) we get for (2.1.2.46)

$$a_{NS} = \frac{d-2}{16}. \tag{2.1.2.48}$$

We conclude that the critical dimension for the $(1,1)$ worldsheet supersymmetric string is

$$d = 10. \tag{2.1.2.49}$$

Like in the bosonic string there are more rigid calculations giving the same result.

The massless spectrum from the NSNS sector is identical to the massless spectrum of the closed bosonic string. Again, we have a tachyon: the NSNS groundstate. Here, however this can be consistently projected out. This is done by imposing the GSO (Gliozzi-Scherk-Olive) projection. To specify what this projection does in the NS sector we introduce fermion number operators F ($\tilde{F}$) counting the number of worldsheet fermionic NS right (left) handed creation operators acting on the vacuum. In addition, we assign to the right (left) handed NS vacuum an F ($\tilde{F}$) eigenvalue of one[13] . Now, the GSO projection is carried out by multiplying states with the GSO projection operator

$$P_{GSO} = \frac{1+(-1)^F}{2}\frac{1+(-1)^{\tilde{F}}}{2}. \tag{2.1.2.50}$$

Obviously this does not change the first excited state (2.1.2.44) but removes the tachyonic NSNS ground state. There are several reasons why this projection is consistent. At tree level[14] for example one may check that the particles which have been projected out do not reappear as poles in scattering amplitudes. Imposing the GSO projection becomes even more natural when looking at the one loop level. In the Euclidean version this means that the worldsheet is a torus. Summing over all possible spin structures (the periodicities of worldsheet fermions when going around the two cycles of the torus) leads naturally to the appearance of (2.1.2.50) in the string partition function [415] (see also [331]). The NSNS spectrum subject to the GSO projection looks as follows. The number operator (2.1.2.43) is quantized in half-integer steps. The GSO projection removes half of the states, the groundstate, the first massive states, the third massive states and so on. The NSNS spectrum of the type II strings is summarized in figure 2.2.

We have achieved our goal of removing the tachyon from the spectrum while keeping the graviton. We also want to have target space spinors. We will see that those

[13] This means that we can write $F = 1 + \sum_{r>0} b^i_{-r} b^i_r$, and an analogous expression for $\tilde{F}$.

[14] The worldsheet has the topology of a cylinder, or a sphere when Wick rotated to the Euclidean $2d$ signature.

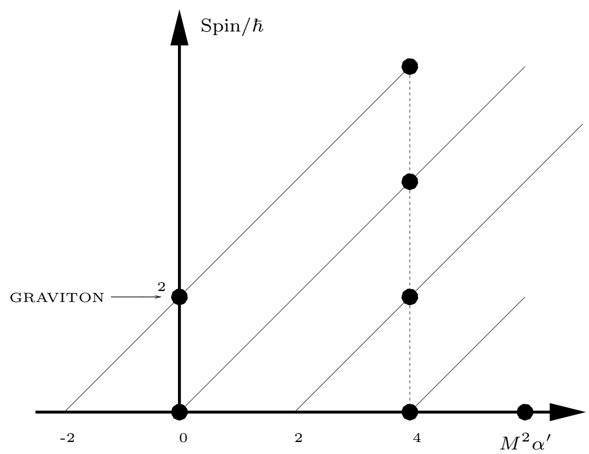

Figure 2.2: NS-NS spectrum of the type II string. In comparison to figure 2.1 the horizontal axis has been stretched by a factor of two.

come by including the R sector into the discussion. The most important issue to be addressed here is the action of the zero modes on the R groundstate. By going to the light-cone gauge, we can again eliminate the plus and minus (or the 0 and 1) directions leaving us with eight[15] zero modes for the left and right moving sectors each. We rearrange these modes into four complex modes

$$D_1 = d_0^2 + i d_0^3, \tag{2.1.2.51}$$

$$D_2 = d_0^4 + i d_0^5, \tag{2.1.2.52}$$

$$D_3 = d_0^6 + i d_0^7, \tag{2.1.2.53}$$

$$D_4 = d_0^8 + i d_0^9. \tag{2.1.2.54}$$

The only non-vanishing anti-commutators for these new operators are ($I = 1, \ldots, 4$; no sum over I)

$$\left\{ D_I, D_I^\dagger \right\} = 2. \tag{2.1.2.55}$$

In particular, the D_I and $D_I^\dagger$ are nilpotent. We can now construct the right moving R vacuum by starting with a state which is annihilated by all the D_I,[16]

$$D_I \left| -, -, -, - \right\rangle = 0 \;\text{ for all } I. \tag{2.1.2.56}$$

Acting with a $D_I^\dagger$ on the vacuum changes the Ith minus into a plus, e.g.

$$D_3^\dagger \left| -, -, -, - \right\rangle = \left| -, -, +, - \right\rangle. \tag{2.1.2.57}$$

[15] We use here the previous result that we need to have $d = 10$ in order to preserve target space Lorentz invariance.

[16] In this notation we suppress the eigenvalue k^μ of the bosonic zero modes.

Acting once more with $D_3^\dagger$ will give zero. Acting with D_3 on (2.1.2.57) will give back (2.1.2.56) because of (2.1.2.55). Thus, we have a $2^4 = 16$-fold degenerate vacuum. This gives an on shell Majorana spinor in ten dimensions. For the left movers the construction is analogous. (The above method to construct the state is actually an option to construct (massless) spinor representations when the d_0^i are identified with the target space Gamma matrices.) Without further motivation (which is given in the books and reviews listed in section 6) we state how the GSO projection is performed in the R sector. First, we define

$$(-1)^F = 2^4\, d_0^2 d_0^3 d_0^4 d_0^5 d_0^6 d_0^7 d_0^8 d_0^9\, (-1)^{\sum_{n>0} d_{-n}^i d_n^i}\,, \qquad (2.1.2.58)$$

where the factor of 2^4 has been introduced such that $(-)^{2F} = 1$, ensuring that (2.1.2.59) defines projection operators. Note also that $\Gamma^\mu = \sqrt{2} d_0^\mu$ satisfies the canonically normalized Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. For the groundstate this is just the chirality operator (the product of all Gamma matrices) in the transverse eight dimensional space. Now, we multiply the R states by one of the following projection operators

$$P^{\pm}_{GSO} = \frac{1 \pm (-1)^F}{2} \qquad (2.1.2.59)$$

We perform the analogous construction in the left moving R sector. There are essentially two inequivalent options: we take the same sign in (2.1.2.59) for left and right movers, or different signs. Taking different signs leads to type IIA strings whereas the option with the same signs is called type IIB. Multiplying the R groundstate with one of the operators (2.1.2.59) reduces the 16 dimensional Majorana spinor to an eight dimensional Weyl spinor[17].

To complete the discussion of the R sector we have to combine left and right movers, i.e. to construct the NSR, RNS, and RR sector of the theory. Let us start with the NSR sector. The mass shell condition (2.1.2.42) reads now

$$2p^+p^- - p^i p^i = 8\left(N_{NS} - \frac{1}{2}\right) = 8\tilde{N}_R, \qquad (2.1.2.60)$$

where the number operator in the R sector is given as

$$N_R = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} n d_{-n}^i d_n^i, \qquad (2.1.2.61)$$

and the analogous expression for the left movers. We have put the normal ordering constant in the Ramond sector to zero. This can easily be justified by replacing the

[17] The two different choices in (2.1.2.59) give either the $\mathbf{8_s}$ or the $\mathbf{8_c}$ representation of $SO(8)$ mentioned in section 2.1.1.3

half integer modded sum over r by an integer modded one in (2.1.2.46). Level matching implies that the lowest allowed state in the NSR sector is massless and given by

$$b^i_{-\frac{1}{2}}\left|k\right\rangle u_a \tag{2.1.2.62}$$

where u_a denotes the eight component Majorana-Weyl spinor comming from the R ground states surviving the GSO projection. The 64 states contained in (2.1.2.62) decompose into an eight dimensional and a 56 dimensional representation of the target space little group $SO(8)$. The 56 dimensional representation gives a gravitino of fixed chirality, whereas the eight dimensional one gives a dilatino of fixed chirality.

The discussion of the RNS sector goes along the same line giving again a gravitino and a dilatino either of opposite (to the NSR sector) chiralities corresponding to type IIA theory, or of the same chiralities when the type IIB GSO projection is imposed.

Finally, in the RR sector the lowest state is obtained by combining the left with the right moving vacuum. This state is massless due to the normal ordering constant $a_R = 0$. It has 64 components. The irreducible decompositions of the RR state depend on whether we have imposed GSO conditions corresponding to type IIA or type IIB. In the type IIA case the 64 states decompose into an eight dimensional vector representation and a 56 dimensional representation. Thus in the type IIA theory, the RR sector gives a massless $U(1)$ one-form gauge potential A_μ and a three-form gauge potential $C_{\mu\nu\rho}$. In the type IIB theory the 64 splits into a singlet, a 28 and a 35 dimensional representation of $SO(8)$. This corresponds to a "zero-form" Φ', a two-form $B'_{\mu\nu}$, and a four-form gauge potential with selfdual field strength $C^*_{\mu\nu\rho\sigma}$. The particle content of the type II theories can be arranged in to $N = 2$ supermultiplets of chiral (type IIB) or non-chiral (type IIA) ten dimensional supergravity. The (target space bosons of the) massless spectrum of the type II strings is summarized in table 2.1.

2.1.2.3 The heterotic string

Since the heterotic string is a bit out of the focus of the present review we will briefly state the results. The starting point is the action (2.1.1.32). Without the λ^A_+ this looks like the type II theories with the left handed worldsheet fermions removed.

Indeed, this part of the theory leads to the spectrum of the type II theories with only the NS and R sector. The massless spectrum corresponds to $N = 1$ chiral supergravity in ten dimensions. It corresponds to the states (the $\tilde{\alpha}^i_n$ are the Fourier coefficients for the left moving bosons, and the b^i_r for the right moving fermions in the NS sector)

$$\tilde{\alpha}^i_{-1} b^j_{-\frac{1}{2}}\left|k\right\rangle , \tag{2.1.2.63}$$

in the NS sector, and

$$\tilde{\alpha}^i_{-1}\left|k\right\rangle u_\alpha \tag{2.1.2.64}$$

in the R sector, where we denoted again the GSO projected R vacuum with u_α. The above states must be massless since they form irreducible representations of $SO(d-2)$. Focusing on the right moving sector we can deduce that the right moving normal ordering constant must be $\frac{1}{2}$ like in the type II case. Hence, the number of dimensions (range of μ) is ten. As it stands the above spectrum leads to an anomalous theory. But there is still the option of switching on the λ_+^A. Let us first deduce the number of additional directions (labeled by A) needed. In the sector where the vacuum is non degenerate due to the presence of the λ_+^A, we know that we need the left moving normal ordering constant to be one. (Otherwise the states (2.1.2.63) would not be massless, but still form $SO(d-2)$ representations.) The vacuum does not receive further degeneracy in the sector where all of the λ_A^+ have anti-periodic boundary conditions. In this sector the normal ordering constant is (see also (2.1.2.46)), the label A stands for anti-periodic

$$\tilde{a}_A = \frac{d-2}{24} + \frac{D}{48}, \tag{2.1.2.65}$$

where we have called the number of additional directions D ($A = 1, \ldots, D$). The consistency condition $\tilde{a}_A = 1$ tells us that there must be 32 additional directions,

$$D = 32. \tag{2.1.2.66}$$

Let us first discuss the simplest option, namely that all of the λ_+^A have always identical boundary conditions, either periodic or antiperiodic. In the periodic sector one easily computes that the normal ordering constant $\tilde{a}_P$ is negative ($-\frac{1}{3}$). Hence, there are no massless states in this sector. In the NS sector we find in addition to (2.1.2.63) the massless states (denoting with $\tilde{b}_r^A$ the Fourier coefficients of λ_+^A in the anti-periodic sector)

$$\tilde{b}^A_{-\frac{1}{2}}\tilde{b}^B_{-\frac{1}{2}}b^i_{-\frac{1}{2}}\left|k\right\rangle. \tag{2.1.2.67}$$

Since the $\tilde{b}^A$ anti-commute this is an anti-symmetric 32×32 matrix. In addition it is a target space vector (because of the index i). Therefore, the state (2.1.2.67) is an $SO(32)$ gauge field. The corresponding R sector provides (after imposing the GSO projection) fermions filling up an $N = 1$ supermultiplet in ten dimensions. Together, with this $SO(32)$ Yang-Mills part the ten dimensional field theory with the same massless content is anomaly free. The GSO projection in the periodic sector is such that only states with an even number of left moving fermionic creators survive. In the

	# of Q's	# of ψ_μ's	massless bosonic spectrum	
IIA	32	2	NSNS	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ
			RR	A_μ, $C_{\mu\nu\rho}$
IIB	32	2	NSNS	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ
			RR	$C^*_{\mu\nu\rho\sigma}$, $B'_{\mu\nu}$, Φ'
heterotic $E_8 \times E_8$	16	1	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ A^a_μ in adjoint of $E_8 \times E_8$	
heterotic $SO(32)$	16	1	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ A^a_μ in adjoint of $SO(32)$	

Table 2.1: Consistent closed string theories in ten dimensions.

P sector it removes half of the groundstates (leaving only spinors of definite chirality with respect to the internal space spanned by the A directions).

Another option is to group the λ^A_+ into two groups of 16 directions. Then we would naturally split the state (2.1.2.67) into three groups: $(\mathbf{120}, \mathbf{1})$, $(\mathbf{1}, \mathbf{120})$, and $(\mathbf{16}, \mathbf{16})$, depending on whether A and B in (2.1.2.67) are both in the first half $(1, \ldots, 16)$, both in the second half $(17, \ldots, 32)$, or one of them out of the first half and the other one out of the second half. So far, this gave only a rearrangement of those states. But now we impose the GSO projection such that only states survive where an even number of fermionic left moving creators act in each half separately. This removes the $(\mathbf{16}, \mathbf{16})$ combination. Further, when we split the range of indices into two groups of 16 each, there will be additional massless states. It is simple to check that in the sector where half of the boundary conditions are periodic and the other half is anti-periodic (the AP or PA sector), the left moving normal ordering constant vanishes. Hence, the corresponding ground states give rise to massless fields, provided right moving creation operators act such that level matching is satisfied. This gives (removing half of those states by GSO projection) $(\mathbf{128}, \mathbf{1})$ additional massless vectors from the PA sector, and another $(\mathbf{1}, \mathbf{128})$ from the AP sector. Together with the vectors from the AA sector this gives an $E_8 \times E_8$ Yang-Mills field. The R sector state fills in the fermions needed for $N = 1$ supersymmetry in ten dimensions. This corresponds to the other known $N = 1$ anomaly free field theory.

The bosonic parts of the massless spectra of the consistent closed string theories in ten dimensions is summarized in table 2.1. We have added the number of supercharges Q from a target space perspective, and also the number of worldsheet supersymmetries ψ_μ, in the NSR formulation.

2.1.3 Strings in non-trivial backgrounds

In the previous sections we have seen that all closed strings contain a graviton, a dilaton, and an antisymmetric tensor field in the massless sector. This is called the universal sector. So far, we have studied the situation where the target space metric is the Minkowski metric, the antisymmetric tensor has zero field strength and the dilaton is constant. In order to investigate what happens when we change the background, we need to modify the action (2.1.1.3) as follows (this action is called the string sigma model)

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\sqrt{\gamma}\gamma^{\alpha\beta}G_{\mu\nu}\left(X\right)\partial_\alpha X^\mu\partial_\beta X^\nu + i\epsilon^{\alpha\beta}B_{\mu\nu}\left(X\right)\partial_\alpha X^\mu\partial_\beta X^\nu\right) \\ &\quad -\frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}\Phi\left(X\right)R^{(2)}, \end{aligned} \tag{2.1.3.1}$$

where $R^{(2)}$ is the scalar curvature computed from $\gamma_{\alpha\beta}$. Throughout this section we will consider a Euclidean worlsheet signature. Note, that the dilaton term does not contain α'. In general, the theory (2.1.3.1) cannot be quantized in an easy way. The best one can do is to take a semiclassical approach. Since α' enters like $\hbar$ in ordinary field theories this will result in a perturbative expansion in α'. The term with the dilaton can be viewed as a first order contribution in this expansion. Without this term, (2.1.3.1) has again three local symmetries: diffeomorphisms and Weyl invariance. The dilaton term breaks Weyl invariance in general. We will be interested in the question under which circumstances Weyl invariance remains unbroken in the semiclassically quantized theory. To answer this, first note that $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ can be viewed as couplings from a two dimensional perspective. Weyl invariance in particular implies global scale invariance. But scale invariance is related to vanishing beta functions in field theory. Thus, we will compute the beta functions of $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ as a power series in α'. However, there is a subtlety here. Under field redefinitions (infinitesimal shifts of X by $\chi\left[X\right]$) the couplings change according to

$$\delta G_{\mu\nu} = 2D_{(\mu}\chi_{\nu)}, \tag{2.1.3.2}$$

$$\delta B_{\mu\nu} = \chi^\rho H_{\rho\mu\nu} + \partial_\mu L_\nu - \partial_\nu L_\mu, \tag{2.1.3.3}$$

$$\delta\Phi = \chi^\rho\partial_\rho\phi, \tag{2.1.3.4}$$

where we have defined

$$H_{\rho\lambda\kappa} = \partial_\rho B_{\lambda\kappa} + \partial_\lambda B_{\kappa\rho} + \partial_\kappa B_{\rho\lambda} \tag{2.1.3.5}$$

and

$$L_\kappa = \chi^\rho B_{\kappa\rho}. \tag{2.1.3.6}$$

Expression (2.1.3.5) defines a field strength corresponding to the B field. It is invariant under a $U(1)$ transformation

$$\delta B_{\mu\nu} = \partial_{[\mu} V_{\nu]}, \tag{2.1.3.7}$$

with V_μ being an arbitrary target space vector. It is easy to check that also (2.1.3.1) possesses the invariance (2.1.3.7). The symmetry (2.1.3.7) can be taken care of by allowing for arbitrary L_μ in (2.1.3.3). Thus the couplings and hence the beta functions are not unique. But actually we will be not just interested in vanishing beta functions. This would ensure only global scale invariance. The requirement of Weyl invariance is more strict and will fix the arbitrariness.

In order to compute the beta functions, we need to fix the worldsheet diffeomorphisms. We leave the explicit form of the fixed metric $\gamma_{\alpha\beta}$ unspecified. The gauge fixing procedure introduces ghosts, the diffeomorphism invariance is replaced by BRST invariance. The ghost action depends only on the 2d geometry. Therefore, we expect that the ghosts contribute only to the dilaton beta function. We will not treat them explicitly but guess their contribution in the end of this section. The semiclassical approach means that we start from some background string $\bar{X}^\mu$ satisfying the equations of motion. We study the theory of the fluctuations around this background string. Instead of using the fluctuation in the coordinate field X^μ we will take the tangent vector to the geodesic connecting the background value $\bar{X}^\mu$ with the actual value X^μ. This difference is supposed to be small in this approximation. In order to compute the tangent vectors we connect the background value and the actual position of the string by a geodesic. The line parameter t is chosen such that at $t = 0$ we are at the background position and at $t = 1$ at the actual position. The geodesic equation is (the dot denotes the derivative with respect to t),

$$\ddot{\lambda}^\mu + \Gamma^\mu_{\nu\rho}\dot{\lambda}^\nu\dot{\lambda}^\rho = 0 \tag{2.1.3.8}$$

and the boundary conditions are

$$\lambda^\mu(0) = \bar{X}^\mu \ , \ \lambda^\mu(1) = X^\mu. \tag{2.1.3.9}$$

Note that the target space Christoffel connection $\Gamma^\mu_{\nu\rho}$ depends on X^μ. The first non-trivial effects should come from terms second order in the fluctuations in the action. (First order terms vanish when the background satisfies the equations of motion.) We call the tangent vector to the geodesic (at $\bar{X}^\mu$)

$$\xi^\mu = \dot{\lambda}^\mu(0)\,. \tag{2.1.3.10}$$

One can solve (2.1.3.8) iteratively leading to a power series in t,

$$\lambda^\mu(t) = \bar{X}^\mu + \xi^\mu t - \frac{1}{2}\Gamma^\mu_{\nu\rho}\xi^\nu\xi^\rho t^2 - \frac{1}{3!}\Gamma^\mu_{\nu\rho\kappa}\xi^\nu\xi^\rho\xi^\kappa t^3 + \dots, \tag{2.1.3.11}$$

where

$$\Gamma^{\mu}_{\nu\rho\kappa} = \nabla_{\nu}\Gamma^{\mu}_{\rho\kappa} = \partial_{\nu}\Gamma^{\mu}_{\rho\kappa} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\kappa} - \Gamma^{\lambda}_{\nu\kappa}\Gamma^{\mu}_{\rho\lambda}. \tag{2.1.3.12}$$

Further, we may choose local coordinates such that only the constant and the term linear in t appears in (2.1.3.11) and all higher order terms vanish in a neighborhood of $\bar{X}^{\mu}$. (This is done by spanning the local coordinate system by tangent vectors to geodesics.) The corresponding coordinates are called Riemann normal coordinates. In these coordinates the Taylor expansion of the various terms in (2.1.3.1) around $\bar{X}^{\mu}$ takes the following form (up to second order in the fluctuations),

$$\partial_{\alpha}X^{\mu} = \partial_{\alpha}\bar{X}^{\mu} + D_{\alpha}\xi^{\mu} + \frac{1}{3}R^{\mu}{}_{\lambda\kappa\nu}\left(\bar{X}\right)\xi^{\lambda}\xi^{\kappa}\partial_{\alpha}\bar{X}^{\nu}, \tag{2.1.3.13}$$

$$G_{\mu\nu}\left(X\right) = G_{\mu\nu}\left(\bar{X}\right) - \frac{1}{3}R_{\mu\rho\nu\kappa}\left(\bar{X}\right)\xi^{\rho}\xi^{\kappa}, \tag{2.1.3.14}$$

$$\begin{aligned} B_{\mu\nu}\left(X\right) = {} & B_{\mu\nu}\left(\bar{X}\right) + D_{\rho}B_{\mu\nu}\left(\bar{X}\right)\xi^{\rho} + \frac{1}{2}D_{\lambda}D_{\rho}B_{\mu\nu}\left(\bar{X}\right)\xi^{\lambda}\xi^{\rho} \\ & -\frac{1}{6}R^{\lambda}{}_{\rho\mu\kappa}B_{\lambda\nu}\left(\bar{X}\right)\xi^{\rho}\xi^{\kappa} + \frac{1}{6}R^{\lambda}{}_{\rho\nu\kappa}B_{\lambda\mu}\left(\bar{X}\right)\xi^{\rho}\xi^{\kappa}, \end{aligned} \tag{2.1.3.15}$$

$$\Phi\left(X\right) = \Phi\left(\bar{X}\right) + D_{\mu}\Phi\left(\bar{X}\right)\xi^{\mu} + \frac{1}{2}D_{\mu}D_{\nu}\Phi\left(\bar{X}\right)\xi^{\mu}\xi^{\nu}, \tag{2.1.3.16}$$

where D_{ρ} denotes the usual covariant derivative in target space, and $R^{\mu}{}_{\nu\rho\sigma}$ is the target space Riemann tensor

$$R^{\mu}{}_{\nu\rho\lambda} = \partial_{\rho}\Gamma^{\mu}_{\nu\lambda} - \partial_{\lambda}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\omega}_{\nu\lambda}\Gamma^{\mu}_{\omega\rho} - \Gamma^{\omega}_{\nu\rho}\Gamma^{\mu}_{\omega\lambda}. \tag{2.1.3.17}$$

Note that in the Riemann normal coordinates the contributions quadratic in the Christoffels vanish. Further, we have defined

$$D_{\alpha}\xi^{\mu} = \partial_{\alpha}\xi^{\mu} + \Gamma^{\mu}_{\lambda\nu}\xi^{\lambda}\partial_{\alpha}\bar{X}^{\nu}. \tag{2.1.3.18}$$

Collecting everything, one can expand the action (2.1.3.1) in a classical contribution S_0 and a contribution due to fluctuations. There will be no part linear in ξ^{μ} as long as $\bar{X}^{\mu}$ satisfies the equations of motion. The first non-trivial part is quadratic in the ξ^{μ}. We denote it by

$$S^{(2)} = S^{(2)}_{G} + S^{(2)}_{B} + S^{(2)}_{\Phi}, \tag{2.1.3.19}$$

with (the background fields G, B and Φ are taken at $\bar{X}^\mu$)

$$\begin{aligned} S_G^{(2)} &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{\alpha\beta}\left(G_{\mu\nu}D_\alpha\xi^\mu D_\beta\xi^\nu\right. \\ &\qquad \left.+R_{\rho\mu\kappa\nu}\partial_\alpha\bar{X}^\mu\partial_\beta\bar{X}^\nu\xi^\rho\xi^\kappa\right), \qquad (2.1.3.20) \\ S_B^{(2)} &= -\frac{1}{4\pi\alpha'}\int d^2\sigma i\epsilon^{\alpha\beta}\left(\partial_\alpha\bar{X}^\rho H_{\rho\mu\nu}\xi^\nu D_\beta\xi^\mu\right. \\ &\qquad \left.+\frac{1}{2}D_\lambda H_{\rho\mu\nu}\xi^\lambda\xi^\rho\partial_\alpha\bar{X}^\mu\partial_\beta\bar{X}^\nu\right) \qquad (2.1.3.21) \\ S_\Phi^{(2)} &= -\frac{1}{4\pi}\int d^2\sigma\sqrt{\gamma}R^{(2)}\frac{1}{2}D_\mu D_\nu\Phi\xi^\mu\xi^\nu. \qquad (2.1.3.22) \end{aligned}$$

The next step is to redefine the fields ξ^μ in terms of a vielbein,

$$\xi^\mu = E_A^\mu\xi^A, \qquad (2.1.3.23)$$

with

$$\begin{aligned} G_{\mu\nu} &= E_\mu^A E_\nu^B\eta_{AB}, \qquad (2.1.3.24) \\ E_A^\mu E_{\mu B} &= \eta_{AB}. \qquad (2.1.3.25) \end{aligned}$$

In what follows, capital latin indices will be raised and lowered with the Minkowski metric. The normal coordinate expansion is useful not only to get the expressions (2.1.3.20), (2.1.3.21), (2.1.3.22) in a covariant looking form. An important advantage of this method is that the functional measure (in a path integral approach) for the ξ^A is the usual translation invariant measure. This will simplify the computation of the partition function. In order to be able to do the field redefinition (2.1.3.23) in a meaningfull way we have to ensure that the fluctuations are parameterized by target space vectors. The tangent vectors to geodesics connecting the background with the actual value are a natural choice. Before writing down the action in terms of the ξ^A, we will absorb the first term in (2.1.3.21) in an additional connection in the kinetic term (the first term in (2.1.3.20)). That can be done by adding and subtracting a term looking like

$$\partial_\alpha\bar{X}^\rho\partial^\alpha\bar{X}^\kappa H_\rho{}^\lambda{}_\mu H_{\kappa\lambda\nu}\xi^\mu\xi^\nu.$$

We define the covariant derivative on ξ^A by plugging (2.1.3.23) into (2.1.3.18) and introducing an additional connection

$$\mathcal{D}_\alpha\xi^A = D_\alpha\xi^A + \frac{i}{2}\frac{\epsilon_\alpha{}^\beta}{\sqrt{-\gamma}}\partial_\beta\bar{X}^\rho E_\mu^A H_\rho{}^\mu{}_\nu E_B^\nu\xi^B, \qquad (2.1.3.26)$$

where $D_\alpha\xi^A$ corresponds to the contribution from (2.1.3.18). The part of the action quadratic in fluctuations finally takes the form

$$S^{(2)} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\left(\gamma^{\alpha\beta}\mathcal{D}_\alpha\xi^A\mathcal{D}_\beta\xi_A + M_{AB}\xi^A\xi^B\right), \qquad (2.1.3.27)$$

where the potential is

$$M_{AB} = \gamma^{\alpha\beta}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \mathcal{G}_{\mu\nu AB} + i\frac{\epsilon^{\alpha\beta}}{\sqrt{\gamma}}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \mathcal{B}_{\mu\nu AB} + \alpha' R^{(2)} \mathcal{F}_{AB}. \quad (2.1.3.28)$$

The matrices $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{F}$ do not have an explicit dependence on the worldsheet coordinates and are given by

$$\begin{aligned} \mathcal{G}_{\mu\nu AB} &= E_A^\rho E_B^\kappa \left(R_{\rho\mu\kappa\nu} - \frac{1}{4} H_\mu{}^\lambda{}_\rho H_{\nu\lambda\kappa} \right), && (2.1.3.29) \\ \mathcal{B}_{\mu\nu AB} &= \frac{1}{2} D_\lambda H_{\rho\mu\nu} E_A^\lambda E_B^\rho, && (2.1.3.30) \\ \mathcal{F}_{AB} &= \frac{1}{2} E_A^\mu E_B^\nu D_\mu D_\nu \Phi. && (2.1.3.31) \end{aligned}$$

Since the action (2.1.3.27) is quadratic in the fluctuations, integrating over the fluctuations will result in the determinant of an operator. For the general form of the operator in (2.1.3.27) it is very covenient to use known formulæ from the heat kernel technique. In the heat kernel approach the partition function

$$Z = \int \mathcal{D}\xi^A \, e^{iS^{(2)}}$$

can be expressed as a formal sum[202, 83]

$$\log Z = \frac{1}{2} \int \frac{dt}{t} e^{-\mathcal{O}t} = \frac{1}{2} \int_{\epsilon\mu^{-2}}^\infty \frac{dt}{t} \sum_{n=-2}^\infty a_n t^{\frac{n}{2}-1}, \quad (2.1.3.32)$$

where ϵ is a dimensionless UV cutoff and μ is a mass scale introduced for dimensional reasons. The symbol $\mathcal{O}$ stands for the operator whose determinant is of interest. We rescale t by α' such that $\mathcal{O}$ has mass dimension 2.[18] In order to compute the beta functions, we are interested in the logarithmically divergent piece, i.e. in a_2. This can be found in the literature[202, 83]

$$a_2 = \frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma} \left(-M_A^A + \frac{d}{6} R^{(2)} \right). \quad (2.1.3.33)$$

The divergence can be cancelled by adding appropriate counterterms to the action. This amounts to a replacement of the bare (infinite) couplings $G_{\mu\nu}$, $B_{\mu\nu}$, Φ in the

[18] The appearance of a power series in α' is more obvious in a Feynman diagramatic treatment. There, the propagator goes like α' whereas vertices go like $1/\alpha'$. This relates directly the order of α' in logarithmically divergent diagrams to the number of loops. The disadvantage of this approach is that the discussion for a general worldsheet metric γ is more involved. Fixing γ to be the Minkowski metric results in problems when computing the dilaton beta function since $R^{(2)}$ vanishes for this choice.

following way,

$$G_{\mu\nu} = G^{ren}_{\mu\nu} - \frac{\alpha'}{2}\log\left(\mu^2/\epsilon\right)\mathcal{G}_{\mu\nu A}{}^{A}, \tag{2.1.3.34}$$

$$B_{\mu\nu} = B^{ren}_{\mu\nu} - \frac{\alpha'}{2}\log\left(\mu^2/\epsilon\right)\mathcal{B}_{\mu\nu A}{}^{A}, \tag{2.1.3.35}$$

$$\Phi = \Phi^{ren} - \frac{1}{2}\log\left(\mu^2/\epsilon\right)\left(-\frac{d+c_g}{6} + \alpha'\mathcal{F}_A{}^A\right), \tag{2.1.3.36}$$

where $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{F}$ are as defined in (2.1.3.29), (2.1.3.30) and (2.1.3.31) but now in terms of the renormalized couplings. Further, we have included a possible contribution of the diffeomorphism fixing ghosts. Their action depends only on the intrinsic two dimensional geometry and neither on the embedding in the target space nor on the form of the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Therefore, the ghosts can contribute only a constant renormalization of the dilaton Φ which we have parameterized by c_g in (2.1.3.36). The beta functions can be computed by taking the derivative of the renormalized couplings with respect to $\log\mu$ using the μ independence of the bare couplings. Up to order α' this leads to (they are all expressed in terms of renormalized quantities and we supress the corresponding superscript in the following)

$$\beta^{(G)}_{\mu\nu} = \alpha'\left(R_{\mu\nu} - \frac{1}{4}H_\mu{}^{\lambda\rho}H_{\nu\lambda\rho}\right), \tag{2.1.3.37}$$

$$\beta^{(B)}_{\mu\nu} = \frac{\alpha'}{2}D^\lambda H_{\lambda\mu\nu}, \tag{2.1.3.38}$$

$$\beta^{(\Phi)} = -\frac{d+c_g}{6} + \frac{\alpha'}{2}D^2\Phi. \tag{2.1.3.39}$$

Because of the ambiguities related to the field redifinitions (2.1.3.2) – (2.1.3.4) we cannot just set the beta functions to zero but only deduce that the model is Weyl invariant (in first approximation) if

$$\bar\beta^{(G)}_{\mu\nu} = \beta^{(G)}_{\mu\nu} + D_{(\mu}M_{\nu)} = 0, \tag{2.1.3.40}$$

$$\bar\beta^{(B)}_{\mu\nu} = \beta^{(B)}_{\mu\nu} + \frac{1}{2}H_{\mu\nu}{}^{\lambda}M_\lambda + \partial_{[\mu}L_{\nu]} = 0, \tag{2.1.3.41}$$

$$\bar\beta^{(\Phi)} = \beta^{(\Phi)} + \frac{1}{2}\partial_\mu\Phi\, M^\mu = 0 \tag{2.1.3.42}$$

The vectors M_μ and L_μ are not fixed by just checking for global scale invariance. In order to compute them we would need to impose (local) Weyl invariance. This could be done by computing the expectation value of the trace of the energy momentum tensor. Instead of doing so, we will choose a rather indirect way of fixing the ambiguity. Implicitly, we will be using a theorem stating that $\bar\beta^{(\Phi)}$ is constant if (2.1.3.40) and (2.1.3.41) are satisfied[111]. In other words this means that up to a constant contribution (2.1.3.42) should be an integrability condition for the other two equations. Before deriving such a condition we need to study of which form the vectors M_μ and L_μ could be at the given order in α'. We want to express a vector in terms of our background fields $G_{\mu\nu}$, Φ and $B_{\mu\nu}$. The field $B_{\mu\nu}$ should enter only via the gauge invariant field strength $H_{\mu\nu\lambda}$ (since we have performed partial integegrations in $S^{(2)}$ such that the beta functions come out in a gauge invariant form). With this information it is easy to check that the only option we have is (a is some constant)

$$M_\mu = a\partial_\mu\Phi \quad , \quad L_\mu = 0, \tag{2.1.3.43}$$

where we do not consider a gradient contribution in L_μ since this would not be relevant. The next step is to take the divergence of (2.1.3.40). Using the Bianchi identity (i.e. the vanishing divergence of the Einstein tensor), the identity

$$H^{\rho\lambda\mu}D_\rho H_{\lambda\mu\nu} = \frac{1}{6}D_\nu\left(H_{\mu\lambda\rho}H^{\mu\lambda\rho}\right),$$

and equations (2.1.3.40) and (2.1.3.41) one obtains

$$D_\nu\left(\frac{a}{2}D^2\Phi + \frac{\alpha'}{12}H^2 + \frac{a^2}{2\alpha'}\left(\partial\Phi\right)^2\right) = 0, \tag{2.1.3.44}$$

where we have defined

$$H^2 = H_{\rho\nu\lambda}H^{\rho\nu\lambda}. \tag{2.1.3.45}$$

On the other hand, equation (2.1.3.42) implies

$$D_\nu\left(\frac{\alpha'}{2}D^2\Phi + \frac{a}{2}\left(\partial\Phi\right)^2\right) = 0. \tag{2.1.3.46}$$

Without the H^2 term, (2.1.3.44) and (2.1.3.46) would be the same. The H^2 term in the dilaton beta function is actually missing in our computation since we took into account only one loop contributions. Any counterterm in the action leading to an order α' contribution in the dilaton beta function should be linear in α'. Since, at tree level the B field enters with a factor $\frac{1}{\alpha'}$, the H^2 term in $\beta^{(\Phi)}$ corresponds to a two loop contribution. In our implicit approach we obtained this term (and in fact all order α' terms in $\beta^{(\Phi)}$) without doing a two loop analysis.

We were not able to fix the value of the constant a, however. This is because it could be absorbed in a rescaling of the field Φ. But this would change the ratio of the constant contribution to the dilaton beta function to the other contributions. Therefore, a is not arbitrary. The constant a can be fixed for example by studying models with trivial metric and B field and a linear dilaton. These models are much easier to treat than the generic one. The result is

$$a = 2\alpha'. \tag{2.1.3.47}$$

Let us discuss the case of trivial metric and vanishing B field a bit further. For a linear dilaton, the $\bar{\beta}^{(G)}$ and $\bar{\beta}^{(B)}$ vanish identically. According to the previously stated theorem (and to our result) the $\bar{\beta}^{(\Phi)}$ function is constant in this case. Models with that feature are known as conformal field theories. The constant dilaton $\bar{\beta}$ function is related to an anomaly of the transformation of the energy momentum tensor under conformal coordinate changes (while keeping the worldsheet metric fixed). If we fix the worldsheet metric to be the Minkowski metric the anomalous transformation of the energy momentum tensor with respect to (2.1.1.9) reads

$$\tilde{T}_{\tilde{\sigma}^+\tilde{\sigma}^+} = \left(\frac{d\sigma^+}{d\tilde{\sigma}^+}\right)^2 T_{\sigma^+\sigma^+}\left(\sigma^+\right) + \frac{c}{12} S\left(\tilde{\sigma}^+, \sigma^+\right) \tag{2.1.3.48}$$

where the second term denotes the Schwarz derivative

$$S\left(w, z\right) = \frac{z' z''' - \frac{3}{2}\left(z''\right)^2}{\left(z'\right)^2}, \tag{2.1.3.49}$$

where z is a function of w and the primes denote derivatives. The Schwarz derivative has the following chain rule

$$S\left(w\left(v\left(z\right)\right), z\right) = \left(\frac{\partial v}{\partial w}\right)^2 S\left(v, z\right) + S\left(w, v\right). \tag{2.1.3.50}$$

The transformation law (2.1.3.48) is the most general possibility such that associativity holds. An analogous consideration applies to T_{--}. Now, from (2.1.3.48) one can deduce part of the operator product expansion (OPE) of two energy momentum tensors. To this end, one considers infinitesimal transformations and uses the fact that they are generated by T_{++}. One obtains the following OPE

$$\begin{aligned} T_{\tilde{\sigma}^+\tilde{\sigma}^+}\left(\tilde{\sigma}^+\right) T_{\sigma^+\sigma^+}\left(\sigma^+\right) \quad = \quad & \frac{c/2}{\left(\tilde{\sigma}^+ - \sigma^+\right)^4} + \frac{2}{\left(\tilde{\sigma}^+ - \sigma^+\right)^2} T_{\sigma^+\sigma^+}\left(\sigma^+\right) \\ & + \frac{1}{\tilde{\sigma}^+ - \sigma^+} \partial_{\sigma^+} T_{\sigma^+\sigma^+}\left(\sigma^+\right) + \ldots, \end{aligned} \tag{2.1.3.51}$$

where the dots stand for terms which are regular for $\tilde{\sigma}^+ = \sigma^+$. For linear dilaton backgrounds this OPE can easily be computed directly[19], leading to (2.1.3.47).

It remains to fix the contribution coming from the gauge fixing ghosts c_g. This can of course be calculated directly[375, 376]. Here, we will guess it correctly, instead. From our discussion of the quantized bosonic string in the light cone gauge in 2.1.2 we remember that the classical Lorentz covariance was preserved in $d = 26$. Comparing with (2.1.3.48) we observe that our gauge fixing procedure was justified only if $c = 0$. Since, we did not have a linear dilaton background there, this can happen only if

$$c_g = -26. \tag{2.1.3.52}$$

[19] One should first compute $T_{\alpha\beta}$ by varying the action with respect to $\gamma_{\alpha\beta}$, and gauge fix $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$ afterwards.

Equation (2.1.3.52) can be confirmed by an explicit computation (which can also be viewed as an alternative way of deriving the critical dimension).

Given the fact, that a linear dilaton contributes to c, one may want to go directly to $d = 4$ by switching on a linear dilaton. One obvious problem with this is however that target space Lorentz covariance is broken explicitly – there is a distinguished direction in which the dilaton derivative points. The more useful way of getting away from a 26 dimensional target space is to replace 22 of the string coordinates by a conformal field theory with central charge $d \to c = 22$.

To summarize, up to order α' the action (2.1.3.1) is Weyl invariant provided that the following set of equations holds,

$$R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\lambda}H_{\nu}{}^{\rho\lambda} + 2D_\mu\partial_\nu\Phi = 0, \tag{2.1.3.53}$$

$$-\frac{1}{2}D^\lambda H_{\lambda\mu\nu} + H_{\lambda\mu\nu}D^\lambda\Phi = 0, \tag{2.1.3.54}$$

$$\frac{1}{6}(d-26) - \frac{1}{2}\alpha' D^2\Phi + \alpha'(\partial\Phi)^2 - \frac{\alpha'}{24}H^2 = 0. \tag{2.1.3.55}$$

2.1.4 Perturbative expansion and effective actions

In the previous section we have seen that imposing Weyl invariance provides us with constraints on the background in which the string propagates. These constraints can be viewed as equations of motion for the background fields. Lifting those up to an action would then yield an effective field theory description for the string theory. We have discussed only the bosonic string, but an extension to the superstring is possible. It may however be problematic. In the NSR formalism it is for example not possible to include terms into the string sigma model which would correspond to non-trivial RR backgrounds. Therefore, we will sketch an alternative method of computing an effective action here. We will not present any explicit calculations but just describe the strategy. Starting from the spectrum and the amount of supersymmetries belonging to a certain string theory one can write down a general ansatz for an effective field theory action of the string excitation modes. This ansatz can be further fixed by comparing scattering amplitudes computed from the effective description to amplitudes obtained from a string computation. The string amplitudes can be described in a diagramatic fashion as depicted in figure 2.3.

The external four legs (hoses) correspond to the two incoming particles scattering into two outgoing particles. The expansion is in terms of the number of holes (the genus) of the worldsheet. The first diagram in 2.3 correponds to two incoming strings joining into one string which in turn splits into two outgoing strings. In that sense it contains two vertices. Analogously the second diagram contains four vertices and so on. Assigning to each vertex one power of the string coupling g_s, this gives a formal

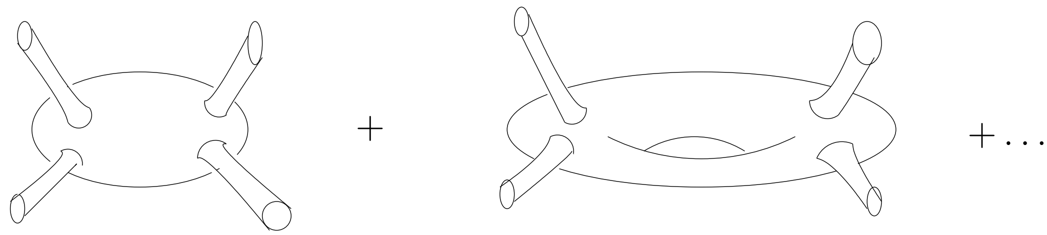

Figure 2.3: Perturbative expansion of the four point function in a string computation

power series

$$\mathcal{A} = \sum_{n=0}^{\infty} g_s^{2n+2} \mathcal{A}^{(n)}. \tag{2.1.4.1}$$

This power series can be truncated after the first contributions as long as $g_s \ll 1$. It remains to specify what g_s is. To this end, we first observe that the power of g_s in the expansion terms in (2.1.4.1) is nothing but minus the Euler number of a worldsheet with n handles and four boundaries. From (2.1.3.1) we know that the dilaton Φ couples to the Euler density of the worldsheet. This follows immediately from the Gauss-Bonnet theorem

$$\frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma} R^{(2)} = \chi \equiv 2\,(1-n) - b, \tag{2.1.4.2}$$

where n is the number of handles and b is the number of boundaries of the two dimensional worldsheet. (The calculation of the scattering amplitudes is performed after the worldsheet signature has been Wick rotated to the Euclidian one.) Thus, one can identify

$$g_s = e^{\langle \Phi \rangle}, \tag{2.1.4.3}$$

where $\langle \Phi \rangle$ denotes a constant vacuum expectation value (VEV) of the dilaton. (Remember from the previous section that a constant contribution to Φ was not fixed by the conformal invariance conditions. This is true for all string theories as can be easily seen by noticing that a constant shift in Φ shifts the action (2.1.3.1) by a constant.) Therefore, the string coupling is an arbitrary parameter in the perturbative approach to string theory. It is only restricted by the consistency requirement that the perturbative expansion in figure 2.3 should not break down, i.e. $g_s \ll 1$.

There is also a second approximation in the computation of the scattering amplitudes. Since the massive string states have masses of the order of the Planck mass, they are "integrated out". This means that we are interested in effects below the Planck scale where those fields do not propagate. The effective field theory actions contain only the massless modes. For consistency, one should then restrict to processes where the momentum transfer is $p^2 \ll 1/\alpha'$.

The (bosonic part of the) effective Lagrangians with at most two derivatives and only the massless fields turn out to be

$$S = S_{univ} + S_{model}, \tag{2.1.4.4}$$

where S_{univ} does not depend on which of the superstring theories we are looking at, and S_{model} is model dependent. The universal sector has as bosonic fields the metric, the dilaton, and the B field. The corresponding action is

$$S_{univ} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\left(\partial\Phi\right)^2 - \frac{1}{12}H^2\right), \tag{2.1.4.5}$$

where $\kappa^2 \sim (\alpha')^4$ is the ten dimensional gravitational constant. Note that the set of equations of motion obtained from this action coincides with the conformal invariance conditions (2.1.3.53)– (2.1.3.55), with the difference that for the superstring the constant contribution in the dilaton equation vanishes for $d = 10$.

For type II strings there are additional contributions giving the kinetic terms for the RR gauge forms and Chern-Simons interactions,

$$S^{II}_{model} = -\frac{1}{2\kappa^2}\int d^{10}x\sum_p \frac{1}{2(p+2)!}F^2_{p+2}, \tag{2.1.4.6}$$

where F_{p+2} is the fieldstrength of a $p+1$ form RR gauge field (plus –in some cases– additional contributions which we will discuss later),

$$F_{p+2} = dA_{p+1} + \ldots . \tag{2.1.4.7}$$

The number p is the spatial extension of an object which couples electrically to the $p+2$ form gauge field. In the worldvolume action of the corresponding p dimensional object this coupling is

$$\int d^{p+1}\sigma \;\; i\epsilon^{\alpha_1\cdots\alpha_{p+1}}\partial_{\alpha_1}X^{\mu_1}\cdots\partial_{\alpha_{p+1}}X^{\mu_{p+1}}A_{\mu_1\cdots\mu_{p+1}}. \tag{2.1.4.8}$$

For a point particle ($p = 0$) the above expression reads for example

$$i\int d\tau\frac{dX^\mu}{d\tau}A_\mu.$$

From expression (2.1.3.1) we observe that the fundamental string is electrically charged under the NSNS B field. We will meet objects which are charged under the RR gauge forms when discussing D-branes in section 2.3. For the type IIA theory we have

$p = 0, 2, 4$. Alternatively we could replace the field strength in (2.1.4.6) by its Hodge dual[20]

$$F_{8-p} = \star F_{p+2}. \tag{2.1.4.9}$$

In the type IIA theory the definition (2.1.4.7) is modified for the four form field strength

$$F_4 = dA_3 + A_1 \wedge H, \tag{2.1.4.10}$$

leading to a non standard Bianchi identity for the four form field strength

$$dF_4 = F_2 \wedge H. \tag{2.1.4.11}$$

Finally, the Chern-Simons interaction for type IIA is

$$S_{CS}^{IIA} = -\frac{1}{8\kappa^2} \int F_4 \wedge F_4 \wedge B. \tag{2.1.4.12}$$

For type IIB theories one has $p = -1, 1, 3$. For $p = -1$ the gauge form is a scalar, which is called axion. The object which is electrically charged under this zero form is localized in space and time. This is an instanton. The definition of the field strength (2.1.4.7) receives further contributions for $p = 1$ and $p = 3$

$$\begin{aligned} F_3 &= dA_2 - A_0 \wedge H, && (2.1.4.13)\\ F_5 &= dA_4 - \frac{1}{\sqrt{3}} A_2 \wedge H + \frac{1}{\sqrt{3}} B \wedge F_3. && (2.1.4.14) \end{aligned}$$

The Chern-Simons interaction for the type IIB theory is

$$S_{CS}^{IIB} = -\frac{9}{4\kappa^2} \int A_4 \wedge H \wedge F_3. \tag{2.1.4.15}$$

The five form field strength has to be selfdual. This is not encoded in the action (2.1.4.6) but has to be added as an additional constraint,

$$F_5 = \star F_5. \tag{2.1.4.16}$$

In the heterotic string we have gauge fields transforming in the adjoint of $SO(32)$ or $E_8 \times E_8$. Their field strength is defined as (we assign mass dimension one to the gauge fields A – this is related to a $\sqrt{\alpha'}$ rescaling of A)

$$F = dA + A \wedge A. \tag{2.1.4.17}$$

[20] This can be done by adding the Bianchi identity $dF_{p+2} = d(\cdots)$ with a Lagrange multiplier to the action and integrating out A_{p+1}. Because of covariance the Lagrange multiplier is a $7-p$ form and its field strength is an $8-p$ form.

The definition of H in (2.1.4.5) needs to be modified[21]

$$H = dB - \frac{\alpha'}{4}\left(\omega_Y - \omega_L\right). \tag{2.1.4.18}$$

The Yang-Mills Chern-Simons form ω_Y is

$$\omega_Y = trA \wedge dA + \frac{2}{3} trA \wedge A \wedge A, \tag{2.1.4.19}$$

where A is the gauge connection of either $E_8 \times E_8$ or $SO(32)$. The modification (2.1.4.18) implies that the B field transforms under gauge transformations and under local Lorentz rotations in a non-trivial way such that H is gauge invariant. The Yang-Mills Chern-Simons form has the property that its exterior derivative gives the instanton density (in a four dimensional subspace with Euclidean signature),

$$d\omega_Y = trF \wedge F. \tag{2.1.4.20}$$

The Lorentz Chern-Simons form is constructed from the spin connection ω,

$$d\omega_L = tr\omega \wedge d\omega + \frac{2}{3} tr\omega \wedge \omega \wedge \omega. \tag{2.1.4.21}$$

If its exterior derivative takes values only on a four dimensional submanifold with Euclidean signature it corresponds to the Euler density of that manifold,

$$d\omega_L = trR \wedge R. \tag{2.1.4.22}$$

If we take the ten dimensional geometry to consist of a direct product of a six dimensional non compact and a four dimensional compact space (with Euclidean signature) the modification (2.1.4.18) implies restrictions on the allowed gauge bundles on the four dimensional compact space. The integration of dH over a compact space should vanish. It follows that the Euler number of this space must be equal to the instanton number of the gauge bundle.

In addition to the universal piece (2.1.4.5), the heterotic action contains a gauge kinetic term and also the Green-Schwarz term which ensures anomaly cancellation

$$S^{heterotic}_{model} = S_{gauge} + S_{GS}, \tag{2.1.4.23}$$

with

$$S_{gauge} = -\frac{1}{2\kappa^2} \int d^{10}x e^{-2\Phi} \frac{\alpha'}{8} trF^2, \tag{2.1.4.24}$$

[21] We present the effective action for the heterotic string just for completeness, more details on differential geometry and anomaly cancelation in the context of the effective heterotic theory can be found e.g. in [223].

where again, the trace is taken over the gauge group ($E_8 \times E_8$ or $SO(32)$). The Green-Schwarz term is

$$S_{GS} = \frac{8\pi i}{\alpha'} \int B \wedge X_8, \tag{2.1.4.25}$$

with (here, a power is meant with respect to the wedge product, e.g. $F^4 \equiv F \wedge F \wedge F \wedge F$)

$$X_8 = \frac{1}{2}\frac{1}{(2\pi)^6}\frac{1}{48}\left(\frac{5}{4}trF^4 - \frac{1}{8}\left(trF^2\right)^2 - \frac{1}{8}trF^2 trR^2 + \frac{1}{8}trR^4 + \frac{1}{32}\left(trR^2\right)^2\right) \tag{2.1.4.26}$$

To close this section on effective actions we identify the different contributions with the worldsheet topologies they correspond to. First, we observe that all the terms appearing in the effective actions are of a structure such that they contain some power of e^{Φ} times a factor which is invariant under constant shifts in Φ. In (2.1.4.3) we have identified the string coupling as a constant part of e^{Φ}. Thus, the leading term in the perturbative expansion in figure 2.3 enters the effective action accompanied with a factor of $e^{-2\Phi}$. These are all terms in (2.1.4.5) and the gauge kinetic term in the heterotic theory (2.1.4.24). One may be tempted to interpret the other terms (containing no $e^{-2\Phi}$ factor) as one loop contributions. This is, however, misleading. In order to simplify the Bianchi identities for the RR gauge forms we have rescaled the RR gauge potentials by e^{Φ}. Undoing this rescaling means that for example the RR form F_2 receives a further contribution

$$A_1 = e^{-\Phi}A_1' \quad \longrightarrow \quad F_2 = e^{-\Phi}\left(dA_1' - d\Phi \wedge A_1'\right) \equiv e^{-\Phi}F_2', \tag{2.1.4.27}$$

and similar relations for the other RR field strengths. (If the terms denoted by dots in (2.1.4.7) contain other RR field strengths additional Φ derivatives will be picked up. But no relative power of e^{Φ} will appear, since those terms always contain one RR field strength or potential and an NSNS field strength or potential. The NSNS fields are not rescaled.) After this rescaling all terms in the type II thoeries are of the structure

$$e^{-2\Phi}\left(\text{invariant under } \Phi \to \Phi + constant\right).$$

Since the rescaled (primed) fields correspond to the actual string excitations, the effective type II actions given here contain only tree level contributions.

As (implicitly) stated above, one loop contributions are multiplied by g_s^2 and hence enter the effective action with a factor of $e^{-2\Phi+2\Phi} = 1$. In the type II examples, we have seen that due to field redefinitions this correspondence may be changed. In the heterotic case, however, there is no field redefinition such that all the terms in the effective action are multiplied by the same power of e^{Φ}. Indeed, the appearance of the Green-Schwarz term corresponds to a torus amplitude from the string theory perspective. We excluded also higher orders in α' which would lead to higher derivative terms and contributions with massive string excitations.[22] As long as the string scale is much shorter (in length) than the scale of the process we are interested in those terms can be neglected.

2.1.5 Toroidal Compactification and T-duality

In the previous sections we argued that perturbative superstring theories are consistent provided that the target space is ten dimensional. As it stands, this cannot describe our observable (four dimensional) world. At the end of section (2.1.3), we sketched as a possible resolution to this problem the option to replace six of the target space dimensions by a conformal field theory with the desired central charge. One simple way to do so, is to replace a six dimensional subspace of the ten dimensional Minkowski space by a compact manifold. The coordinates of that compact manifold should belong to a conformal field theory with a consistent central charge. This restricts the set of possible compactifications. The easiest option is to compactify the additional directions on circles (by periodic identification of the corresponding coordinates). This clearly does not change the central charge contribution of those directions, since the central charge depends only on local features of the target space.

2.1.5.1 Kaluza-Klein compactification of a scalar field

Before discussing some details of torus compactifications of string theories we recall the Kaluza-Klein compactification of a free massless scalar field. This will enable us to appreciate new "stringy" features which we will study afterwards. Let us start with a free massless scalar living in a five dimensional Minkowski space. We label the first four coordinates with a greek index $\mu = 0, \ldots, 3$ and call the fifth direction y. The five dimensional field equation for the scalar φ is

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \partial_y^2\right)\varphi\left(x^\mu, y\right) = 0. \tag{2.1.5.1}$$

Now, we compactify the fifth direction on a circle of radius R

$$y \equiv y + 2\pi R. \tag{2.1.5.2}$$

Solutions to (2.1.5.1) have to respect the periodicity (2.1.5.2). Therefore, the y dependent part of φ can be expanded into a Fourier series of periodic functions. Focusing

[22] An exception is the ω_L correction in (2.1.4.18) and the Green Schwarz term. They can be deduced by using supersymmetry and anomaly cancellation.

on the nth Fourier mode, we find

$$\varphi_n\left(x^\mu, y\right) = \varphi_n\left(x^\mu\right) e^{i\frac{n}{R}y}, \tag{2.1.5.3}$$

with integer n, i.e. the momentum in the fifth direction is quantized. Plugging (2.1.5.3) back into (2.1.5.1) leads to

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m_n^2\right)\varphi_n\left(x^\mu\right) = 0, \tag{2.1.5.4}$$

with

$$m_n = \frac{n}{R}, \tag{2.1.5.5}$$

i.e. the nth Fourier mode leads in the effective four dimensional description to a Klein-Gordon field with mass (2.1.5.5). Since the general solution of (2.1.5.1) is a superposition of all Fourier modes the four dimensional description contains an infinite Kaluza-Klein tower of massive four dimensional fields (depending only on the x^μ). There are two limits to be discussed. The decompactification limit is $R \to \infty$. In this case all the Kaluza-Klein masses (2.1.5.5) vanish. The four dimensional description breaks down. The other limit is $R \to 0$ (or the compactification radius becomes much shorter than the experimental distance resolution). In this case, the KK masses (2.1.5.5) become infinite except for $n = 0$. Only the massless mode survives and no trace from the fifth dimension is left. This picture is very different in string theories as we will see now.

2.1.5.2 The bosonic string on a circle

Even though the bosonic string is inconsistent because it contains a tachyon, we will first study the compactification of the bosonic string on a circle. The essential stringy properties will be visible in this toy model. We compactify the 26th coordinate (the 25th spatial direction),

$$x^{25} \equiv x^{25} + 2\pi R. \tag{2.1.5.6}$$

In the point particle limit string theory is just quantum mechanics of a free relativistic particle. The plane wave solution contains the factor $e^{ip_{25}x^{25}}$ where p_{25} is the center of mass momentum in the 25th direction. This wave function should be periodic under (2.1.5.6). This leads to a quantization condition for the center of mass momentum in the compact direction

$$p_{25} = \frac{n}{R}, \tag{2.1.5.7}$$

with integer n (the momentum number). So far, everything is analogous to the free scalar field discussed above. The new stringy property arises by observing that the string can wind around the compact direction. Technically, this means that the periodic boundary condition for the closed string is modified

$$X^{25}\left(\tau, \sigma + \pi\right) = X^{25}\left(\tau, \sigma\right) + 2\pi m R, \tag{2.1.5.8}$$

where the integer m denotes the winding number. With this ingredients the mode expansions (2.1.2.3) and (2.1.2.4) are

$$\begin{aligned} X_R^{25} &= \frac{1}{2}x^{25} + \left(\frac{n}{2R} - mR\right)\sigma^- + \frac{i}{2}\sum_{k\neq 0}\frac{1}{k}\alpha_k^{25}e^{-2ik\sigma^-}, && (2.1.5.9)\\ X_L^{25} &= \frac{1}{2}x^{25} + \left(\frac{n}{2R} + mR\right)\sigma^+ + \frac{i}{2}\sum_{k\neq 0}\frac{1}{k}\tilde{\alpha}_k^{25}e^{-2ik\sigma^+}. && (2.1.5.10)\end{aligned}$$

Taking into account the compact direction, the mass shell condition has to be modified in a straightforward way,

$$\sum_{\mu=0}^{24} p_\mu p^\mu = -M^2. \tag{2.1.5.11}$$

Comparison with the constraints $T_{++} = T_{--} = 0$ (2.1.1.10) gives

$$M^2 = 4\left(\frac{n}{2R} - mR\right)^2 + 8N - 8 = 4\left(\frac{n}{2R} + mR\right)^2 + 8\tilde{N} - 8 \tag{2.1.5.12}$$

where we have used the result of section 2.1.2 for the normal ordering. In particular, the level matching condition (the second equality in (2.1.5.12)) implies that

$$N - \tilde{N} = nm. \tag{2.1.5.13}$$

Thus, for zero winding and momentum number the spectrum coincides with the spectrum of the uncompactified string (see section 2.1.2). In the massless sector we have again a graviton, antisymmetric tensor and dilaton which are obtained from the state

$$\alpha_{-1}^i\tilde{\alpha}_{-1}^j\left|k\right\rangle \quad , \quad i,j \neq 25. \tag{2.1.5.14}$$

The target space interpretation of the remaining excitations (containing creator(s) in 25th direction) is different. The two states

$$\alpha_{-1}^i\tilde{\alpha}_{-1}^{25}\left|k\right\rangle \quad , \quad \alpha_{-1}^{25}\tilde{\alpha}_{-1}^i\left|k\right\rangle \tag{2.1.5.15}$$

are target space vectors. They correspond to gauge fields of a $U(1) \times U(1)$ gauge symmetry. Finally, the state

$$\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}\left|k\right\rangle \tag{2.1.5.16}$$

describes a target space scalar. The spectrum is supplemented by a Kaluza-Klein and winding tower of additional states as n, m run through the integer numbers. An interesting question is wether some of these additional states are massless. For massless states the mass shell condition (2.1.5.12) reads

$$2N - 2 + \left(\frac{n}{2R} - Rm\right)^2 = 2\tilde{N} - 2 + \left(\frac{n}{2R} + Rm\right)^2 = 0. \qquad (2.1.5.17)$$

These equations can be solved for nonvanishing n or m only at special values of R. The most interesting case is[23]

$$R^2 = \frac{1}{2} = \alpha'. \qquad (2.1.5.18)$$

One obtains the additional solutions listed in table 2.2.

n	m	N	$\tilde{N}$
1	1	1	0
-1	-1	1	0
1	-1	0	1
-1	1	0	1
2	0	0	0
-2	0	0	0
0	2	0	0
0	-2	0	0

Table 2.2: Each line in this table gives a configuration of winding, momentum and occupation numbers leading to massless states at $R^2 = \frac{1}{2}$.

Each of the first four states in table 2.2 contains one creator. This gives four additional massless vectors (if the creator points into a non-compact direction) and four massless scalars (if the creator points into the 25th direction). The latter four states in table 2.2 correspond to massless scalars. Together with (2.1.5.15) and (2.1.5.16) we have six vectors and nine scalars. The vectors combine into an $SU(2) \times SU(2)$ gauge field whereas the scalars form a $(\mathbf{3}, \mathbf{3})$ representation. For the special value (2.1.5.18) the gauge group $U(1) \times U(1)$ is enhanced to the non-abelian group $SU(2) \times SU(2)$. The rank of the gauge group is not changed.

An immediate question is: what is so special about (2.1.5.18)? To answer this, we rewrite (2.1.5.12) in a suggestive way

$$M^2 = 4N + 4\tilde{N} - 8 + \frac{n^2}{R^2} + 4m^2R^2, \qquad (2.1.5.19)$$

[23]Later, in section 2.2.1, we will also discus the case $R^2 = 2$, where less states become massless.

where we already aplied (2.1.5.13). We observe that the spectrum is invariant under

$$n \leftrightarrow m \text{ and } R \leftrightarrow \frac{\alpha'}{R}. \tag{2.1.5.20}$$

Recall that in the previous equations we have set $\alpha' = 1/2$. The symmetry (2.1.5.20) is called T-duality. Winding and momentum numbers are interchanged and simultaneously the compactification radius is inverted. If R takes the value (2.1.5.18), the spectrum is invariant under interchanging winding with momentum. This radius is called the selfdual radius. Because of the symmetry (2.1.5.20) we can restrict the compactifications to radii equal or larger than (2.1.5.18). This is an important difference to the point particle discussed in the previous section. To make this difference clearer let us take the compactification radius to zero. All the Kaluza-Klein momenta diverge and only states with $n = 0$ survive. This is similar to the point particle case. On the other hand, all winding states degenerate. In order to make sense out of this situation one can apply the T-duality tranformation (2.1.5.20). But then $R = 0$ leads to the decompactification limit and we are back at the 26 dimensional string. Therefore, in string theory there are always traces of compact dimensions left.

Compactifying the string on a D dimensional torus, the above considerations lead to a $\mathbb{Z}_2^D$ symmetry in a straightforward way. However, combining the T-duality along circles with basis redefinitions of the torus lattice and integer shifts in the internal B field leads to an enhancement of the T-duality group to $SO(D, D, \mathbb{Z})$.

2.1.5.3 T-duality in non trivial backgrounds

In this section we will argue that the above described T-dualiy is also a symmetry for non-trivial background configurations. We closely follow[392]. Our starting point is the non-linear sigma model (2.1.3.1). Compactification of one target space dimension is possible if the sigma model is invariant under constant shifts in this direction. For the first term in (2.1.3.1) this implies that the tangent to the compactified direction is a Killing vector. The second term is invariant provided that the Lie derivative of $B_{\mu\nu}$ in the Killing direction is an exact two-form. For the last term to be invaraint the Lie derivative of the dilaton Φ must vanish. We now choose coordinates such that the isometry is represented by a translation in the $d-1$ direction

$$X^{d-1} \to X^{d-1} + c. \tag{2.1.5.21}$$

We call the other coordinates X^i. The previously mentioned conditions on $B_{\mu\nu}$ and Φ imply that those fields are independent of X^{d-1} (up to gauge transformations).

The next step is to gauge the symmetry (2.1.5.21) and to “undo” this by constraining the gauge fields to be of pure gauge. The constraint is implemented with the help of a Lagrange multiplier λ which finally will replace X^{d-1} in the T-dual model. We introduce two dimensional gauge fields A_α changing under (2.1.5.21) as

$$A_\alpha \to A_\alpha - \partial_\alpha c, \tag{2.1.5.22}$$

and replace

$$\partial_\alpha X^{d-1} \to D_\alpha X^{d-1} \equiv \partial_\alpha X^{d-1} + A_\alpha. \tag{2.1.5.23}$$

Together with the above mentioned constraint (implemented by a Lagrange multiplier) this amounts to adding to (2.1.3.1) (for simplicity we choose $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$)[24] a term

$$\begin{aligned} S_A &= -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(G_{d-1,d-1}A_\alpha A^\alpha + 2G_{d-1,\nu}A^\alpha\partial_\alpha X^\nu\right. \\ &\qquad \left. +2\epsilon^{\alpha\beta}B_{d-1,\nu}A_\alpha\partial_\beta X^\nu + 2\lambda\epsilon^{\alpha\beta}\partial_\alpha A_\beta\right). \end{aligned} \tag{2.1.5.24}$$

Integrating over λ will result in the constraint of vanshing field strength for the A_α which in turn imposes

$$A_\alpha = \partial_\alpha\varphi, \tag{2.1.5.25}$$

with φ being a worldsheet scalar. Shifting X^{d-1} by φ gives back the original sigma model (2.1.3.1). Thus, adding (2.1.5.24) does not change anything. However, there is a subtlety here. Compactifying the $d-1$ direction means that we identify X^{d-1} with $X^{d-1}+2\pi$ (this time we put the compactification radius into the target space metric). In order to be able to absorb φ into X^μ, φ should respect the same periodicity. This can be ensured as follows. We continue the worldsheet to Euclidean signature and study the sigma model for a torus worldsheet. Then we can assign two winding numbers (corresponding to the two cycles of the torus) to the Lagrange multiplier λ. Summing over these winding numbers (in a path integral approach) will impose the required periodicity on the gauge fields A_α. Going through the details of this prescription leads to the conclusion that the λ "direction" is compact $\lambda \equiv \lambda + 2\pi$.

Instead of integrating out λ (to check that we did not change the model) we can integrate out A_α. (Since A_α is not a propagating field this can be done by solving the equations of motion. As well, one can integrate out A_α in a path integral, which is Gaussian.) This procedure leads us to a dual model

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\tilde{G}_{\mu\nu}\partial_\alpha\tilde{X}^\mu\partial_\beta\tilde{X}^\nu + \epsilon^{\alpha\beta}\tilde{B}_{\mu\nu}\partial_\alpha\tilde{X}^\mu\partial_\beta\tilde{X}^\nu\right) \\ &\quad -\frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}\tilde{\Phi}R^{(2)}. \end{aligned} \tag{2.1.5.26}$$

[24]For the Minkowskian worldsheet signature the i in front of the $B_{\mu\nu}$ coupling in (2.1.3.1) is replaced by one.

The set of dual coordinates is $\left\{\tilde{X}^\mu\right\} = \left\{\lambda, X^i\right\}$, and the dual background fields are,

$$\tilde{G}_{d-1,d-1} = \frac{1}{G_{d-1,d-1}}, \quad \tilde{G}_{d-1,i} = \frac{B_{d-1,i}}{G_{d-1,d-1}}, \tag{2.1.5.27}$$

$$\tilde{G}_{ij} = G_{ij} - \frac{G_{i,d-1}G_{d-1,j} + B_{i,d-1}B_{d-1,j}}{G_{d-1,d-1}}, \tag{2.1.5.28}$$

$$\tilde{B}_{d-1,i} = \frac{G_{d-1,i}}{G_{d-1,d-1}}, \quad \tilde{B}_{ij} = B_{ij} + \frac{G_{i,d-1}B_{d-1,j} + B_{i,d-1}G_{d-1,j}}{G_{d-1,d-1}}. \tag{2.1.5.29}$$

To find the dual expression for the dilaton is a bit more complicated. One can compute $\tilde{\Phi}$ in a perturbative way. To this end, one requires that $\tilde{\Phi}$ is such that the conformal invariance conditions (2.1.3.53), (2.1.3.54), (2.1.3.55) are satisfied whenever the original background satisfies them. This leads to the following formula for the dual dilaton

$$e^{-2\Phi}\sqrt{-G} = e^{-2\tilde{\Phi}}\sqrt{-\tilde{G}}. \tag{2.1.5.30}$$

From a path integral perspective the dilaton transformation can be motivated as follows. The path integral measure for the X^μ is covariant with respect to the metric $G_{\mu\nu}$. In the dual model one would naturally use a measure which is covariant with respect to the dual metric $\tilde{G}_{\mu\nu}$. The change of the measure introduces a Jacobian which leads to (2.1.5.30). To our knowledge this is a rather qualitative statement which is difficult to prove explicitly.

To make contact with the simple case discussed in the previous section we should take $G_{d-1,d-1} = R^2/\alpha'$, $G_{ij} = \eta_{ij}$, $B_{\mu\nu} = 0$ and $\Phi = const.$ Then the T-duality formulæ (especially (2.1.5.27)) imply that the compactification radius is inverted. The dilaton receives a constant shift and nothing else changes. This dilaton shift was not visible in the discussion of the previous section. On the other hand, in the present section we did neither see that T-duality interchanges winding with momentum nor that there is an enhancement of gauge symmetry at the selfdual radius, because we did not study the spectrum of the general string theory.

2.1.5.4 T-duality for superstrings

In extending the discussion of section 2.1.5.2 to the superstring we will be sketchy and omit the technical details. Most of the statements from section 2.1.5.2 can be directly taken over to the superstring. In the Ramond sector, some new ingredients occur. First, we consider the type II superstring. Instead of the 26th direction we compactify the tenth direction. Combining the T-duality transformation (2.1.5.20) with the mode expansions (2.1.5.9) and (2.1.5.10) one realizes that (2.1.5.20) can be achieved by assigning $X^9 = X_L^9 - X_R^9$, instead of $X^9 = X_L^9 + X_R^9$, or equivalently change

$$(\text{right movers}) \longrightarrow -(\text{right movers}) \tag{2.1.5.31}$$

while keeping the original prescription of combining left with right movers. Carrying this prescription over to the fermionic sector we observe that in the right moving Ramond sector (see (2.1.2.58) for the definition)

$$(-)^F \longrightarrow -(-)^F . \tag{2.1.5.32}$$

This in turn implies that the T-duality transformation takes us from the type IIA to type IIB string and vice versa. Hence, T-duality is not a symmetry in type II superstrings but relates the type IIA string with type IIB. (This is true, whenever we perform T-duality in an odd number of directions.) The type IIA string compactified on a circle with radius R is equivalent to the type IIB string compactified on a circle with radius α'/R. This is consistent with the observation that the massless spectra of circle compactified type IIA and type IIB theories are identical as depicted in table 2.3, where $\mu, \nu = 0, \ldots, 8$.

	NS-NS	R-R
IIA	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ, $G_{\mu 9}$, $B_{\mu 9}$	A_μ, A_9, $C_{\mu\nu\rho}$, $C_{\mu\nu 9}$
IIB	$G_{\mu\nu}$, $B_{\mu\nu}$, Φ, $G_{\mu 9}$, $B_{\mu 9}$	$B'_{\mu 9}$, Φ', $C_{\mu\nu\rho 9}$, $B'_{\mu\nu}$

Table 2.3: Massless type II fields in nine dimensions

In order to discuss compactifications of the heterotic string, it is useful to employ a formulation where the additional 32 left moving fermions are bosonized into 16 left moving bosonic degrees of freedom. We will not carry out this construction here. It can be found in the books listed in section 6.1.1. The result which is of interest in the current context is that those 16 left moving bosons are compactified on an even selfdual lattice[25]. That is, that even without further compactifications from ten to less dimensions the heterotic string contains already a left-right asymmetric compactification. The theory does not depend on changing the basis of the 'internal' lattice. Compactifying the tenth dimension one observes another new feature which is present in the heterotic string. In the previously discussed cases, there was one modulus in circle compactfications, *viz.* the radius of the circle. For the heterotic string we have 16 more moduli. These are called Wilson lines. They arise from the

[25]There exist exactly two such lattices $\Gamma_8 \times \Gamma_8$ and Γ_{16} giving rise to the $E_8 \times E_8$ and the $SO(32)$ string, respectively.

possibility that the non-abelian gauge fields can take constant vacuum expectation values (vev) in the Cartan subalgebra of the gauge group. The fact that (at least) one of the ten directions has been compactified is important here. Otherwise, a constant vev could be gauged away. To see this explicitly, let us assume that the gauge field vev is (proportional to a generator in the Cartan subalgebra)

$$A_9 = \frac{\Theta}{R} = e^{-\frac{x^9\Theta}{R}}\partial_9 e^{\frac{x^9\Theta}{R}}, \tag{2.1.5.33}$$

where the second part of the equation shows that a constant vev is a pure gauge configuration. However, in the compact case we have identified x^9 with $x^9 + 2\pi R$ and hence only gauge transformations which are periodic under this shift are allowed. This implies that the Wilson line (2.1.5.33) can be gauged away only if Θ is an integer. From this discussion it follows that generically the gauge group is broken to $U(1)^{16}$ in the compactification process. In addition there are the (abelian) Kaluza-Klein gauge fields $G_{9\mu}$ and $B_{9\mu}$ corresponding to a $U(1)\times U(1)$ gauge symmetry. Thus, generically there is a $U\left(1\right)^{18}$ gauge symmetry in the circle compactified heterotic string. Depending on the moduli (Wilson lines and compactification radius) there are special points of stringy gauge group enhancement.

It can be proven that the $E_8 \times E_8$ heterotic string and the $SO(32)$ string are continuously connected in moduli space once they have been compactified to nine dimensions. This can be shown by observing that for a certain configuration of Wilson lines (where the gauge group is broken to $SO(16) \times SO(16)$ in either theory) T-duality maps the two compactifications on each other. (For details see e.g. Polchinski's book[371].) All other compactifications can be reached by continuously changing the moduli. Including the original ten dimensional theories as decompactification limits we see that the two different heterotic strings belong to the same set of theories sitting at different corners in moduli space. For completeness we should mention that for compactifications of the heterotic string on a D dimensional torus one finds the T-duality group $SO\left(16 + D, D, \mathbb{Z}\right)$.

2.2 Orbifold fixed planes

In the previous sections we have studied the theory of a one dimensional extended object – the string. One of the striking features of this theory is that it automatically also describes objects which are extended along more than one space direction. As the simplest example we will study now the orbifold fixed planes. Here, one compactifies the string on a torus whose lattice has a discrete symmetry, and gauges this symmetry[26]. Thus, the compact manifold is a D dimensional torus divided by some discrete

[26] With gauging of a discrete symmetry we mean that only invariant states are kept.

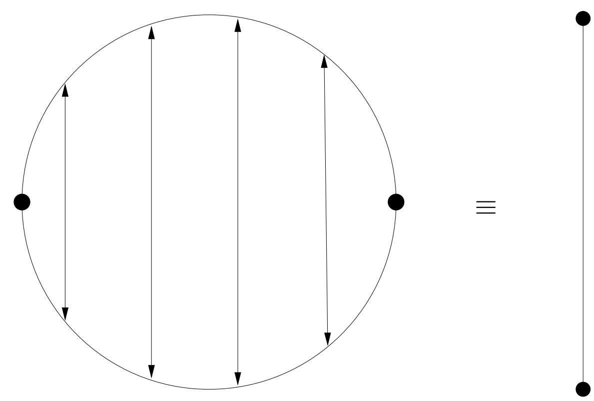

Figure 2.4: The interval as an orbicircle. The fixed points (black dots) form the ends of the interval.

group. (We will consider $\mathbb{Z}_2$ as such a group. It leaves an arbitrary lattice invariant.) There are some points or —when combined with the other directions— planes which are invariant under the discrete group. These are the orbifold fixed planes. They present singularities in the compact part of the space time. String theory gives a physical meaning to orbifold fixed planes. We will see that certain string excitations (particles or gauge fields from the target space perspective) are confined to be located at the orbifold fixed planes. By looking at an example where the orbifold can be reached as a singular limit of a smooth manifold we will see that for string theory the singular nature of this limit is not "visible". Instead of discussing the general setups for orbifold compactifications we will present two examples: the bosonic string on an orbicircle and the type IIB string on $T^4/\mathbb{Z}_2$. We hope that this will demonstrate the general idea with a minimal amount of technical complications. For more details (and also orbifold compactifications of the heterotic string) we recommend the review[354].

2.2.1 The bosonic string on an orbicircle

Let us start by describing the target space geometry. We compactify the 25th dimension on a circle like in section 2.1.5.2. In addition, we identify opposite points on this circle. If we choose the "fundamental domain" to be $-\pi R < x^{25} < \pi R$ this is done by the $\mathbb{Z}_2$ identification: $x^{25} \equiv -x^{25}$. The resulting target space is an interval in the 25th direction as depicted in figure 2.4. Taking into account the uncompactified dimensions, the end points of the interval (the fixed points of the $\mathbb{Z}_2$) correspond to planes with 24 spatial directions. Therefore, we call them orbifold-24-planes.

We proceed by constructing the untwisted spectrum. The term untwisted (in contrast to twisted) will become clear later. It means that we construct the spectrum

State	$\mathbb{Z}_2$	$24+1$ dim. rep.
$\alpha^i_{-1}\|0\rangle$	$+$	1 vector
$\alpha^{25}_{-1}\|0\rangle$	$-$	1 scalar

Table 2.4: Untwisted right moving states

which is invariant under the orbifold projection $x^{25} \to -x^{25}$. Since in the bosonic string the right moving sector is identical to the left moving one, we first write down the right moving states only. The result is collected in table 2.4 ($i = 2, \ldots, 24$; the zeroth and first direction are eliminated by the light-cone gauge).

Now we need to combine left with right movers such that the resulting state is invariant under the $\mathbb{Z}_2$. This is the case for the product of the vector with the vector and the scalar with the scalar. Thus, we obtain the metric G_{ij}, the antisymmetric tensor B_{ij} and the dilaton Φ. The additional $U(1)$ vectors G_{i25} and B_{i25} are projected out in contrast to section 2.1.5.2. The combination of the scalar from the left moving sector with the scalar from the right moving sector yields a target space scalar $G_{25\,25}$. Since the groundstate is $\mathbb{Z}_2$ invariant, the tachyon will survive the projection. If we are at the selfdual radius, there might be additional massless states (without imposing $\mathbb{Z}_2$ invariance these are listed in table 2.2). The action of the $\mathbb{Z}_2$ takes winding number to minus winding number and momentum number to minus momentum number as can be seen from the mode expansion (2.1.5.9), (2.1.5.10). This means that we can keep only invariant superpositions of states. From the first four entries in table 2.2 we obtain two additional massless vectors. These arise as follows. We add the first state to the second state of the listing and act with α^i_{-1}, or we add the third to the fourth state and act with $\tilde{\alpha^i}_{-1}$. We can also subtract the second from the first state and act with α^{25}_{-1}, or we subtract the fourth from the third state and act with $\tilde{\alpha}^{25}_{-1}$. This gives two massless scalars. Adding the fifth to the sixth entry and the seventh to the eighth, we obtain two more scalars at the selfdual radius. This looks very unusual. Since we do not have any $U(1)$ gauge fields away from the selfdual radius, now also the rank of the gauge group is enhanced at the selfdual radius. There are also additional tachyons at the selfdual radius. These are the two states which are obtained by adding the $n = 0, m = 1$ vacuum to the $n = 0, m = -1$ vacuum. The second state is the same with m and n interchanged. These two additional tachyons have mass squared $M^2 = -6$, as can be easily computed from (2.1.5.19).

Now, we come to a new feature which is unique to string theory. There are additional twisted sector states. These states are periodic under shifting σ by π only up to a (non-trivial) $\mathbb{Z}_2$ transformation. In our case this implies for the string that its center of mass position has to be located at a fixed plane and that the integer Fourier modes are replaced by half-integer ones in the 25th direction. In the twisted sector we need to compute the groundstate energy. This can be done by first modifying equation (2.1.2.46) in a straightforward way

$$a^{twisted} = -\frac{23}{2}\sum_{n=1}^{\infty} n - \frac{1}{2}\sum_{r=\frac{1}{2}}^{\infty} r. \tag{2.2.1.1}$$

Regularizing this expression according to the prescription (2.1.2.47) gives $a = \frac{15}{16}$. This implies that the groundstate is tachyonic and also that there is no massless state coming from this twisted ground state. There is one more tachyonic state at the first level in the twisted sector. This is obtained by acting with $\alpha^{25}_{-\frac{1}{2}}\tilde{\alpha}^{25}_{-\frac{1}{2}}$. Collecting the results, we obtain one tachyon with $M^2 = -\frac{15}{2}$ and one with $M^2 = -\frac{7}{2}$ at each fixed plane. Altogether, there are four tachyons (and states with positive mass squared) located at the fixed planes.

The singular nature of the fixed points does not raise any problem in string theory. It introduces twisted sector states which result in additional particles which are located at the orbifold-24-planes in target space.

It is interesting to note that the orbifold at the selfdual radius is equivalent to the toroidally compactified bosonic string at twice the selfdual radius.[27] For a detailed disussion of this equivalence we refer to Polchinski's book[371]. Here, we will only check that the light (tachyonic and massless) spectra coincide. Obviously, the gauge groups $U(1) \times U(1)$ are the same. For the bosonic string on the circle (with $R^2 = 2$) these are the off-diagonal metric and B-field components G_{i25}, B_{i25}, whereas for the orbicircle compactification at $R^2 = \frac{1}{2}$ these come from states with non-trivial winding and momenta as discussed above. It remains to identify the four additional massless scalars and the two tachyons (found in the non-trivial winding-momenta sector) of the orbicircle compactification (at selfdual radius) and the four additional tachyons from the twisted sector in the circle compactification. Here, the special choice $R^2 = 2$ for the circle compactification comes into the game. With (2.1.5.19) evaluated at $R^2 = 2$ we find exactly these missing states. At first, there are four massless scalars: the vacua with $m = 0$ and $n = \pm 4$, or $m = \pm 1$ and $n = 0$. The two tachyons with $M^2 = -6$ are obtained from the two vacua with $m = 0$ and $n = \pm 2$. The two twisted sector tachyons with $M^2 = -\frac{15}{2}$ correspond to the two vacua with $m = 0$ and $n = \pm 1$. The other two twisted sector tachyons with $M^2 = -\frac{7}{2}$ can be identified in the circle compactification as the vacua with $m = 0$ and $n = \pm 3$.

The equivalence of the $S^1/\mathbb{Z}_2$ compactification at the selfdual radius and the S^1 compactification at twice the selfdual radius shows that the moduli spaces of both

[27] Alternatively, we could use the T-dual version at half the selfdual radius.

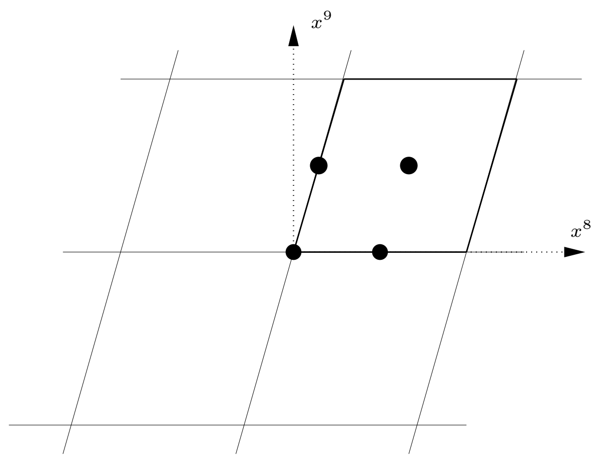

Figure 2.5: The orbifold $T^2/\mathbb{Z}_2$

compactifications are connected at this point. This feature has a stringy origin. From the target space perspective this is quite surprising. A field theory on $24+1$ dimensional Minkowski space times an interval with certain fields constrained to live at the endpoints of the interval is smoothly connected to a field theory on $24+1$ dimensional Minkowski space times a circle with all fields living in the whole space. However, due to the tachyons both vacua are unstable. In the next section we will see that similar things happen for the superstring which does not have tachyons in its spectrum.

2.2.2 Type IIB on $T^4/\mathbb{Z}_2$

Again, we start by describing the target space geometry. We compactify the six, seven, eight and nine direction on a four dimensional torus. We view this four dimensional torus as the product of two two-dimensional tori. The coordinates are labeled such that the sixth and seventh direction form one T^2 and the eighth and ninth a second T^2. Let us focus on this second T^2 with the understanding that the same applies to the first T^2. In figure 2.5 this is depicted by drawing a lattice in the eight-nine plane. The fundamental cell is the parallelogram with edges drawn with stronger lines. The lattice vectors are the lower and the left edge of the fundamental cell. A two dimensional torus is obtained by gluing together the opposite edges of the fundamental cell. Shifts by lattice vectors connect identified points.

"Dividing" the T^4 by $\mathbb{Z}_2$ means that in addition we identify points via the prescription

$$\left\{x^6, x^7, x^8, x^9\right\} \to \left\{-x^6, -x^7, -x^8, -x^9\right\}. \tag{2.2.2.1}$$

This $\mathbb{Z}_2$ action leaves the four points indicated by black dots in figure 2.5 times the

Sector	State	$\mathbb{Z}_2$	5 + 1 dim. rep.
NS:	$\psi^i_{-\frac{1}{2}}\left\|0\right\rangle$	$+$	1 vector
	$i = 2, \ldots, 5$		
	$\psi^{6,7,8,9}_{-\frac{1}{2}}\left\|0\right\rangle$	$-$	4 scalars
R:	$\left\|s_1 s_2 s_3 s_4\right\rangle$	$e^{\frac{i\pi}{2}(s_3+s_4)}$	
	$s_1 = s_2,\ s_3 = -s_4$	$+$	2 (anti-chiral) spinors
	$s_1 = -s_2\ ,\ s_3 = s_4$	$-$	2 (chiral) spinors

Table 2.5: Untwisted right moving states

four points in the first torus invariant. Thus, we obtain sixteen orbifold five-planes. At first, we construct the untwisted spectrum. Since in the type IIB case the left moving sector is identical to the right moving one, we first write down the right moving states, only. The result is collected in table 2.5. We choose the GSO projection such that —in the notation of (2.1.2.56)— states with an odd number of minus signs survive. The projection (2.2.2.1) can be viewed as rotations by 180° in the eight-nine plane and simultaneously in the six-seven plane. This is useful for the identification of the behavior of the R-sector under $\mathbb{Z}_2$ transformations. We consider only states which lead to massless particles when combined with the left movers.

In the NSNS sector, we can combine the left moving vector with the right moving one leading to the six dimensional graviton G_{ij}, the antisymmetric tensor B_{ij} and the dilaton Φ. Further, we can combine scalars from the left moving sector with scalars from the right moving one. This gives sixteen massless scalars corresponding to G_{ab} and B_{ab} where the indices a, b are internal, i.e. $a, b = 6, \ldots, 9$. The target space vectors G_{ia} and B_{ia} are projected out. In the RR sector, we can combine the chiral spinor with the chiral one, and the anti-chiral with the anti-chiral one. This leads to 32 massless (on-shell) degrees of freedom in the RR sector. Tensoring a chiral spinor with a chiral spinor gives a selfdual two-form potential (3 on shell components) and a scalar. The tensor product of two antichiral spinors yields an anti-selfdual two-form potential and a scalar. We can perform four of those combinations, each. With the notation of table 2.1 the RR states can be identified as follows:

- C^*_{ijkl} (or C^*_{abcd}) gives $\begin{pmatrix} 4 \\ 4 \end{pmatrix} = 1$ degree of freedom (one scalar),
- C^*_{abij} gives 3 anti-selfdual two-forms and 3 selfdual two-forms (18 degrees of freedom),
- B'_{ij} gives a two-form (6 degrees of freedom),
- B'_{ab} gives six scalars,

- Φ' gives a scalar .

All other fields from the RR sector are projected out. Fermionic degrees of freedom are obtained from the NSR and RNS sector. Combing the vector with the anti-chiral spinors gives four times[28]a $(\mathbf{2},\mathbf{2}) \otimes (\mathbf{2},\mathbf{1}) = (\mathbf{3},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2})$ representation of the six dimensional little group $SO(4) = SU(2) \times SU(2)$.[29] Therefore, this tensor product provides us with four chiral gravitini and four chiral fermions.

Combining the NS sector scalars with the chiral R sector spinors gives 16 chiral spinors. From the existence of the four chiral gravitini we can guess that the resulting low energy effective field theory has $N = 4$ chiral supersymmetry in six dimensions. (For a collection of supersymmetries in various dimension see[399].)

Before checking that also the rest of the massless states fit into supersymmetric multiplets we should construct the twisted sector states. The construction does not depend on the location of the fixed plane. Therefore, we restrict the construction to one plane and multiply the result by 16. In the twisted sector, the NS fermions with an index corresponding to a compact dimension are integer modded whereas the R sector fermions are half integer modded. Now, there are NS sector zero modes forming a four dimensional Clifford algebra. The twisted NS ground state is two-fold degenerate after imposing the GSO projection. (We modify the notation of (2.1.2.56) in a straightforward way. Since we have only two creators and two anihilators, the twisted NS groundstate has two entries. Performing the GSO projection means that we keep those states with an odd number of minus signs.) In the twisted R sector we do not have zero modes in the compact direction. This lifts some of the vacuum degeneracy as compared to the untwisted sector. The twisted R ground state is labeled only by the first two entries. Again, we keep only states with an odd number of minus signs. In order to deduce the masses of the states in the twisted NS sector, we observe from (2.1.2.46) (and its regularisation) that replacing four integer moded bosons by half integer moded ones changes the normal ordering constant by $-\frac{4}{24} - \frac{4}{16} = -\frac{5}{12}$. Changing the modding of four worldsheet fermions from half-integer to integer gives another shift of $-\frac{4}{16} + \frac{4}{24} = -\frac{1}{12}$. Thus, we arrive at

$$a_{NS}^{twisted} = a_{NS}^{untwisted} - \frac{1}{2} = 0. \tag{2.2.2.2}$$

The twisted NS sector groundstate is massless. The R sector groundstate is always massless, since fermions have the same modding as bosons. The analogon of table 2.5 for the twisted sector is table 2.6. Since all the twisted sector groundstates are

[28] One factor of two arises because the NSR and RNS sector yield such a tensor product, each. The second factor of two is due to the two anti-chiral spinors in table 2.5.

[29] A vector is in the $(\mathbf{2},\mathbf{2})$, an anti-chiral spinor in the $(\mathbf{2},\mathbf{1})$, a chiral spinor in the $(\mathbf{1},\mathbf{2})$, a selfdual twoform in the $(\mathbf{3},\mathbf{1})$ and an anti-selfdual twoform in the $(\mathbf{1},\mathbf{3})$.

Sector	State	$\mathbb{Z}_2$	$5+1$ dim. rep.
NS:	$\vert s_3 s_4\rangle$	$e^{\frac{i\pi}{2}(s_3+s_4)}$	
	$s_3 = -s_4$	$+$	2 scalars
R:	$\vert s_1 s_2\rangle$		
	$s_1 = -s_2$,	$+$	1 (chiral) spinors

Table 2.6: Twisted right moving states

invariant under the $\mathbb{Z}_2$ we can form all possible left-right tensor products. Multiplying with 16 (the number of fixed planes) we obtain 64 scalars from the NSNS sector. The RR sector leads to 16 anti-selfdual two-forms and 16 scalars. The RNS and NSR sector give rise to 64 chiral spinors.

After we have obtained the full massless spectrum of the type IIB string on $T^4/\mathbb{Z}_2$ we can fit it into super-multiplets of $N = 4$ chiral supergravity in six dimensions. The possible supermultiplets are the gravity multiplet and the tensor multiplet. The gravity multiplet contains the graviton and four chiral gravitini from the untwisted sector. In addition, five selfdual two-forms are in the gravity multiplet. A tensor multiplet is made out of an anti-selfdual two-form, five scalars and four chiral fermions. The five selfdual two-forms in the gravity multiplet we take from B_{ij}, B'_{ij} and C^*_{ijab}. After filling the gravity multiplet, we are left with 21 anti-selfdual two-forms, 105 scalars and 84 chiral fermions. Thus, the remaining degrees of freedom fit into 21 tensor multiplets.

To summarize, the massless spectrum of the type IIB string on $T^4/\mathbb{Z}_2$ consists of one gravity multiplet and 21 tensor multiplets of $D = 6$ chiral $N = 4$ supersymmetry. Some of the degrees of freedom are confined to live on the orbifold-5-planes which fill the $5+1$ dimensional non-compact space but are located in the four dimensional compact space. In the next section we will argue, that this setup is smoothly connected to compactifications without orbifold-5-planes.

2.2.3 Comparison with type IIB on $K3$

In the previous section we compactified the type IIB string on $T^4/\mathbb{Z}_2$. Among others, we obtained four chiral gravitini. If we compactified on T^4 instead, the two ten dimensional gravitini would give rise to four non-chiral gravitini in six dimensions. Thus, our orbifolding removes half of the supersymmetries. This is due to the fact, that the $T^4/\mathbb{Z}_2$ manifold belongs to a larger class of manifolds which are called Calabi-Yau n-folds. Here, n denotes the number of complex dimensions, i.e. $n = 2$ in our case. The Calabi-Yau twofolds are all connected by smooth deformations and commonly denoted by $K3$. One important feature of Calabi-Yau n-folds is that they possess

$SU(n)$ holonomy. This means that (for K3) going around closed (non-contractable) curves induces an $SU(2)$ transformation. In a toroidal compactification we split the ten dimensional spinor into a couple of lower dimensional spinors. The possible values of the internal spinor indices count the number of resulting lower dimensional spinors. In a torus compactification, each value of the internal indices gives rise to a massless spinor. This is because the internal homogenous Dirac equation has always a solution – any constant spinor. If instead of a torus with trivial holonomy we compactify on $K3$ with $SU(2)$ holonomy, only spinors which do not transform under the holonomy group give rise to massless six dimensional spinors. This removes half of the internal components and thus breaks half of the supersymmetry. Indeed, all $K3$ compactifications yield the same massless spectra. This is a consequence of the fact that the number of zero-modes (of Laplace and Dirac operators) does not change as we move from one $K3$ to another one. The number of zero modes of the Laplace operators[30] are usualy listed in Hodge diamonds. The Hodge diamond for $K3$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{2.2.3.1}$$

In the following we explain (roughly) how to read (2.2.3.1). The $K3$ is a complex manifold. Therefore, we can choose complex coordinates (and so we do). Then a tensor can have a couple of holomorphic indices and a couple of anti-holomorphic indices. In other words, there are (p, q) forms on $K3$, where p corresponds to the number of holomorphic indices and q to the number of anti-holomorphic ones. Since the complex dimension of $K3$ is two, the values of p, q can be zero, one or two. We denote the number of zero modes of a (p, q) form with $h^{(p,q)}$. These (Hodge) numbers are arranged into a Hodge diamond as follows[31]

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} . \tag{2.2.3.2}$$

From (2.2.3.1) we can deduce that an object which represents a zero or a four form in the internal space, has one zero mode. Such an object gives rise to one massless

[30] There are several different Laplace operators whose form depends on the tensor structure of the object they act on.

[31] The symmetry of (2.2.3.1) is not accidental. The vertical symmetry is related to Hodge duality and the horizontal one to interchanging holomorphic with anti-holomorphic coordinates.

six-dimensional field. A $p+q=2$ form possesses 22 zero modes, thus leading to 22 massless fields in six dimensions. In order to write down the massless spectrum of the $K3$ compactified type IIB string we need to know another feature of the family of $K3$ manifolds. All Calabi-Yau manifolds (and in particular $K3$) are Ricci flat. This means that the Ricci tensor vanishes and hence we do not need any non-trivial background configuration in order to satisfy the conformal invariance conditions derived in section 2.1.3. This remains true under certain deformations of the metric of $K3$. The space of such non-trivial (not related to coordinate changes) metric deformations is 58 dimensional for the family of $K3$s.

Now, we are ready to derive the bosonic massless spectrum of the $K3$ compactified type IIB string. At first, we collect all zero forms of $K3$. From the NSNS sector these are G_{ij},B_{ij},Φ, and from the RR sector B'_{ij}, Φ', C^*_{ijkl}. Since $h^{(0,0)}=1$ these appear once in the lower dimensional spectrum. The G_{ab} are not differential forms on $K3$ but metric deformations. They result in 58 massless scalars. Since $h^{(p,q)}=0$ for $p+q$ odd, the Kaluza-Klein fields G_{ia} and B_{ia} do not give rise to massless six dimensional fields.

It remains to count the two-forms on $K3$. (The four form C^*_{abcd} we have already counted as C^*_{ijkl} because of selfduality.) The two-forms are B_{ab}, B'_{ab} and C^*_{ijab}. B_{ab} and B'_{ab} lead to 44 scalars in six dimensions. The 22 zero-modes of C^*_{ijab} can be decomposed into three selfdual and 19 anti-selfdual twoforms in six dimensions[38]. Taking into account that the $SU(2)$ holonomy breaks half of the supersymmetry (as compared to T^4 compactifications) and that the fermionic zero modes are all of the same chirality, we obtain the same massless spectrum as in the $T^4/\mathbb{Z}_2$ case.

Indeed, $T^4/\mathbb{Z}_2$ corresponds to a limit in the space of $K3$ manifolds where the $K3$ degenerates. As long as one considers $K3$s very close to that point one obtains the same massless spectrum. In string theory even the limit to the point where the $K3$ degenerates is well defined.

Let us see what happens when we repeat the $K3$ analysis for the orbifold $T^4/\mathbb{Z}_2$. We will focus only on the bosonic spectrum. First, we need to know the Hodge diamond for $T^4/\mathbb{Z}_2$. This can be easily "computed" without much knowledge of algebraic geometry. On T^4 we obtain the Hodge numbers just by counting independent components of the corresponding differential forms,

$$h^{p,q}=\left(\begin{array}{c}2\\ p\end{array}\right)\left(\begin{array}{c}2\\ q\end{array}\right). \tag{2.2.3.3}$$

The $\mathbb{Z}_2$ action is taken into account by removing forms which are odd under the $\mathbb{Z}_2$.

Thus, the Hodge diamond for $T^4/\mathbb{Z}_2$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 4 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} . \tag{2.2.3.4}$$

In six dimensions we obtain two twoforms B_{ij} and B'_{ij} and three scalars Φ, Φ', C^*_{ijkl}. The internal metric components G_{ab} yield ten scalars. (Note, that constant rescalings of the coordinates would change the range of those coordinates, and hence are non-trivial deformations equivalent to a constant change of the corresponding metric components.) From the metric deformations we obtain 48 less scalars than in the $K3$ compactification. It remains to take into account the twoforms on $T^4/\mathbb{Z}_2$: B_{ab}, B'_{ab} and C^*_{ijab}. We obtain 12 massless scalars from B_{ab} and B'_{ab}, together. On the $K3$ there were 32 more massless scalars coming from this sector. The C^*_{ijab} combine into three selfdual and three anti-selfdual twoforms. In the $K3$ compactification we obtained 16 more anti-selfdual two-forms.

The $T^4/\mathbb{Z}_2$ spectrum we computed here, would correspond to the one which we had obtained in a field theory compactification. It has 80 massless scalars and 16 anti-selfdual twoforms less than the $K3$ compactified theory. In field theory, the spectrum jumps when we take the singular orbifold limit in the family of $K3$s. From the above construction it is obvious that we counted only untwisted states from a string perspective. Indeed, the missing 80 scalars and 16 anti-selfdual twoforms are exactly what we obtained from the twisted sector in the previous section. In string theory the spectrum of the compactified theory does not feel the singular nature of the orbifold limit. All that happens is that some part of the spectrum is localized to the orbifold fixed planes. This localization appears in internal space and is not visible at experiments which cannot resolve the distances of the size of the compact manifold. The energy scale of such experiments depends on the type of interactions fields propagating into the compact directions carry. For purely gravitational interactions it needs to be much higher than e.g. for electro-magnetic interactions. We will come back to this later.

To summarize we recall that string theory can be compactified on singular manifolds. The moduli spaces of such compactifications can be connected to compactifications on smooth manifolds. There are massless states which are localized at singularities of the compact manifold. These are the twisted sector states. They are of truly stringy origin.

2.3 D-branes

In this section we will present another kind of extended objects resulting from string theory — the D-branes. They are different from the previously studied orbifold planes. D-branes can exist also in uncompactified theories. They are dynamical objects, i.e. they interact with each other and can move independent of the size of some compact space. (The orbifold planes could move only if we changed the size or shape of the compact manifold.) When we discussed the fundamental string we did not consider the possibility of open strings. We will catch up on that in the following.

2.3.1 Open strings

2.3.1.1 Boundary conditions

We recall the action of the superstring

$$S = \frac{1}{2\pi\alpha'}\int d\sigma^+ d\sigma^- \left(\partial_- X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_- \psi_{+\mu} + \frac{i}{2}\psi_-^\mu \partial_+ \psi_{-\mu}\right). \qquad (2.3.1.1)$$

Now, we view the values $\sigma = 0, \pi$ as true boundaries of the string worldsheet. Varying (2.3.1.1) with respect to X^μ gives apart from the equations of motion (which are identical to closed strings) boundary terms which should vanish separately,

$$\delta X^\mu \partial_\sigma X_\mu|_{\sigma=0}^{\pi} = 0. \qquad (2.3.1.2)$$

For the closed string we have solved this equation by relating the values at $\sigma = 0$ with the ones at $\sigma = \pi$. This procedure was local because we took the string to join to a closed string at $\sigma = \pi$. Now, we proceed differently by not correlating the two ends of the string, i.e.

$$\delta X^\mu \partial_\sigma X_\mu\big|_{\sigma=0} = \delta X^\mu \partial_\sigma X_\mu\big|_{\sigma=\pi} = 0. \qquad (2.3.1.3)$$

Let us focus on the boundary at $\sigma = 0$. We have two choices to satisfy the boundary condition. If —for $i = 0, \ldots, p$— we allow for free varying ends (δX^i arbitrary at the boundary) we obtain Neumann boundary conditions at those ends[32]

$$\partial_\sigma X^i = 0 \quad , \quad i = 0, \ldots, p. \qquad (2.3.1.4)$$

For the remaining $d - p - 1$ coordinates X^a we choose to freeze the end of the string — the variation vanishes at the boundary. Hence, in those directions the end of the string is confined to some constant position. The resulting boundary conditions are Dirichlet conditions (c^a is a constant vector),

$$X^a = c^a \quad , \quad a = p + 1, \ldots, d. \qquad (2.3.1.5)$$

[32] Here, we use i to label Neumann directions. After fixing the light cone gauge we take $i = 2, \ldots, p$.

The end of the open string defines a surface which extends along $p+1$ dimensions and is located in $d-p-1$ dimensions. This object is called D-brane, where the "D" refers to the Dirichlet boundary condition specifying its position. If we choose identical boundary conditions for the other end of the open string (at $\sigma = \pi$), we describe an open string starting and ending on the same D-brane. For different boundary conditions the open string stretches between two different D-branes. The Neumann conditions imply that no momentum can flow out of the ends of the open string. In the Dirichlet directions momentum can leave the string through its end — it is absorbed by the D-brane. The target space Lorentz group is broken to $SO(p,1)$.

Varying the action with respect to the worldsheet fermions results in the same equations of motions as in the closed string case and in the boundary conditions

$$\left(-\psi_{+\mu}\delta\psi_+^\mu + \psi_{-\mu}\delta\psi_-^\mu\right)\Big|_{\sigma=0}^{\pi} = 0. \tag{2.3.1.6}$$

In the closed string case we have solved this by assigning either periodic or anti-periodic boundary conditions to the worldsheet fermions. Since now the ends of the string are separated in the target space this would imply some non-locality. Therefore, we impose the boundary conditions (2.3.1.6) at each end separately

$$\left(-\psi_{+\mu}\delta\psi_+^\mu + \psi_{-\mu}\delta\psi_-^\mu\right)\Big|_{\sigma=0} = \left(-\psi_{+\mu}\delta\psi_+^\mu + \psi_{-\mu}\delta\psi_-^\mu\right)\Big|_{\sigma=\pi} = 0. \tag{2.3.1.7}$$

Let us focus again on the boundary at $\sigma = 0$. We can solve (2.3.1.7) by one of the options

$$\psi_+^\mu = \pm\psi_-^\mu \quad \text{at} \quad \sigma = 0. \tag{2.3.1.8}$$

However, there is a correlation with the bosonic boundary conditions via worldsheet supersymmetry. To be specific, we choose the plus sign for Neumann conditions

$$\psi_+^i = \psi_-^i \quad \text{at} \quad \sigma = 0. \tag{2.3.1.9}$$

The supersymmetry transformations (in particular (2.1.1.22) and (2.1.1.23)) should not change this boundary condition. Since $\partial_\tau X^i$ is not specified by the boundary conditions this yields

$$\epsilon_+ = -\epsilon_- \quad \text{at} \quad \sigma = 0, \tag{2.3.1.10}$$

which implies that due to the boundary (at least) half of the worldsheet supersymmetry is broken. (If we had started with $(1,0)$ worldsheet supersymmetry —as we did for the heterotic string— the boundary would break worldsheet supersymmetry completely.) In order to ensure that not all of the supersymmetry is broken we have to choose the opposite (compared to (2.3.1.9)) boundary conditions for worldsheet fermions in Dirichlet directions

$$\psi_+^a = -\psi_-^a \quad \text{at} \quad \sigma = 0. \tag{2.3.1.11}$$

We could also interchange the fermionic boundary conditions in Dirichlet and Neumann directions. Then another worldsheet supersymmetry would survive. There is no physical difference between the two choices. Nevertheless it is important, that we take the boundary conditions in the Neumann directions to be "opposite" to the ones in Dirichlet directions. One may also check that the open string action is invariant under the worldsheet supersymmetries (2.1.1.20), (2.1.1.22) and (2.1.1.23) provided that the worldsheet fermions satisfy the boundary conditions (2.3.1.9), (2.3.1.11) and (2.3.1.10) is fulfilled. (Partial integration introduces boundary integrals which vanish if these additional constraints hold.) Recall also that the functional form of the worldsheet supersymmetry parameter is restricted by the chirality conditions (2.1.1.28).

In the following we are going to discuss the boundary conditions at the other end of the open string at $\sigma = \pi$. Going back to the closed string we deduce from (2.1.1.22) that for anti-periodic supersymmetry parameter ϵ_+ the fermions are anti-periodic for periodic bosons and vice versa. This means that anti-periodic ϵ_+ belongs to the NS sector and periodic ones to the R sector. From the discussion of the boundary conditions at $\sigma = 0$ we infer that the supersymmetry parameter has to satisfy one of the following conditions,

$$\epsilon_+ = \pm\epsilon_- \quad \text{at} \quad \sigma = \pi. \tag{2.3.1.12}$$

In order to relate this to something like periodicity or anti-periodicity we perform the so called doubling trick. This means that we define a function ε on the interval $0 \leq \sigma < 2\pi$. This is done in the following way (we indicate only the σ dependence),

$$\varepsilon = \begin{cases} \epsilon_+(\sigma) & , \quad 0 \leq \sigma < \pi \\ \pm\epsilon_-(2\pi - \sigma) & , \quad \pi \leq \sigma < 2\pi \end{cases} . \tag{2.3.1.13}$$

The sign in the second line of (2.3.1.13) is correlated to the sign in (2.3.1.12) by the requirement of continuity at $\sigma = \pi$. Hence, ε is (anti)-periodic for the lower (upper) sign in (2.3.1.12) (taking into account the sign in (2.3.1.10)).

Now, let us perform this doubling trick also on the worldsheet bosons and fermions. For the bosons it is useful to rewrite the boundary conditions. Dirichlet boundary conditions mean that

$$\partial_+ X^a = -\partial_- X^a \quad \text{at} \quad \sigma^+ - \sigma^- = 0. \tag{2.3.1.14}$$

Neumann conditions can be written as

$$\partial_+ X^i = \partial_- X^i \quad \text{at} \quad \sigma^+ - \sigma^- = 0. \tag{2.3.1.15}$$

The next step is to specify the boundary conditions at $\sigma = \pi$. After having done this, one can perform the doubling trick, i.e. define a boson ∂X^μ on the interval $0 \leq \sigma < 2\pi$ analogously to the definition of ε in (2.3.1.13) where $\partial_\pm X$ take the role of $\epsilon_\pm$. As the reader can easily verify, the outcome is that ∂X^μ is periodic whenever we have chosen the same type of boundary conditions at the two ends of the string in the x^μ direction. The corresponding open string sectors are called DD (NN) according to the choice of Dirichlet (Neumann) boundary conditions at the two ends. For an opposite choice of boundary conditions (ND or DN strings) ∂X^μ will turn out to be anti-periodic. In analogy to the closed string we call the sector with periodic ε R sector and the one with anti-periodic ε NS sector. For DD or NN strings this implies that in the NS sector we take the boundary conditions at $\sigma = \pi$ to be

$$\psi^i_+ = -\psi^i_- \quad , \psi^a_+ = \psi^a_- \quad \text{at} \quad \sigma = \pi. \tag{2.3.1.16}$$

Defining a "doubled" worldsheet fermion Ψ^μ in analogy to ε (where the role of $\epsilon_\pm$ is taken over by the $\psi^\mu_\pm$) we find that for DD or NN strings Ψ is anti-periodic. In the R sector we take the boundary conditions

$$\psi^i_+ = \psi^i_- \quad , \psi^a_+ = -\psi^a_- \quad \text{at} \quad \sigma = \pi \tag{2.3.1.17}$$

and obtain periodic boundary conditions. In the above we have used that, for example, in the R-sector periodicity of ε implies that the boundary conditions of $\epsilon_\pm$ at $\sigma = \pi$ are identical to the ones at $\sigma = 0$. Plugging this back into the supersymmetry transformations (2.1.1.22) and (2.1.1.23) evaluated at $\sigma = \pi$ and taking into account the boundary conditions for the bosons, we obtain the boundary conditions of the worldsheet fermions at $\sigma = \pi$. This in turn determines the periodicity of Ψ^μ. Performing the same procedure for ND or DN boundary conditions, one finds that Ψ^μ is periodic in the NS sector and anti-periodic in the R sector whenever x^μ is a direction with ND or DN boundary conditions. The ND or DN directions are somewhat similar to the twisted sectors we met when discussing $\mathbb{Z}_2$ orbifolds.

At first, we will consider only the case of a single D-brane. This means that we can have only DD or NN boundary conditions depending on whether we are looking at a direction transverse or longitudinal to the D-brane. Then ∂X^μ will always be periodic and Ψ^μ (anti-)periodic in the (NS) R sector.

2.3.1.2 Quantization of the open string ending on a single D-brane

The quantization of the open string is very similar to the closed superstring. In the following we will point out the differences. At first, we solve the equations of motion again by taking a superposition of left-moving and right-moving fields. For the bosons, these are given in (2.1.2.3) and (2.1.2.4). The boundary conditions relate this two solutions. (In addition, we need to replace $e^{-2in\sigma^\pm} \to e^{-in\sigma^\pm}$.)[33] In Neumann directions, they impose

$$\alpha_n^i = \tilde{\alpha}_n^i. \tag{2.3.1.18}$$

For Dirichlet directions, we obtain a similar relation and constraints on the zero modes,

$$x^a = c^a \quad , \quad p^a = 0 \quad , \quad \alpha_n^a = -\tilde{\alpha}_n^a. \tag{2.3.1.19}$$

The general solutions for the bosonic directions read

$$X^i = x^i + p^i\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^i e^{-in\tau}\cos n\sigma, \tag{2.3.1.20}$$

$$X^a = c^a - \sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin n\sigma. \tag{2.3.1.21}$$

The mode expansions for the NS sector fermions look as follows

$$\psi_-^i = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^i e^{-ir\sigma^-}, \tag{2.3.1.22}$$

$$\psi_+^i = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^i e^{-ir\sigma^+}, \tag{2.3.1.23}$$

$$\psi_-^a = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^a e^{-ir\sigma^-}, \tag{2.3.1.24}$$

$$\psi_+^a = -\frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^a e^{-r\sigma^+}. \tag{2.3.1.25}$$

For the R sector fermions, one obtains

$$\psi_-^i = \sum_{n\in\mathbb{Z}} d_n^i e^{-in\sigma^-}, \tag{2.3.1.26}$$

$$\psi_+^i = \sum_{n\in\mathbb{Z}} d_n^i e^{-in\sigma^+}, \tag{2.3.1.27}$$

$$\psi_-^a = \sum_{n\in\mathbb{Z}} d_n^a e^{-in\sigma^-}, \tag{2.3.1.28}$$

$$\psi_+^a = -\sum_{n\in\mathbb{Z}} d_n^a e^{-in\sigma^+}. \tag{2.3.1.29}$$

[33] If we had chosen the open string half as long as the closed one we would not need this replacement.

The next step is to eliminate two directions by performing the light cone gauge. We take this to be the time-like (Neumann) direction and a space-like Neumann direction, which we choose to be x^1.[34] For the open string we have only an NS sector and an R sector. Since the right movers are not independent from the left movers, the right and left moving sectors do not decouple anymore. The constraints that the expressions (2.1.1.24) – (2.1.1.27) vanish are not all independent. The zero mode part of vanishing energy momentum tensor again yields the mass shell condition (since the mode expansions differ by factors of two, there is a difference of a factor of four as compared to the closed string (2.1.2.42)),

$$M^2 = 2\left(N - a\right), \tag{2.3.1.30}$$

where a is the normal ordering constant and the number operator N is defined in (2.1.2.43) for the NS sector and in (2.1.2.61) for the R sector. In the NS sector the GSO projection operator is as in (2.1.2.50) with the second factor removed. The lowest GSO invariant states in the NS sector are

$$b^i_{-\frac{1}{2}}\left|k\right\rangle \quad , \quad b^a_{-\frac{1}{2}}\left|k\right\rangle, \tag{2.3.1.31}$$

where we have indicated again the momentum eigenvalue of the vacuum by k. The first set of these states transforms in the vector representation of $SO(p-1)$ – the little group of the unbroken Lorentz group. Hence, this state should be massless leading to the consistency condition[35]

$$a_{NS} = \frac{1}{2}. \tag{2.3.1.32}$$

Like in the closed string, this translates into a condition on the number of target-space dimensions

$$d = 10. \tag{2.3.1.33}$$

The first states in (2.3.1.31) (with label i) form a $U\left(1\right)$ gauge field. The states with label a are scalars transforming in the adjoint of $U\left(1\right)$ (here, this appears just as a pompous way of saying that they are neutral under $U(1)$, however, later we will discuss

[34]We do not consider an open string ending on a $D0$ brane, here. As in the case of the closed string, it is useful to take Lorentz invariance as a guiding principle for a consistent quantization. For a $D0$ brane, the Lorentz group is broken down to time reparameterizations which is too small for our purposes. Later, we will see that we can obtain the $D0$ brane by T-dualizing a higher dimensional D-brane.

[35]In principle, we could combine the first states in (2.3.1.31) with one of the second states in order to form a massive vector representation as long as $p < d - 1$. However, later we will see that the case $p = d - 1$ is related by T-duality to the other cases. With this additional ingredient it follows that the states in (2.3.1.31) must be massless.

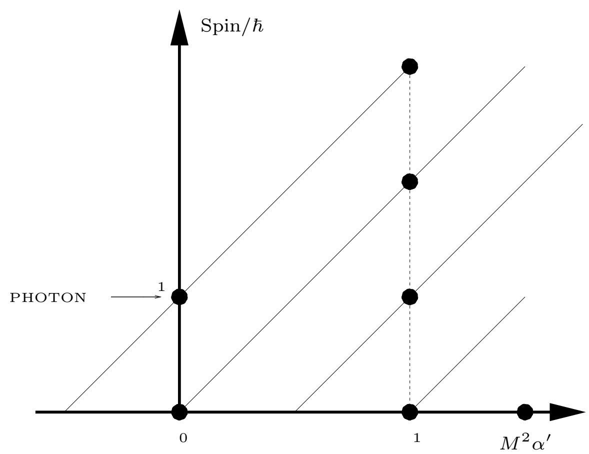

Figure 2.6: NS mass spectrum of the open superstring

non-abelian gauge groups where those fields are adjoints rather than singlets). Since the center of mass position of the open string is confined to be within the world volume of the D-brane, all the open string states correspond to target-space particles which are confined to live on the D-brane. The NS mass-spectrum is depicted in figure 2.6.

The construction of the R sector vacuum state goes along the same lines as in the closed string. The ten dimensional Majorana spinor decomposes into a couple of spinors with respect to the unbroken Lorentz group $SO(p, 1)$. We impose the GSO projection by multiplying the states with one of the projection operators defined in (2.1.2.59). The sign is a matter of convention. The R-vacuum is massless on its own. It leads to target space spinors providing all fermionic degrees of freedom which are needed to obtain the maximal rigid supersymmetry in $p + 1$ dimensions.[36]

In the following sections we will investigate systems with more than one D-brane. This will lead to non-abelian field theories on a stack of D-branes. But before doing so, we will briefly discuss the possible D-brane setups which are in agreement with supersymmetry.

2.3.1.3 Number of ND directions and GSO projection

At first, consider the case that we have an open string with an odd number of ND directions. Thus, we will have an odd number of directions where the worldsheet

[36]The maximally possible amount of supersymmetry differs for rigid and local supersymmetry. In rigid supersymmetry, the highest occurring spin should not exceed one, whereas in locally supersymmetric field theories, spin two fields (the gravitons) are allowed. In $3 + 1$ dimensions, the maximal supersymmetry is $N = 4$ ($N = 8$) for rigid (local) supersymmetry. From this one can deduce the maximally allowed amount of supersymmetry in higher dimensions by viewing the $3 + 1$ dimensional theory as a toroidally compactified higher dimensional theory.

fermions have zero-modes. For example in the R sector, the zero-modes form a Clifford algebra in $p+1 = odd$ dimensions. The representation of this algebra by the R ground state will be irreducible (there is no notion of chirality in odd dimensions). Therefore, we cannot perform the GSO projection on those states. The theory will not lead to target-space supersymmetry.

Let us now discuss the case of an even number of ND directions, taken to be $8-2n$. Then the GSO projection operator on the R sector ground state will be of the form

$$P_{GSO} = 1 \pm 2^n \, d_0^2 \ldots d_0^{2n+1}. \tag{2.3.1.34}$$

Using some algebra this can be written as

$$P_{GSO} = 1 \pm e^{i\pi(J_{23}+\ldots+J_{2n,2n+1})}, \tag{2.3.1.35}$$

where the J_{kl} are the generators of rotations in the kl plane

$$J_{kl} = -\frac{i}{2}\left[d_0^k, d_0^l\right]. \tag{2.3.1.36}$$

The eigenvalue of the ND Ramond groundstate under a 180° rotation in one plane is $\pm i$. Thus, the eigenvalues of the R groundstate $|R\rangle$ under P_{GSO} will be

$$P_{GSO}\,|R\rangle = (1 \pm i^n)\,|R\rangle\,. \tag{2.3.1.37}$$

From this we deduce that the GSO projection is possible only if the number of ND directions is an integer multiple of four. This means for example that, if a lower dimensional D-brane lives inside the worldvolume of a higher dimensional D-brane, the higher dimensional D-brane has to extend in four or eight more directions. We could have deduced this result faster by noting that (2.3.1.34) defines a projection operator only if n is a multiple of four since otherwise the second term in (2.3.1.34) squares to -1.

2.3.1.4 Multiple parallel D-branes – Chan Paton factors

In this section we will discuss sets of parallel Dp-branes.[37] First, let us have a look at two parallel Dp-branes which are separated by a vector δc^a in the transverse space. (Later we will see that the distance between parallel D-branes is a modulus, i.e. any value is consistent.) From strings ending with both ends on the same D-brane we obtain the same spectrum as discussed in the section 2.3.1.2. In particular, we obtain a $U(1) \times U(1)$ gauge symmetry where the corresponding gauge fields live on the first brane for the first $U(1)$ and on the second brane for the second $U(1)$ factor.

[37] Recall that the worldvolume of a Dp-brane has p space like and one time like dimension.

In addition, we have strings stretching between the two branes. There are two such strings with opposite orientations. As compared to section 2.3.1.2, only the mode expansion for the bosons in Dirichlet directions is modified. The string starting on the brane at c^a and ending on the brane at $c^a + \delta c^a$ has the mode expansion

$$X^a = c^a + \frac{\delta c^a}{\pi}\sigma - \sum_{n\neq 0} \frac{1}{n}\alpha_n^a e^{-in\tau} \sin n\sigma. \qquad (2.3.1.38)$$

The string with the opposite orientation is obtained by replacing $\sigma \to \pi - \sigma$. We rewrite the term with δc^a in a suggestive way

$$\delta c^a \sigma = \frac{1}{2}\delta c^a \left(\sigma^+ - \sigma^-\right) \qquad (2.3.1.39)$$

and compare with the expressions (2.1.5.9) and (2.1.5.10). The finite distance between the D-branes enters the mode expansion in a very similar way as the winding number in the toroidally compactified closed string does. This is also intuitively expected – the winding closed string is stretched around a compact dimension. As a finite winding number also the finite distance contributes to the mass of the stretched string state, it results in a shift of

$$\delta M^2 = \frac{\left(\delta c^a\right)^2}{\pi^2}. \qquad (2.3.1.40)$$

The strings stretching between the branes transform under $U(1) \times U(1)$ with the charges $(1, -1)$ and $(-1, 1)$ depending on the orientation. We will see below that these charge assignments are necessary for consistency. Pictorially, they are obtained by the rule that a string starting at a brane has charge $+1$ with respect to the $U(1)$ living on that brane whereas it has charge -1 if it ends on the brane. (The photon which starts and ends on the same brane has net charge zero.) The $U(1) \times U(1)$ can be also rearranged into a diagonal and a second $U(1)$ such that all states are neutral under the diagonal $U(1)$.

The lightest GSO-even states in the NS sector of the stretched string are

$$\psi^i_{-\frac{1}{2}} \left|12\right\rangle \quad , \quad \psi^i_{-\frac{1}{2}} \left|21\right\rangle , \qquad (2.3.1.41)$$

$$\psi^a_{-\frac{1}{2}} \left|12\right\rangle \quad , \quad \psi^a_{-\frac{1}{2}} \left|21\right\rangle \qquad (2.3.1.42)$$

where $\left|12\right\rangle$ and $\left|21\right\rangle$ denote the NS vacua for strings stretched between the two D-branes and we have dropped the zero mode momentum eigenvalues k in the notation. In this sector, the lightest states form a vector and $d-(p+2)$ scalars. (Note that, in the light cone gauge, we have to combine one of the transverse excitations (2.3.1.42) with the longitudinals (2.3.1.41) in order to obtain a massive vector.) The R sector states provide the fermions needed to fill up supermultiplets. (The amount of supersymmetry is the same as in the single brane case.)

Now, take the inter-brane distance to zero. We obtain two massless vectors and $2\,(d-p-1)$ massless scalars. Together with the massless fields coming from strings ending on identical branes, the vectors combine into a $U(2)$ gauge field, and the scalars combine into $d-p-1$ scalars transforming in the adjoint of $U(2)$. (One can split $U(2)$ into a diagonal $U(1)$ times an $SU(2)$. All fields are $SU(2)$ adjoints and neutral under $U(1)$.) Moving the D-branes apart from each other can be viewed as a Higgs mechanism from the target space perspective. The amount of supersymmetry leads to flat directions for the scalars in the adjoint of $U(2)$. This means that a scalar can have some non zero vev breaking $U(2)$ to $U(1)\times U(1)$. With the given amount of supersymmetry, all massless fields transform in the same representation of the gauge group as the vector bosons (*viz.* in the adjoint). Therefore, the Higgs mechanism can only work for non-abelian gauge groups. Our charge assignments of the open strings stretched between two different D-branes thus lead to a consistent picture.

An economic way of studying systems with N parallel D-branes is to replace all the different sectors corresponding to the possibilities of strings stretching among the N D-branes by one matrix valued state

$$|\cdot\rangle \to |\cdot, ij\rangle\, \lambda_{ji} \tag{2.3.1.43}$$

where λ is an $N\times N$ matrix. The component λ_{ji} corresponds to a string stretching between the ith and the jth brane. The matrix λ is called Chan-Paton factor. Consider again the case where all the N D-branes are separated in the transverse space. For the lightest NS sector states the diagonal elements λ_{ii} are N $U(1)$ gauge fields and $d-p-2$ scalars. They are neutral under $U(1)^N$, i.e.

$$\lambda_{ii} = \lambda_{ii}^\dagger . \tag{2.3.1.44}$$

The off-diagonal elements correspond to massive vectors and scalars. Open string sectors with opposite orientation have opposite charges under $U(1)^N$, i.e.

$$\lambda_{ij} = \lambda_{ji}^\dagger . \tag{2.3.1.45}$$

The Chan-Paton factor is a unitary $N\times N$ matrix. Maximal gauge symmetry is obtained when all N D-branes sit at the same point in the transverse space. The diagonal and off-diagonal elements of λ combine and give rise to a $U(N)$ vector multiplet.

2.3.2 D-brane interactions

Already at an intuitive level, one can deduce that D-branes interact. This comes about as follows. The two ends of an open string ending on the same D-brane can join to form a closed string. The closed string is no longer bound to live on the D-brane,

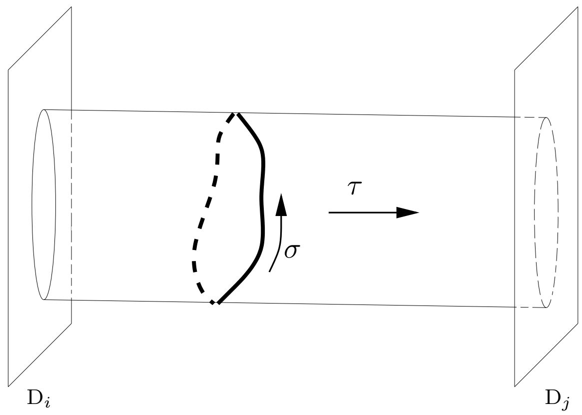

Figure 2.7: D-brane D_i and D-brane D_j talking to each other by exchanging a closed string.

it can escape into the bulk of the target space. In particular, it may reach another D-brane by which it is absorbed. The absorption process is inverse to the emission process. The closed string hits the D-brane where it can split into an open string which is constrained to live on the second D-brane. D-branes talk to each other by exchanging closed strings. In figure 2.7 we have drawn such a process. In order to make contact to conventions of the standard reviews on D-brane physics we take the closed string twice as long as the open string (see also the footnote 33). This implies that in the closed string mode expansions we replace $e^{-2in\sigma^\pm} \rightarrow e^{-in\sigma^\pm}$.

We will compute the process depicted in figure 2.7 in Euclidean worldsheet signature. The range of the worldsheet coordinates is

$$0 \leq \sigma < 2\pi \quad , \quad 0 \leq \tau < 2\pi l. \tag{2.3.2.1}$$

The Euclidean worldtime τ is taken to be compactified on a a circle of radius $2l$. The worldtime taken by a string to get from one brane to the other one is $2\pi l$ – this process can be periodically continued such that one period lasts $4\pi l$. (The factor of 2π is a matter of convention. It is introduced because compact directions are usually specified by the radius of the compactification circle rather than its circumference.) Note also, that l has nothing to do with the distance of the D-branes. The distance in target space will appear later and will be denoted by y.

Since we have defined the D-branes in terms of open strings it will be useful to compute also the D-brane interactions in terms of open strings. To this end, we perform a so called worldsheet duality transformation, i.e. we interchange σ with τ. The resulting picture is an open string one-loop vacuum amplitude. It is described by the annulus diagram drawn in figure 2.8.

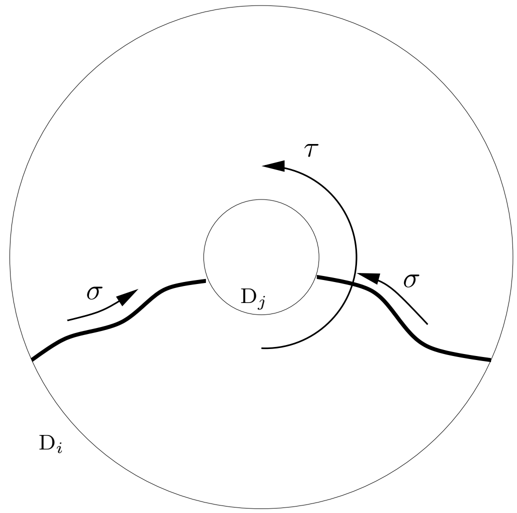

Figure 2.8: D-brane D_i and D-brane D_j talking to each other by a pair of virtual open strings stretching between them.

The parameter ranges for the open string are

$$0 \leq \tau < 2\pi \quad , \quad 0 \leq \sigma < 2\pi l. \tag{2.3.2.2}$$

The periodicity of closed string worldsheet fermions is related to the behavior of open strings under shifts in τ by 2π. The diagram 2.8 corresponds to a vacuum amplitude and thus to a trace in the open string sector. This trace is actually a supertrace with respect to worldsheet (and target space) supersymmetry. The additional sign in the trace over worldsheet fermions is imposed by specifying the boundary condition

$$\psi^\mu_\pm(\tau + 2\pi, \sigma) = (-)^F \, \psi^\mu_\pm(\tau, \sigma) \, . \tag{2.3.2.3}$$

Thus, a $(-)^F$ insertion (canceling the $(-)^F$ in (2.3.2.3)) corresponds to closed string RR sector exchange whereas no $(-)^F$ insertion yields the closed string NSNS sector exchange. From the open string perspective there is no exchange of NSR or RNS sector closed strings between the D-branes. In the picture 2.7, the D-brane appears as a boundary state of the closed string.[38] This boundary state is a superposition of an NSNS sector state and an RR sector state. There are no NSR or RNS sector contributions. This can be explained by the fact that the D-brane is a target space boson. It is specified by the target space vector c^a in (2.3.1.5) and hence transforms as a vector and not as a spinor under rotations in the space transverse to the brane. We will not present the details of the boundary state formalism here, and recommend

[38] It will turn out that the closed strings which are exchanged in figure 2.7 are type II superstrings.

the review [195] instead. To be slightly more specific let us just present the defining equation for a boundary state in closed string theory (as usual we label the Neumann directions by $i = 0, \ldots, p$ and the Dirichlet directions by $a = p+1, \ldots, 9$)

$$\partial_\tau X^i \left|\text{D-brane}\right\rangle = (X^a - c^a) \left|\text{D-brane}\right\rangle = 0. \tag{2.3.2.4}$$

This relates the right moving and left moving bosonic excitations the boundary state can carry. Applying the worldsheet supersymmetry transformations (2.1.1.22) and (2.1.1.23) on (2.3.2.4) and requiring that there is a combination of the two supersymmetries which annihilates the boundary state tells us that the boundary state should have the same number of right moving and left moving fermionic excitations. Also, when ϵ_+ is taken to be (anti)-periodic then ϵ_- should have the same periodicity. The boundary state cannot be excited by NS fermions in, say, the right moving sector and R fermions in the left moving sector. It has only an NSNS and an RR sector.

Instead of the non-standard range for the open string worldsheet coordinates we would like to have the standard range

$$0 \leq \tau < 2\pi t \quad , \quad 0 \leq \sigma < \pi. \tag{2.3.2.5}$$

In order to achieve this we redefine $\tau \to \tau t$ and $\sigma \to \frac{\sigma}{2l}$. Under this redefinition, the Hamiltonian (which is obtained by integrating the $\tau\tau$ component of the energy momentum tensor over σ) transforms according to

$$H \to 2lt^2 H. \tag{2.3.2.6}$$

Further, we want the time evolution operator when going once around the annulus to transform as ($2\pi t$ should be identified with the worldsheet time it takes the open string to travel around the annulus once)

$$e^{-2\pi H} \to e^{-2\pi t H}. \tag{2.3.2.7}$$

This yields the relation

$$tl = \frac{1}{2}. \tag{2.3.2.8}$$

The annulus vacuum amplitude in figure 2.8 yields the vacuum energy of an open string starting on the D-brane D_i and ending on the D-brane D_j. This can be expressed as

$$\begin{aligned} -\frac{1}{2}\log\det H = -\frac{1}{2}\mathrm{tr}\log H = \frac{1}{2}\lim_{\epsilon\to 0}\mathrm{tr}\frac{dH^{-\epsilon}}{d\epsilon} = \\ \mathrm{tr}\left(\lim_{\epsilon\to 0}\frac{d}{d\epsilon}\left(\epsilon\int_0^\infty \frac{dt}{2t}t^\epsilon e^{-2\pi t H}\right) + \frac{1}{2}\log 2\pi - \frac{1}{2}\Gamma'(1)\right). \end{aligned} \tag{2.3.2.9}$$

At a formal level this expression is correct. However, the next step is to take the limit of $\epsilon \to 0$ before performing the integral over t. This would be allowed only if the integral were converging. This is not the case in most of the applications (for example, the integral diverges if H is just a number and no trace is taken). But the error done is some unknown additive constant contribution which is not of interest for us[39]. Together with this unknown constant we also drop the $\frac{1}{2}\left(\log 2\pi - \Gamma'(1)\right)$ and obtain for the amplitude in figure 2.8 (reinstalling α')

$$\int_0^\infty \frac{dt}{2t}\,\mathrm{Str}\, e^{-2\pi\alpha' t H} \tag{2.3.2.10}$$

(here we have replaced the trace by a supertrace. It refers to target space supersymmetry, i.e. the trace receives an additional minus sign for target space spinors. The integral over t is usually regulated by a UV cutoff near $t = 0$.) The expression (2.3.2.10) has also an intuitive interpretation. The supertrace describes a process where a pair of open strings is created from the vacuum, then propagates for a time $2\pi t$ and annihilates. This corresponds to the diagram drawn in figure 2.8. Further, we integrate over all possible moduli t of the annulus with the measure $\frac{dt}{2t}$. The Hamiltonian is $p^2 + M^2$ which can be expressed by use of (2.3.1.30) and (2.3.1.40) as follows

$$H = p^2 + \frac{y^2}{\pi^2} + 2\left(N - a\right), \tag{2.3.2.11}$$

where y is the distance between the two D-branes, and a is the normal ordering constant ($\frac{1}{24}$ per bosonic direction, $\frac{1}{48}$ per fermionic direction in the NS sector, and $-\frac{1}{24}$ per fermionic direction in the R sector). Recalling that in this expression we have set $\alpha' = \frac{1}{2}$ gives (just multiply with appropriate powers of $2\alpha'$ to get the mass dimension right)

$$\alpha' H = \alpha' p^2 + \frac{y^2}{4\pi^2\alpha'} + \left(N - a\right). \tag{2.3.2.12}$$

It is useful to split (2.3.2.10) into several contributions

$$\begin{aligned} \int \frac{dt}{2t}\mathrm{Str}\, e^{-2\pi\alpha' t H} = \\ \int \frac{dt}{2t}\,\mathrm{tr}_{\substack{\text{ZERO}\\\text{MODES}}}\left(e^{-2\pi t\alpha' H_0}\right)\mathrm{tr}_{\text{BOSONS}}\left(e^{-2\pi t H_B}\right) \\ \left(\mathrm{tr}^{GSO}_{\substack{\text{NS}\\\text{FERMIONS}}}\left(e^{-2\pi t H_{NS}}\right) - \mathrm{tr}^{GSO}_{\substack{\text{R}\\\text{FERMIONS}}}\left(e^{-2\pi t H_R}\right)\right). \end{aligned} \tag{2.3.2.13}$$

We have split the Hamiltonian into

$$\alpha' H = \alpha' H_0 + H_B + H_{NS/R}, \tag{2.3.2.14}$$

[39] In our case the trace is actually a supertrace which vanishes when taken over a constant. However, since the corresponding series does not converge absolutely the result depends on the ordering in which we take the trace.

with

$$H_0 = p^2 + \frac{y^2}{4\alpha'^2\pi^2}, \tag{2.3.2.15}$$

$$H_B = \sum_{i=1}^{8}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i - \frac{1}{24}\right), \tag{2.3.2.16}$$

$$H_{NS} = \sum_{i=1}^{8}\left(\sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i - \frac{1}{48}\right), \tag{2.3.2.17}$$

$$H_R = \sum_{i=1}^{8}\left(\sum_{n=1}^{\infty} n d_{-n}^i d_n^i + \frac{1}{24}\right). \tag{2.3.2.18}$$

An additional minus sign in the R sector contribution is due to the fact that we take the supertrace with respect to space time supersymmetry (R-sector states are space time fermions). The superscript GSO indicates that the trace is taken over GSO even states. We will clarify this point later.

The trace over the zero modes is

$$\mathrm{tr}_{\substack{\text{ZERO}\\ \text{MODES}}} = 2V_{p+1}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}\, e^{-2\pi t\alpha' k^2 - \frac{ty^2}{2\pi\alpha'}} = 2V_{p+1}\left(8\pi^2\alpha' t\right)^{-\frac{p+1}{2}}\, e^{-\frac{ty^2}{2\pi\alpha'}}, \tag{2.3.2.19}$$

where the factor of two counts the possible orientations of the open string traveling through the annulus. The factor V_{p+1} denotes formally the worldvolume of the parallel D_p branes. It arises due to the normalization of states with continuous momentum ($\langle p|p\rangle = \delta^{(p+1)}(0) = V_{p+1}/(2\pi)^{p+1}$). To express oscillator traces, it is useful to define the following set of functions[40]

$$f_1(q) = q^{\frac{1}{12}}\prod_{n=1}^{\infty}\left(1-q^{2n}\right) \quad , \quad f_2(q) = q^{\frac{1}{12}}\sqrt{2}\prod_{n=1}^{\infty}\left(1+q^{2n}\right),$$
$$f_3(q) = q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right) \quad , \quad f_4(q) = q^{-\frac{1}{24}}\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right). \tag{2.3.2.20}$$

These functions satisfy the identity

$$f_3^8(q) = f_2^8(q) + f_4^8(q). \tag{2.3.2.21}$$

In order to translate the open string calculation back to the closed string process (figure 2.7) we will make use of the modular transformation properties,

$$f_1\left(e^{-\frac{\pi}{s}}\right) = \sqrt{s}f_1\left(e^{-\pi s}\right), \quad f_3\left(e^{-\frac{\pi}{s}}\right) = f_3\left(e^{-\pi s}\right), \quad f_2\left(e^{-\frac{\pi}{s}}\right) = f_4\left(e^{-\pi s}\right). \tag{2.3.2.22}$$

[40] These are related to the Jacobi theta functions, which are also often used in the literature.

Next, we are going to compute the trace over the worldsheet bosons. The sum over the coordinate label i in (2.3.2.16) can be written in front of the exponential as a product. Nothing depends explicitly on the direction i, therefore this gives a power of eight to the result for a single bosonic direction. The second sum can be decomposed into a level part and an occupation number part, giving the result

$$\mathrm{tr}_{\mathrm{BOSONS}}\, e^{-2\pi t H_B} = \left(e^{\frac{\pi t}{12}} \prod_{l=1}^{\infty} \sum_{k=0}^{\infty} e^{-2\pi t l k} \right)^8 . \qquad (2.3.2.23)$$

Here, l denotes the level of a creator α_{-l} acting on the ground state and k is the occupation number (the number of times this creation operator acts). The sum over k is just a geometric series, and we obtain the result

$$\mathrm{tr}_{\mathrm{BOSONS}}\, e^{-2\pi t H_B} = \frac{1}{f_1^8\left(e^{-\pi t}\right)}. \qquad (2.3.2.24)$$

The calculation of the traces over the fermionic sectors is similar. Let us just point out the differences. First of all, we have to take the trace only over GSO even states. This is done by inserting the GSO projection operator into the trace and summing over all states

$$\mathrm{tr}^{GSO}\left(\cdots\right) = \frac{1}{2}\mathrm{tr}\left(\cdots\right) + \frac{1}{2}\mathrm{tr}\left((-)^F \cdots\right). \qquad (2.3.2.25)$$

The second difference – as compared to the bosonic calculation – is that for world-sheet fermions the occupation number can be only zero or one (since the creators anticommute). The NS trace without the $(-)^F$ insertion comes out to be

$$\frac{1}{2}\mathrm{tr}_{\mathrm{NS}}\left(e^{-2\pi t H_{NS}}\right) = \frac{1}{2} f_3^8\left(e^{-\pi t}\right). \qquad (2.3.2.26)$$

Since the NS vacuum is GSO odd we assign an additional minus to states with even and zero occupation if $(-)^F$ is inserted into the trace,

$$\frac{1}{2}\mathrm{tr}_{\mathrm{NS}}\left((-)^F e^{-2\pi t H_{NS}}\right) = -\frac{1}{2} f_4^8\left(e^{-\pi t}\right). \qquad (2.3.2.27)$$

For the R-sector trace without the $(-)^F$ insertion one obtains

$$-\frac{1}{2}\mathrm{tr}_{\mathrm{R}}\left(e^{-2\pi t H_R}\right) = -\frac{1}{2} f_2^8\left(e^{-\pi t}\right), \qquad (2.3.2.28)$$

where the 16-fold degeneracy of the R vacuum has been taken into account by the factor of $\sqrt{2}$ in the definition of f_2 (2.3.2.20). The R sector trace with a $(-)^F$ insertion vanishes identically. Half of the R sector groundstates have eigenvalue $+1$ whereas the other half has eigenvalue -1. Adding up all the results and using the identity (2.3.2.21) we find that the net result for the annulus amplitude vanishes. This implies also that

Sector	$\int_{l\to\infty} dl \times$
RR	$-\frac{1}{2}\left(4\alpha'\pi^2\right)^{-\frac{p+1}{2}}\ V_{p+1}\, l^{\frac{p-9}{2}}\ e^{-\frac{y^2}{4\pi\alpha' l}}$
NSNS	$\frac{1}{2}\left(4\alpha'\pi^2\right)^{-\frac{p+1}{2}}\ V_{p+1}\, l^{\frac{p-9}{2}}\ e^{-\frac{y^2}{4\pi\alpha' l}}$

Table 2.7: Contributions of massless RR and NSNS sector closed strings to figure 2.7.

the closed string diagram 2.7 vanishes, and a hasty interpretation of this would lead to the conclusion that D-branes do not interact (at least not via the exchange of closed strings). However, as we will argue now, this is not the case. The situation rather is that repulsive and attractive interactions average up to zero. In order to see this[41], let us translate the annulus result back to the tree channel. Further, we would like to filter out the contributions of massless closed string excitations. To this end, we replace t in terms of l using (2.3.2.8), and afterwards apply (2.3.2.22). The contribution of massless closed string excitations is obtained by focusing on the leading contribution in the $l \to \infty$ limit. (Massless interactions have infinite range whereas the interactions carried by massive bosons have finite range.) We collect the result of this straightforward calculation in table 2.7. We separate closed RR contributions from NSNS sector contributions. The former ones correspond to the $(-)^F$ insertion and the latter to the 1 insertion in the annulus amplitude.

From table 2.7 we deduce that interactions carried by closed strings in the RR sector cancel interactions mediated by closed strings in the NSNS sector. One can take the diagram 2.7 to the field theory limit. In that limit the 'hose' connecting the two D-branes becomes particle propagators (lines). In the NSNS sector we find propagators for the metric (fluctuations), the dilaton and the anti-symmetric tensor $B_{\mu\nu}$. The D-branes appear as source terms for those fields. The NSNS contribution to diagram 2.7 tells us the strength of this coupling. In particular, it yields the strength of the gravitational coupling which is given by the tension T_p. A detailed analysis of the effective field theory and comparison with table 2.7 leads to the result [371]

$$T_p^2 = \frac{\pi}{\kappa^2}\left(4\pi^2\alpha'\right)^{3-p} e^{-2\Phi_0}, \tag{2.3.2.29}$$

[41] Already in the annulus computation, there are signs for such a cancellation. For the result, the minus sign in front of the R sector contribution was essential. Since R sector states are target space fermions, this indicates that the result is due to target space supersymmetry.

where κ is the gravitational coupling in the effective theory (see section 2.1.4) and Φ_0 denotes the constant vev of the dilaton. Even though we did not derive this explicitly here, let us make a few comments to motivate the expression qualitatively. In a field theory calculation the propagator of an NSNS field is accompanied by a power of $\kappa^2 e^{2\Phi_0}$. With T_p defined as in (2.3.2.29) the κ and Φ_0 dependence drop out. Since $\kappa^2 \sim (\alpha')^4$, the mass dimension of T_p is correct. (The α' dependence in the string calculation yields agreement with (2.3.2.29) after substituting for the integration variable l such that the α' dependence in the exponent vanishes.) Further, the exchange of massless particles should lead to a Coulomb interaction in field theory. For the interaction between the two D-branes this means that the potential should be given by the distance to the power of two minus the number of transverse dimensions

$$V \sim y^{p-7}. \tag{2.3.2.30}$$

In the string result (table 2.7), we can extract the y dependence after rescaling the integration parameter l such that the y dependence in the exponent disappears. The result agrees with (2.3.2.30). In order to fix the numerical coefficient, one needs to do a more detailed analysis of the field theory calculation. More details on this can be found in Polchinski's book[371].

The second line in table 2.7 tells us that and how string RR fields couple to the brane. We find the same Coulomb law as for the gravitational interaction. The RR field should be a $p+1$ form. The $p = even$ branes interact via closed type IIA strings and the $p = odd$ branes via closed type IIB strings. The value of the RR contribution is exactly minus the value of the NSNS contribution. This provides us with the RR charge of the D-brane[42]

$$\mu_p^2 = 2\kappa^2 T_p^2 e^{2\Phi_0}, \tag{2.3.2.31}$$

where we have taken into account the dilaton dependent RR field redefinition performed in section 2.1.4. The signs are undetermined at this level.

2.3.3 D-brane actions

In the following we will specify the actions for the field theory on the D-brane. We will also argue that the D-brane interaction with the bulk field is obtained by adding the action for fields living on the D-brane (the D-brane action) to the effective type II action of section 2.1.4. The previous calculation fixes the coefficient in front of the D-brane action.

[42] The factor of $2\kappa^2$ has been introduced in order to match the charge definition to be used later in equ. (3.3.0.12).

2.3.3.1 Open strings in non-trivial backgrounds

In this section we will modify the calculation presented in section 2.1.3 such that it accommodates open string excitations. We perform a Wick rotation such that the worldsheet is of Euclidean signature. Further, we map the parameter space of the open string worldsheet from a strip to the upper half plane via the conformal transformation

$$z = e^{\tau + i\sigma} = z^1 + iz^2. \tag{2.3.3.1}$$

The discussion is performed for bosonic strings but later we will also give the result for superstrings. Since the open string couples naturally to closed strings (via joining its ends) we also switch on non-trivial closed string modes. These are the metric $G_{\mu\nu}$ and the antisymmetric tensor $B_{\mu\nu}$. We will take those background fields to be constant. For the open string we take Neumann boundary conditions in all directions at first. (For the superstring this is not consistent with RR conservation. At the moment we will ignore this problem and return to it later.) Later, we will discuss that T-duality maps Neumann to Dirichlet boundary conditions. Hence, the restriction to Neumann boundary conditions will not result in a loss of generality. At first, we consider a single brane setup. The massless open string excitation is now a $U(1)$ gauge field A_μ. We restrict ourself to the special case that the $U(1)$ field strength $F_{\mu\nu}$ is slowly varying, i.e. we neglect contributions containing second or higher derivatives of $F_{\mu\nu}$. Under these conditions we will be able to perform the calculation to all orders in α' (in difference to section 2.1.3). The nonlinear sigma model reads

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'}\left[\int d^2z \frac{1}{2}\left(\partial_\alpha X_\mu \partial^\alpha X^\mu + i\epsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)\right. \\ &\quad \left. + i\int_{z^2=0} dz^1 A_\mu \partial_1 X^\mu\right]. \end{aligned} \tag{2.3.3.2}$$

Here, A has been rescaled such that α' appears as an overall factor in front of the action. The target space indices μ,ν are raised and lowered with the constant background metric $G_{\mu\nu}$. The worldsheet metric is taken to be the identity in the z^α coordinates (2.3.3.1).[43] Using Stoke's theorem the term with the constant B field can be rewritten as a surface integral

$$\int d^2z \epsilon^{\alpha\beta} B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu = 2\int_{z^2=0} dz^1 B_{\mu\nu} X^\mu \partial_1 X^\nu. \tag{2.3.3.3}$$

The term with the B-field can be absorbed into a redefinition of the gauge field A_μ and we can put it to zero without loss of generality. (It can be recovered by replacing $F \to F - 2B$.)

[43] In principle this choice introduces gauge fixing ghosts as stated in section 2.1.3. Since their effect is not altered by the presence of the boundary we will not discuss the ghosts here. We should, however, mention that there are technical subtleties when taking into account the dilaton in worldsheets with boundaries, see e.g. [49]. (Recall that the ghosts contribute to the dilaton beta function.)

In order to proceed we specify a classical configuration around which we are going to expand. We denote this again by $\bar{X}^\mu$. For freely varying ends the equation of motion and boundary conditions read

$$\begin{aligned} \partial_z \partial_{\bar{z}} \bar{X}^\mu &= 0, \qquad (2.3.3.4)\\ \left(\partial_2 \bar{X}^\mu + i F_\nu{}^\mu \partial_1 \bar{X}^\nu\right)_{|z^2=0} &= 0. \qquad (2.3.3.5) \end{aligned}$$

The presence of the $U(1)$ gauge field A_μ results in inhomogeneous Neumann boundary conditions. Since we have restricted ourselves to the case where the target space metric is constant, the background field expansion simplifies in comparison to the computation of section 2.1.3. The fields can simply be Taylor expanded. Neglecting second and higher derivatives of $F_{\mu\nu}$, the background field expansion terminates at the third order in the fluctuations,

$$\begin{aligned} S\left[\bar{X}+\xi\right] = S\left[\bar{X}\right] &+ \frac{1}{2\pi\alpha'}\int d^2z \frac{1}{2}\partial^\alpha \xi_\mu \partial_\alpha \xi^\mu \\ &+ \frac{i}{2\pi\alpha'}\int_{z^2=0} dz^1 \left(\frac{1}{2}F_{\mu\nu}\xi^\nu\partial_1\xi^\mu + \frac{1}{2}\partial_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_1\bar{X}^\mu + \frac{1}{3}\partial_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_1\xi^\mu\right). \qquad (2.3.3.6) \end{aligned}$$

Since we have chosen the worldsheet metric to be the identity (and the geodesic curvature of the boundary to vanish) a suitable technique to integrate out the fluctuations ξ^μ is given by a Feynman diagrammatic approach. This means that we split the action into a free and an interacting piece. The free piece determines the propagator whereas the interacting one leads to vertices. As the free part of the action we take

$$S_{free} = \frac{1}{4\pi\alpha'}\int d^2z \partial^\alpha \xi_\mu \partial_\alpha \xi^\mu + \frac{i}{4\pi\alpha'}\int_{z^2=0} dz^1 F_{\mu\nu}\xi^\nu \partial_1 \xi^\mu. \qquad (2.3.3.7)$$

Hence, the interacting part is given by the rest

$$S_{int} = \frac{i}{2\pi\alpha'}\int_{z^2=0} dz^1 \left(\frac{1}{2}\partial_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_1\bar{X}^\mu + \frac{1}{3}\partial_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\partial_1\xi^\mu\right). \qquad (2.3.3.8)$$

In order to compute the propagator we have to invert the two dimensional Laplacian and to satisfy the (inhomogeneous Neumann) boundary conditions arising from the variation of S_{free} with respect to ξ^μ (with free varying ends of the ξ^μ),

$$\begin{aligned} \frac{1}{2\pi\alpha'}\Box\Delta_{\mu\nu}\left(z,z'\right) &= -\delta\left(z-z'\right)G_{\mu\nu}, \qquad (2.3.3.9)\\ \left(\partial_2\Delta_{\mu\nu}\left(z,z'\right) + iF_\mu{}^\lambda\partial_1\Delta_{\lambda\nu}\left(z,z'\right)\right)_{|z^2=0} &= 0. \qquad (2.3.3.10) \end{aligned}$$

This boundary value problem can be solved (for example by borrowing the method of mirror charges from electro statics) with the result

$$\begin{aligned} &\Delta^{\mu\nu}\left(z,z'\right) = \\ &\quad -\alpha'\left[G^{\mu\nu}\log\frac{|z-z'|}{|z-\bar{z}'|} + \left(\hat{G}^{-1}\right)^{\mu\nu}\log\left|z-\bar{z}'\right|^2 + \theta^{\mu\nu}\log\frac{z-\bar{z}'}{\bar{z}-z'}\right]. \qquad (2.3.3.11) \end{aligned}$$

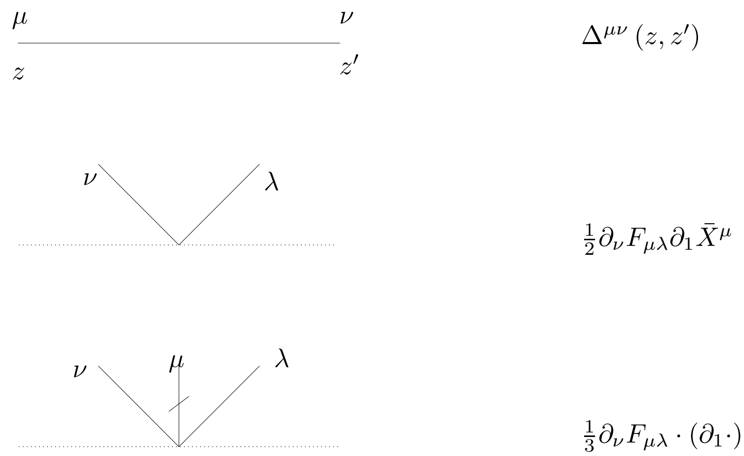

Figure 2.9: Feynman rules: The dotted line denotes the worldsheet boundary. The slash on the leg means that a derivative acts on the corresponding leg.

Here, we have introduced the following matrices (an index $S(A)$ stands for (anti-)symmetrization and $G^{\mu\nu}$ are the components of the inverted target space metric G^{-1}, as usual)

$$\left(\hat{G}^{-1}\right)^{\mu\nu} = \left(\frac{1}{G+F}\right)_S^{\mu\nu} = \left(\frac{1}{G+F} G \frac{1}{G-F}\right)^{\mu\nu}, \tag{2.3.3.12}$$

$$\left(\hat{G}\right)_{\mu\nu} = G_{\mu\nu} - \left(F G^{-1} F\right)_{\mu\nu}, \tag{2.3.3.13}$$

$$\theta^{\mu\nu} = \left(\frac{1}{G+F}\right)_A^{\mu\nu} = \left(\frac{1}{G+F} F \frac{1}{G-F}\right)^{\mu\nu}. \tag{2.3.3.14}$$

The interaction piece S_{int} gives rise to two vertices —one with two and one with three legs— located at the boundary. The Feynman rules are summarized in figure 2.9.

The propagator $\Delta^{\mu\nu}(z, z')$ becomes logarithmically divergent if the arguments coincide. Therefore, we replace the logarithm of zero by its dimensionally regularized version

$$\log\left|z - z'\right|_{|z=z'} = -\int \frac{d^2k}{2k^2} e^{k(z-z')}{}_{|z=z'} = -\lim_{\epsilon\to 0} \mu^\epsilon \int \frac{d^{2-\epsilon}k}{2\left(k^2+m^2\right)}, \tag{2.3.3.15}$$

where the momentum integral extends over a two dimensional plane. We introduced a mass scale μ which is needed in order to keep the mass dimension fixed while changing the momentum space dimension. In the last step we have introduced also an infrared cutoff m^2.[44] With our regularization prescription we obtain

$$\log\left(z-z'\right)_{|z=z'} = \frac{1}{2}\pi^{\frac{2-\epsilon}{2}}\left(\frac{\mu}{m}\right)^{\epsilon}\Gamma\left(\frac{\epsilon}{2}\right), \tag{2.3.3.16}$$

which has a simple pole as ϵ goes to zero.

The bare background field (coupling) A_μ is infinite. By adding counterterms to the action the bare field can be expressed in terms of a renormalized field which is finite as ϵ goes to zero. The only counterterm arises from the diagram in figure 2.10. The action is written in terms of renormalized fields by adding

$$\delta S = -\frac{i}{2\pi\alpha'}\int_{z^2=0} dz^1 \frac{1}{2}\partial_\nu F_{\mu\lambda}\partial_1 \bar{X}^\mu \Delta^{\nu\lambda}\left(z^1, z'^1\right)_{|z'\to z}. \tag{2.3.3.17}$$

(and replacing the bare gauge field by the renormalized one. Hoping that the renormalization program is sufficiently familiar we do not introduce sub- or super-scripts indicating the difference between bare and renormalized couplings). The beta-function of A_μ is obtained by applying $\mu\frac{d}{d\mu}$ on the renormalized couplings and using the fact that the bare couplings are independent of the cutoff. This leads to (now, A_μ denotes the renormalized coupling)

$$\beta^A_\rho = \mu\frac{d}{d\mu}A_\rho = \partial_\nu F_{\rho\lambda}\left(\hat{G}^{-1}\right)^{\lambda\nu}. \tag{2.3.3.18}$$

In this case the vanishing of the beta function ensures conformal invariance. (We do not encounter the subtleties which we met in section 2.1.3. Partially, this is the case because we have written the action always in a manifestly gauge invariant form, i.e. in terms of the gauge field strength. By performing partial integrations differently we could have carried out the calculation in a slightly more complicated way, with the same result.) The equation of motion for the gauge field is

$$\beta^A_\mu = 0. \tag{2.3.3.19}$$

This equation of motion can be lifted to the Dirac-Born-Infeld action

$$S = \frac{\sqrt{\pi}}{\kappa}\left(4\pi^2\alpha'\right)^{\frac{3-p}{2}}\int d^{p+1}x\, e^{-\Phi}\sqrt{\det\left(G+F\right)}, \tag{2.3.3.20}$$

where $p+1$ is the number of Neumann directions (i.e. for our discussion $p+1 = 10(26)$ for the super (bosonic) string[45]). The factor in front of the integral in (2.3.3.20) has not been fixed by our current discussion. We will explain how to fix it below. The same applies to the dilaton dependence. (We discussed only the case of a constant dilaton Φ.) Since we have rescaled A_μ by powers of α' such that the α' dependence

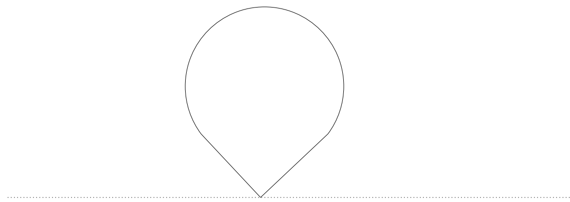

Figure 2.10: The logarithmically divergent Feynman diagram.

appears as an overall factor in (2.3.3.2), the α' expansion of the action (2.3.3.20) is not obvious. Performing rescalings such that A_μ has mass-dimension one (or zero) shows that the α' expansion is a power expansion in $F_{\mu\nu}$. Also, we note that the propagator (2.3.3.17) contains higher orders in α'. Alternatively, we could have chosen a propagator satisfying homogenous Neumann boundary conditions at the price of having an additional vertex operator. This additional vertex does not lead to a one loop (leading order in α') divergence since it is antisymmetric in its legs. The leading order α' equation is

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.3.3.21}$$

Lifting this to an action would give (in the small α' approximation)

$$S \sim \int d^{p+1}x e^{-\Phi}\sqrt{-G}F^2, \tag{2.3.3.22}$$

where the Φ dependence has been taken such that the result coincides with the small α' expansion of (2.3.3.20). Expanding (2.3.3.20) in powers of F and keeping only terms up to F^2 we find in addition to (2.3.3.22) a contribution $\int d^{p+1}xe^{-\Phi}\sqrt{-G}$. From a field theory perspective this is a tree level vacuum energy. So far, we did not properly couple the open string excitations to gravity. We included the effects of bulk fields on the equations of motion for open string excitations, but we did not encounter a back reaction, i.e. that the field A_μ enters the equations of motion for the closed string excitations. The reason is that the back reaction is an annulus effect. We will not present a detailed annulus calculation but sketch the result. (In principle we have done the necessary computations in the previous section.) Since the beta functions depend on local features (short distance behaviors) one would guess that for the beta function it may not matter whether the worldsheet is an annulus or a disc. However, the annulus may degenerate as depicted in figure 2.11.

[44] A different (maybe more elegant) way to deal with infrared divergences is discussed e.g. in [102], see also the appendix of[228].

[45] We view the result of our computation as a result for the bosonic modes of the superstring. Since we did not specify the effective action for the closed bosonic string the factor in front of (2.3.3.20) is irrelevant for the bosonic string (see also the discussion of the Fischler Susskind mechanism below).

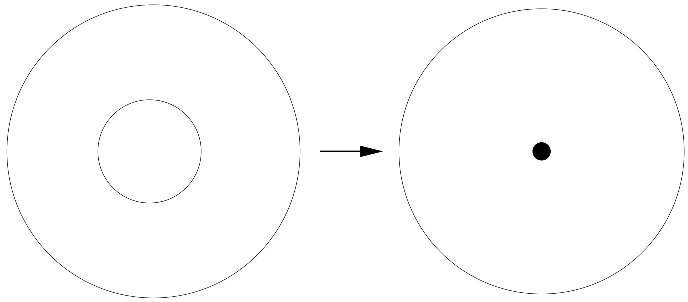

Figure 2.11: Fischler Susskind mechanism: The annulus degenerates into a punctured disc as the inner circle shrinks to zero. This gives rise to a closed string counterterm depending on the open string excitations.

This gives an additional short distance singularity. (The inner circle of the annulus becomes short.) This singularity can be taken care of by adding counterterms to the closed string action. The counterterms depend on open string modes. This in turn leads to terms in the closed string beta functions which depend on the open string modes. This process is known as the Fischler Susskind mechanism. The net effect is that we add the open string effective action to the closed string effective action (2.3.3.20) and obtain the equations of motion by varying the sum. This is the expected back reaction. (In particular the Einstein equation now contains the energy momentum tensor of the open string modes.) After taking the back reaction into account, the coefficient in (2.3.3.20) does matter. In the previous section we have computed the tension of the D-brane (2.3.2.29). This fixes the coefficient and the dilaton dependence[46] as given in (2.3.3.20). According to our discussion in the previous section, the presence of a D-brane should also back-react on the RR background. We could not see this in the present consideration since we did not take into account non trivial RR backgrounds. (In fact, it is rather complicated to switch on non-trivial RR backgrounds in the non-linear sigma model.) We will come back to the discussion of RR contributions to the open string effective action below.

So far, we have studied the case of a single D-brane. How is this discussion modified in the presence of multiple D-branes? We have focused on the case where we have only Neumann boundary conditions. This means that multiple D-branes must sit on top of each other, simply because there is no space dimension left in which they could be separated. The effect of having more than a single brane is that the gauge field A_μ is a $U(N)$ gauge field – it is a matrix. Calling the expression (2.3.3.2) an action does not make much sense anymore since we would have a matrix valued action. Therefore, one

[46]Note also that this dilaton dependence agrees with our general discussion in section 2.1.4. The Euler number of the disc differs by one from the Euler number of the sphere.

takes just the bulk part of the action (the first line in (2.3.3.2)) and computes instead of the partition function the Wilson loop along the string boundary[142],

$$W = \left\langle tr\left(Pe^{i\int_{\partial M} dtA_\mu \dot{X}^\mu}\right)\right\rangle, \qquad (2.3.3.23)$$

where we have denoted the boundary of the worldsheet by ∂M and chosen some t to parameterize this curve. The letter P stands for the path ordered product. The expectation value is computed with respect to the bulk action only. Now, it is problematic to get an expression containing all orders in α'. The leading α' contribution to the beta function results in the Yang-Mills equation

$$\nabla_\mu F^{\mu\nu} = 0, \qquad (2.3.3.24)$$

where ∇ denotes a gauge covariant derivative. The effective action in leading approximation can be obtained as follows. We expand (2.3.3.20) to first order in F^2. We replace F^2 by trF^2. In addition, we multiply the zeroth order term in F by the number N of D-branes (the tension is N times the tension of a single D-brane). The generalization of the Dirac-Born-Infeld action (2.3.3.20) to non-abelian gauge fields is a subject of ongoing research, see e.g. [449, 75].

2.3.3.2 Toroidal compactification and T-duality for open strings

In the previous section we have discussed the case of having Neumann boundary conditions in all directions. This means that the D-branes have been space filling objects. In order to obtain results for D-branes extending along less dimensions we will discuss T-duality for open strings, now.

At first, we focus on the case with trivial background fields. From section 2.1.5 we recall that T-duality interchanges winding with momentum modes. For the open string we have either winding or momentum modes in compact directions. A string with DD boundary conditions along the compact dimension can have non-trivial winding modes. Since the ends of the string are tied to the D-brane it cannot unwrap. On the other hand, the DD string does not have quantized Kaluza Klein momenta. The D-brane can absorb any momentum carried by the string in the compact direction. For NN strings, opposite statements are true. If the string has Neumann boundary conditions along the compact dimension, its ends can move freely in that direction – it can continuously wrap and unwrap the compact dimension. On the other hand, the string cannot transfer Kaluza-Klein momentum to the D-brane. The NN string has non-trivial momentum modes. This consideration suggests that T-duality for open strings interchanges Neumann with Dirichlet boundary conditions.

Let us substantiate these qualitative statements by studying the effect of T-duality on the mode expansions. For the T-duality transformation we use the "recipe" (2.1.5.31).

To be specific we choose the ninth direction to be compact, i.e.

$$x^9 \equiv x^9 + 2\pi R. \tag{2.3.3.25}$$

For the string with NN boundary conditions in the ninth dimension this implies that the center of mass momentum is quantized

$$p^9 = \frac{n}{R}, \tag{2.3.3.26}$$

with n being an integer. There are no integer winding numbers in the case of NN boundary conditions. We rewrite the mode expansion (2.3.1.20) in a suggestive way

$$\begin{aligned} X^9 &= X^9_R + X^9_L, && (2.3.3.27)\\ X^9_R &= \frac{x^9}{2} + \frac{n}{2R}\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^9_n e^{-in\sigma^-}, && (2.3.3.28)\\ X^9_L &= \frac{x^9}{2} + \frac{n}{2R}\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^9_n e^{-in\sigma^+}. && (2.3.3.29) \end{aligned}$$

Applying the recipe (2.1.5.31), we obtain the mode expansion for the T-dual coordinate

$$X^9 \quad \stackrel{\text{T-DUALITY}}{\longrightarrow} \quad \tilde{X}^9 = \frac{n}{R}\,\sigma + \sum_{n\neq 0}\frac{1}{n}\alpha^9_n e^{-in\tau}\sin n\sigma. \tag{2.3.3.30}$$

This mode expansion is zero at $\sigma = 0$ and $2nR'$ at $\sigma = \pi$, where (see (2.1.5.20))

$$R' = \frac{1}{2R} = \frac{\alpha'}{R}. \tag{2.3.3.31}$$

The interpretation is that the open string ends on a D-brane located at $x^9 = 0$[47]. The open string winds n times around a circle of radius R'. It is rather obvious that —starting from a DD string with mode expansion (2.3.3.30)— T-duality will take us to an NN string with mode expansion (2.3.3.27), (the center of mass position depends again on the way we distribute a constant between left and right movers). So, T-duality inverts the compactification radius and interchanges Dirichlet with Neumann boundary conditions. We leave it to the reader to verify that an investigation of the worldsheet fermions and of ND directions is consistent with this picture.

In section 2.3.2 we have noticed that Dp-branes with p even (odd) interact via the exchange of closed type IIA(B) strings. Our present observation that T-duality along a compact direction interchanges Dirichlet with Neumann boundary conditions implies that a Dp-brane with even p is mapped onto a Dq-brane with odd q, and vice

[47]We could have obtained a different position by distributing the center of mass position of the NN string x^9 asymmetrically among left and right movers.

versa (q = p $\pm$ 1). This goes along nicely with our earlier statement (section 2.1.5.4) that T-duality along one circle interchanges type IIA with type IIB strings.

Finally, let us discuss T-duality for open strings in the presence of non-trivial background fields. For the closed string we have done this in section 2.1.5.3. Because the discussion of the closed string background fields is not affected by the open string, we will focus on the special case where only the open string gauge field is non-trivial. For simplicity we also restrict to one D-brane only (for multiple D-branes see e.g. [143])[48]. Let us first outline in words the procedure we are going to carry out. The compactification has to be done in a Killing direction. (Shifts along the compact direction are isometries.) We will take this dimension to be the ninth. The next step is to gauge this isometry and to undo the gauge by forcing the corresponding gauge field to be trivial. This will be done again by adding a Lagrange multiplier times the field strength of the isometry gauge field. The Lagrange multiplier will become the T-dual coordinate in the end. In particular the Lagrange multiplier lives on a circle whose radius is inverse to the original compactification radius. This is derived from the requirement that the radius of the isometry-gauge group ($U(1)$) agrees with the compactification radius. We will not discuss the technical details of this derivation (they are presented for example in the appendix of [15]). Instead, we will focus on a detailed discussion of the boundary conditions. The boundary condition of the isometry gauge fields is constrained by the boundary condition of the open string. This will be implemented by a second Lagrange multiplier which lives only at the boundary of the worldsheet. After integrating out the isometry gauge fields the integration over this second Lagrange multiplier will give the boundary condition for the T-dual coordinate (the "first" Lagrange multiplier)[49].

After having described the strategy, we will now present the details of the procedure. Setting $\alpha' = \frac{1}{2}$ the open string worldsheet action with a non-trivial $U(1)$ gauge field coupling to the boundary reads (for convenience we use a rescaled A_μ as compared to (2.3.3.2) and choose Minkowskian worldsheet signature here)

$$S = \frac{1}{2\pi}\left(\int_{\mathcal{M}} d^2z\, \partial_\alpha X^\mu \partial^\alpha X^\mu + \int_{\partial\mathcal{M}} dt\, (\, A_\mu \partial_t X^\mu + V_\mu \partial_n X^\mu)\right), \tag{2.3.3.32}$$

where $\mathcal{M}$ denotes the worldsheet and $\partial\mathcal{M}$ its boundary (parameterized by t). With ∂_n we denote the derivative into the direction normal to the boundary. We specify the

[48] We will comment briefly on the case of multiple D-branes at the end of this section.

[49] As we will see below integrating over the second Lagrange multiplier κ boils down to setting an argument of a delta function to zero. This in turn implies boundary conditions on the Lagrange multiplier λ.

character of the boundary conditions in X^9 direction by the following assignments[50]

Boundary Condition	δX^9	$\partial_n \delta X^9$
Dirichlet	fixed	free
Neumann	free	fixed

. (2.3.3.33)

This implies that for Dirichlet boundary conditions we set $A_9 = 0$ whereas for Neumann boundary conditions $V_9 = 0$ is chosen. For the Neumann boundary conditions (free varying ends) the variation of S gives the boundary condition (we denote the normal vector with n^α and the tangent vector with t^α)

$$n^\alpha \partial_\alpha X^9 = -\frac{1}{2} F^{9\nu} \partial_t x_\nu, \tag{2.3.3.34}$$

where $F_{\mu\nu}$ is the field strength of the $U(1)$ gauge field A_μ. For Dirichlet conditions we obtain

$$V_9 = 0. \tag{2.3.3.35}$$

Since this should be in agreement with our assignment that the variation of the end of the open string in the ninth direction is fixed (possibly related to the variations in other directions), the function V_μ should be interpreted as a vector which is tangent to the brane. Equation (2.3.3.35) then means that the D-brane is localized in the ninth direction.

Since we have chosen the simplified case of trivial closed string backgrounds any direction (in cartesian target space coordinates) is an isometry. Suppose that in addition the x^9 derivative of the $U(1)$ gauge field is pure gauge, i.e. zero modulo gauge transformations. So, without loss of generality we restrict ourselves to the case that the gauge background is X^9 independent. We also assume that the tangent vector V_μ does not depend on X^9. We specify the boundary condition on X^9 by the equation

$$b^\alpha \partial_\alpha X^9_{|\partial\mathcal{M}} = \text{independent of } X^9, \tag{2.3.3.36}$$

where b^α is a worldsheet vector with a given orientation to the boundary. In case of Dirichlet boundary conditions, b^α is parallel to the boundary ($b^\alpha = t^\alpha$). For free varying ends (Neumann boundary conditions) b^α is normal to the boundary. In the action (2.3.3.32) X^9 does not mix with the other fields. We focus on the X^9 dependent part

$$\begin{aligned} S &= \bar{S} + S^{(9)}, & (2.3.3.37)\\ S^{(9)} &= \frac{1}{2\pi}\left(\int_{\mathcal{M}} d^2z\, \partial_\alpha X^9 \partial^\alpha X^9 + \int_{\partial\mathcal{M}} dt \left(A_9 \partial_t X^9 + V_9 \partial_n X^9\right)\right), & (2.3.3.38)\end{aligned}$$

[50] A fixed boundary condition on a variation means that this variation depends on the boundary values of variations of other fields (or is zero). In particular, if we do not vary the other directions we can replace "fixed" by "zero" in (2.3.3.33).

where $\bar{S}$ stands for the X^9 independent part. The action is invariant under constant shifts in X^9. We transform this into a local symmetry by the replacement

$$\partial_\alpha X^9 \to D_\alpha X^9 = \partial_\alpha X^9 + \Omega_\alpha, \tag{2.3.3.39}$$

where Ω_α is the isometry gauge field. (We use this terminology in order to avoid confusion with the open string excitation mode A_μ.) The isometry gauge field Ω_α transforms under local shifts in X^9 such that $D_\alpha X^9$ is invariant. We introduce a bulk Lagrange multiplier λ in order to constrain the Ω-field strength[51]

$$f = \epsilon^{\alpha\beta}\partial_\alpha\Omega_\beta \tag{2.3.3.40}$$

to vanish. Further, we add a second boundary Lagrange multiplier κ whose task is to fix the boundary condition of Ω_α. Taking into account the Lagrange multipliers, the gauged action reads[52]

$$\begin{aligned} S^{(9)}_{gauged} &= \frac{1}{2\pi}\int_{\mathcal{M}} d^2z\,\left(\partial_\alpha X^9\partial^\alpha X^9 + \Omega_\alpha\Omega^\alpha + 2\Omega_\alpha\partial^\alpha X^9 - 2\epsilon^{\alpha\beta}\Omega_\beta\partial_\alpha\lambda\right) \\ &\quad +\frac{1}{2\pi}\int_{\partial\mathcal{M}} dt\,\left(A_9\partial_t X^9 + V_9\partial_n X^9\right) \\ &\quad +\frac{1}{2\pi}\int_{\partial\mathcal{M}} dt\,\left(A_9 t_\alpha + V_9 n_\alpha + \kappa b_\alpha + 2\lambda t_\alpha\right)\Omega^\alpha, \end{aligned} \tag{2.3.3.41}$$

where for later convenience we have performed partial integrations such that no derivative of Ω_α appears in the action. The worldsheet vector t_α denotes the tangent vector to the boundary. The T-dual model will be obtained by integrating out Ω_α. The T-dual coordinate will be λ. Its boundary conditions are going to be fixed by the integration over κ. Before going through the steps of this prescription, let us verify that the gauged action is equivalent to the ungauged one. Integration over λ leads to

$$\Omega_\alpha = \partial_\alpha\rho, \tag{2.3.3.42}$$

where ρ is an arbitrary worldsheet scalar. Integrating out κ leads to the boundary condition

$$b^\alpha\partial_\alpha\rho = 0. \tag{2.3.3.43}$$

Because neither the background (nor b^α) depend on X^9, the scalar ρ can be absorbed completely into a redefinition of X^9 without spoiling the boundary condition (2.3.3.36). (In addition ρ needs to live on a circle with radius equals the compactification radius.

[51] In two dimensions we can hodge dualize the two form field strength to a scalar f.

[52] As before we write the case of D and N boundary conditions into one formula. Recall that $A_9 = 0$ and $b_\alpha = t_\alpha$ for D boundary conditions, and $V_9 = 0$ and $b_\alpha = n_\alpha$ for N boundary conditions.

This issue has been addressed in [15]. The discussion given there leads to the observation that λ lives on a circle with inverted radius.) Hence, the gauged and ungauged models are equivalent.

In order to construct the T-dual model we first integrate out Ω_α. Because the action (2.3.3.41) does not contain any derivatives of Ω_α (it is ultra local with respect to the isometry gauge field), the functional integral over Ω_α factorises into a bulk integral and a boundary integral

$$\int \mathcal{D}\Omega_{\mathcal{M}\cup\partial\mathcal{M}}\,(\dots) = \int \mathcal{D}\Omega_{\mathcal{M}}\,(\dots) \times \int \mathcal{D}\Omega_{\partial\mathcal{M}}\,(\dots)\,. \tag{2.3.3.44}$$

Integrating out Ω in the bulk leads to the ungauged bulk action with X^9 replaced by λ. This is exactly as in the closed string computation (up to a boundary term)

$$\begin{aligned} \tilde{S}^{(9)}_{bulk} &= \frac{1}{2\pi}\int_{\mathcal{M}} d^2z\,\left(\partial_\alpha\lambda\partial^\alpha\lambda + 2\epsilon^{\alpha\beta}\partial_\alpha\lambda\partial_\beta X^9\right) && (2.3.3.45)\\ &= \frac{1}{2\pi}\int_{\mathcal{M}} d^2z\,\partial_\alpha\lambda\partial^\alpha\lambda + \int_{\partial\mathcal{M}} dt\,2\lambda\partial_t X^9, && (2.3.3.46) \end{aligned}$$

where in the second line we have used Stokes theorem.

The additional ingredient comes from the second factor in (2.3.3.44). This gives a two dimensional delta function

$$\int \mathcal{D}\Omega_{\partial\mathcal{M}} e^{-S_{gauged,\partial\mathcal{M}}} \sim \delta^2\left(A_9 t_\alpha + V_9 n_\alpha + \kappa b_\alpha + 2i\lambda t_\alpha\right). \tag{2.3.3.47}$$

Let us evaluate this delta function for the two cases: X^9 has Dirichlet boundary conditions ($b_\alpha = t_\alpha$) or Neumann conditions ($b_\alpha = n_\alpha$). In the first case, the evaluation of the delta function fixes κ in terms of λ and sets $V_9 = 0$. This means that λ has free varying ends, i.e. Neumann boundary conditions. Taking into account the boundary term in (2.3.3.46) we obtain that the dual $U(1)$ gauge field is determined by the position of the original D-brane,

$$\tilde{A}_\lambda = -2{X^9}_{|\partial\mathcal{M}}\,. \tag{2.3.3.48}$$

Recall that the original Dirichlet boundary condition may depend on the other directions, i.e. the rhs of (2.3.3.48) is some fixed function.

If X^9 satisfies Neumann conditions, the evaluation of the delta function leads to $\kappa = 0$ and the Dirichlet boundary condition

$$\lambda_{|\partial\mathcal{M}} = -\frac{1}{2}A_9. \tag{2.3.3.49}$$

In the T-dual string theory there is a D-brane located in x^9 along the curve A^9 (note that A_9 may depend on the coordinates different from x^9). Note also that plugging the boundary condition (2.3.3.49) into (2.3.3.46) cancels the original A_9 coupling to the boundary.

To summarize, we have seen that T-duality interchanges Dirichlet with Neumann boundary conditions. The position of the D-brane is interchanged with the $U(1)$ gauge field component in the T-dualized directions. Starting with Neumann boundary conditions it is easy to see that gauge transformations do not change the sigma model for the string, i.e. the field equations of the string excitations do not depend on gauge transformations. Via T-duality this translates to changes of the position of a D-brane, in particular constant shifts are moduli of the theory. From the above expressions it is also clear that performing the T-duality twice will result in the original theory.

With these considerations we can go back to the effective action (2.3.3.20) and generalize it to non space filling branes. This is done by simply replacing the A_μ components where μ labels a Dirichlet direction by scalars. These scalars are the collective coordinates of the lower dimensional D-brane. One can also parameterize the worldvolume of the D-brane by an arbitrary set of parameters. In this case one needs to replace bulk fields by the induced quantities. The effective D-brane action for lower dimensional D-branes can be also computed in the sigma model approach directly. This has been done in [315].

Finally, let us comment briefly on the case of multiple branes. We start with Neumann boundary conditions. The gauge field A_9 is now a matrix. Suppose that this matrix is diagonal. In this case the above discussion is valid if we just replace A_9 by a diagonal matrix everywhere. In the T-dual theory, the position of the D-brane is a diagonal matrix. The interpretation is that each entry corresponds to the position of a single D-brane. The matrix describes a set of D-branes. The more general case of non-diagonal gauge fields is rather complicated. It is addressed e.g. in[143, 140, 141].

2.3.3.3 RR fields

So far, we have discussed D-brane effective actions only for trivial RR backgrounds. The reason was mainly of technical origin. It is rather complicated to describe non-trivial RR backgrounds in a sigma model approach. Later in section 4.3, we will use such a description for a particular background. Now, we will not discuss the RR background in a sigma model. Instead we will use our computation of section 2.3.2 and field theoretic arguments.

In section 2.3.2 we have seen that the Dp-brane carries RR charge with respect to a $p+1$ form RR gauge potential of type II theories. In section 2.3.3.1 we argued that the interaction of D-branes via closed strings is obtained by adding the effective D-brane action to the effective type II action (IIA for even p, and IIB for odd p).

Combining these two observations, we infer that the effective D-brane action contains an additional piece

$$S_1 = S_{DBI} + \frac{\sqrt{\pi}}{\kappa}\left(4\pi^2\alpha'\right)^{\frac{3-p}{2}}\int d^{p+1}x A_{1,\dots,p+1}, \tag{2.3.3.50}$$

where we assume that the D-brane worldvolume extends along the first $p+1$ dimensions. (In general, the D-brane can be parameterized by a set of $p+1$ parameters. In this case, the D-brane action is written in terms of induced fields.) We have abbreviated the action (2.3.3.20) with S_{DBI}. Further, we used the result (2.3.2.31) to fix the coefficient in front of the RR coupling.

The label in S_1 has been introduced because now we will argue that there are further couplings to RR fields. These occur if another a D-brane lies within the worldvolume of the considered D-brane, or a D-brane intersects the considered D-brane. In such a case there will be strings starting and ending on different D-branes. They give rise to massless fields transforming in the fundamental representation of the gauge group living on the considered D-brane. Under certain circumstances there may be chiral fermions leading to potential gauge anomalies. Such anomalies can be canceled by assigning anomalous gauge transformations to certain bulk RR fields and adding an interaction term to the effective D-brane action. This procedure has been carried out in detail in[218, 145]. Here, we just briefly give the result.

In cases that there is an anomaly, this anomaly can be canceled by adding a Chern-Simons term to the D-brane action

$$S = S_1 + S_{CS}, \tag{2.3.3.51}$$

with (for N coincident D-branes – for $N > 1$ also the DBI action needs to be modified as discussed in the end of section 2.3.3.1)

$$S_{CS} = \int_{\mathcal{B}_p} C \wedge \left(\mathrm{tr}\, e^{\frac{iF}{2\pi}}\right)\sqrt{\hat{\mathcal{A}}\left(R\right)}. \tag{2.3.3.52}$$

The way of writing the Chern-Simons term needs explanation. The integral is taken over the worldvolume of the Dp-brane which is denoted by $\mathcal{B}_p$. The integral is a formal expression in differential forms. It is understood that only $p+1$ forms out of this expression are kept.

The first form C is an RR $q+1$ form where q is the spatial dimension of the surface in which the two D-branes (or sets of D-branes) overlap. The last term contains the so called A-roof genus. This is a polynomial in the curvature two-form (for an explicit definition see e.g. [218]). In addition to adding S_{CS} to the D-brane action the RR form C receives a contribution under gauge transformations. This comes about as follows.

The definition of the RR field strength receives a correction (the correction is related to a Chern-Simons form whose explicit form is not needed here)

$$H = dC + \text{correction} \tag{2.3.3.53}$$

such that

$$dH = 2\pi\delta\left(\mathcal{B}_p \to M_{10}\right) \mathrm{Tr}_N e^{\frac{iF}{2\pi}} \sqrt{\hat{\mathcal{A}}\left(R\right)}, \tag{2.3.3.54}$$

where the delta function means that this correction is supported on the worldvolume of the D-brane, only. Even though the right hand side of (2.3.3.54) is gauge invariant, C has to change under gauge transformations in order to ensure that H is invariant. The construction is such that the change of S_{CS} under gauge transformations cancels the anomaly.

2.3.3.4 Noncommutative geometry

It is interesting to observe that the D-brane action can be expressed as a noncommutative gauge theory. Here, noncommutative must not be confused with non Abelian. It does not refer to the gauge group but to a property of space. Before sketching the connection to string theory, we will briefly give some basic ingredients of noncommutative field theory. In difference to commutative field theory it is assumed that the coordinates of $\mathbb{R}^n$ do not commute (we indicate this by putting a hat on the coordinate)

$$\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij}, \tag{2.3.3.55}$$

where we restrict to the case that θ^{ij} are c-numbers. Because of the non commuting coordinates one has to specify the ordering in say complex functions. For our purpose the Weyl ordering is appropriate. The Weyl ordering is constructed as follows. The starting point is the pair of the function and its Fourier transform in commutative space (first with commuting coordinates)

$$\phi\left(x\right) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}}} \int d^n k \, e^{ikx} \tilde{\phi}\left(k\right). \tag{2.3.3.56}$$

The Weyl ordered functions are defined by replacing the commuting coordinates x^i with the non commuting ones $\hat{x}^i$ in (2.3.3.56) (but keeping k as a commutative integration variable),

$$\phi_W\left(\hat{x}\right) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}}} \int d^n k \, e^{ik\hat{x}} \tilde{\phi}\left(k\right). \tag{2.3.3.57}$$

A natural prescription to multiply two Weyl ordered functions is

$$(\phi_W \star \psi_W)(\hat{x}) \equiv \phi(\hat{x})\,\psi(\hat{x}) = \frac{1}{(2\pi)^n}\int d^nkd^nq\, e^{-i(k+q)\hat{x}} e^{ik\hat{x}}\tilde{\phi}(q+k)\,\tilde{\psi}(-k)\,. \tag{2.3.3.58}$$

Multiplying the two exponentials on the rhs of (2.3.3.58) using the BCH formula and afterwards dropping the hat on the coordinates leads to a natural way to deform the algebra of ordinary functions (ϕ: commuting $\mathbb{R}_n \to \mathbb{C}$) by replacing the ordinary product by the Moyal product

$$(\phi \star \psi)(x) = e^{i\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}}\phi(x)\,\psi(y)_{|x=y}\,. \tag{2.3.3.59}$$

This deformed algebra is noncommutative but still associative. In the limit $\theta^{ij} \to 0$ it becomes the familiar commuting algebra (ordinary multiplication in $\mathbb{C}$).

Noncommutative field theories – as we will meet them on D-branes– are roughly obtained as follows. One takes the ordinary action for the field theory and replaces products of fields by the Moyal product (2.3.3.59). (This is only a very rough prescription since for example any “zero” can be expressed as the commutator with respect to the ordinary product which becomes something non-trivial after the deformation. An additional principle is for example given by the requirement that the deformed action should posses the same (but possibly deformed) symmetries as the commutative one.)

Our starting point for connecting D-branes to noncommutative field theory is a slightly rescaled version of the non linear sigma model (2.3.3.2)[53]

$$S = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2z\left(G_{ij}\partial_\alpha X^i\partial^\alpha X^j - 2\pi i\alpha' B_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\right). \tag{2.3.3.60}$$

A possible $U(1)$ gauge background could be absorbed into B_{ij} by use of Stoke's theorem. However, we will restrict first to the case that B_{ij} is constant, and add a $U(1)$ gauge field coupling to the boundary later. Note also that B_{ij} has now mass dimension two – the canonical dimension of a gauge field strength. We consider the case that all coordinates X^i have Neumann boundary conditions (coordinates with Dirichlet boundary conditions do not play a role here and may be added as spectators). The propagator for the X^i can be easily obtained from the expressions (2.3.3.11). The

[53] The index i instead of μ indicates here that we focus on space like target space dimensions. The rescalings of fields have been done mainly in order to achieve agreement with the literature[418].

redefined quantities are

$$\left(\hat{G}^{-1}\right)^{ij} = \left(\frac{1}{G+2\pi\alpha' B}\right)_S^{ij} = \left(\frac{1}{G+2\pi\alpha' B} G \frac{1}{G-2\pi\alpha' B}\right)^{ij}, \quad (2.3.3.61)$$

$$\hat{G}_{ij} = G_{ij} - \left(2\pi\alpha'\right)^2 \left(BG^{-1}B\right)_{ij}, \quad (2.3.3.62)$$

$$\begin{aligned} \theta^{ij} &= 2\pi\alpha' \left(\frac{1}{G+2\pi\alpha' B}\right)_A^{ij} \\ &= -\left(2\pi\alpha'\right)^2 \left(\frac{1}{G+2\pi\alpha' B} B \frac{1}{G-2\pi\alpha' B}\right)^{ij}. \end{aligned} \quad (2.3.3.63)$$

In particular the open string ends propagate according to (call $z^1 = \tau$)

$$\left\langle X^i(\tau) X^j(\tau')\right\rangle = -\alpha' \left(\hat{G}^{-1}\right)^{ij} \log\left(\tau - \tau'\right)^2 + \frac{i}{2}\theta^{ij}\, \epsilon\left(\tau - \tau'\right), \quad (2.3.3.64)$$

where the epsilon function is equal to the sign of its argument, and zero for vanishing argument.

Let us pause for a moment and explain how the last term in (2.3.3.64) arises. The propagator (2.3.3.11) contains a term (a factor of α' appears now in the definition of θ^{ij} (2.3.3.63))

$$-\frac{\theta^{ij}}{2\pi} \log \frac{z - \bar{z}'}{\bar{z} - z'}. \quad (2.3.3.65)$$

We take $z = \tau + i\sigma$ (hoping that this does not cause confusion due to the fact that now τ and σ parameterize the upper half plane whereas they parameterized a strip earlier (and will so in later sections)). Ordering with respect to real and imaginary part, one obtains for (2.3.3.65)

$$-\frac{\theta^{ij}}{2\pi} \log\left(\frac{\left(\tau - \tau'\right)^2 + 2i\left(\sigma + \sigma'\right)\left(\tau - \tau'\right)}{\left(\tau - \tau'\right)^2 + \left(\sigma + \sigma'\right)^2}\right). \quad (2.3.3.66)$$

Using the relation

$$\log z = \log|z| + i\, arg\,(z)$$

and taking the limit $\sigma + \sigma' \to +0$ one obtains

$$-\frac{i}{2}\theta^{ij}\left(1 - \epsilon\left(\tau - \tau'\right)\right). \quad (2.3.3.67)$$

Dropping an irrelevant constant, this yields the last term in (2.3.3.64).

In the following we will be interested in the $\alpha' \to 0$ limit (while keeping θ^{ij} fixed), where the propagator (2.3.3.64) takes the form

$$\left\langle X^i(\tau) X^j(0)\right\rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau). \quad (2.3.3.68)$$

With this propagator one can compute the following operator product

$$: e^{ip_i x^i(\tau)} :: e^{iq_i x^i(0)} := e^{-\frac{i}{2}\theta^{ij} p_i q_j \epsilon(\tau)} : e^{ip_i X^i(\tau) + iq_i X^i(0)} :, \tag{2.3.3.69}$$

where the normal ordering means that self contractions within the exponentials are subtracted. By use of Fourier transformation one can deduce the operator product for generic functions

$$: \phi(X(\tau)) :: \psi(X(0)) := : e^{\frac{i}{2}\theta^{ij} \frac{\partial^2}{\partial X^i(\tau)\partial X^j(0)}} \phi(X(\tau)) \psi(X(0)) : . \tag{2.3.3.70}$$

In the limit of coincident arguments the operator product can be related to the Moyal product

$$\lim_{\tau \to +0} : \phi(X(\tau)) \psi(X(0)) := (\phi \star \psi)(X(0)) . \tag{2.3.3.71}$$

This expression suggests that we are likely to obtain noncommutative field theory if we use the limiting procedure on the lhs of (2.3.3.71) as a way to regularize composite operators. This regularization technique is known as point splitting. In composite operators well defined (normal ordered) parts are taken at different points, and then the limit to coinciding points is performed (after adding counterterms if needed).

In the following we are going to argue that we obtain an effective noncommutative theory on the D-brane if we use the point splitting regularization instead of dimensional (or Pauli-Villars) regularization. For a trivial worldsheet metric point splitting simply means that we cut off short distances by keeping

$$\left|\tau - \tau'\right| > \delta, \tag{2.3.3.72}$$

and take δ to zero in the end. First, we add the following interaction term to (2.3.3.60)

$$S_{int} = -i \int d\tau A_i(X) \partial_\tau X^i. \tag{2.3.3.73}$$

Classically this term is invariant under a gauge transformation

$$\delta A_i = \partial_i \lambda. \tag{2.3.3.74}$$

Now, we are going to observe that whether or not the partition function is invariant depends on the regularization prescription. To this end, note that δZ contains a term

$$\delta Z = -\langle \int d\tau A_i(X) \partial_\tau X^i \cdot \int d\tau' \partial_{\tau'} \lambda \rangle + \ldots . \tag{2.3.3.75}$$

Schematically this integral has the form

$$\int d\tau \int d\tau' \partial_{\tau'} f\left(\langle X^i(\tau) X^j(\tau')\rangle\right) = \int d\tau f\left(\langle X^i(\tau) X^j(\tau')\rangle\right)\Big|_{\tau'=\tau+\delta}^{\tau'=\tau-\delta} . \tag{2.3.3.76}$$

If we treat the divergence at $\tau = \tau'$ with dimensional regularization (as we did in section 2.3.3.1) this expression vanishes since it does not matter from which side we approach the singularity. (The epsilon function in the propagator is zero at $\tau = \tau'$ and the logarithms are replaced by the regularized expressions.)

If, however, we choose the point splitting method (2.3.3.72) instead, we obtain

$$\begin{aligned} \delta Z &= -\int d\tau : A_i\left(X\left(\tau\right)\right)\partial_\tau X^i\left(\tau\right) :: \lambda\left(X\left(\tau - 0\right)\right) - \lambda\left(X\left(\tau + 0\right)\right) \\ &= -\int d\tau : \left(A_i \star \lambda - \lambda \star A_i\right)\partial_\tau X^i : + \ldots\,, \end{aligned} \tag{2.3.3.77}$$

where in the second step the connection between the operator product and the Moyal product (2.3.3.71) has been used. Hence, when using the point-splitting regularization (2.3.3.72), the string partition function is not invariant under ordinary gauge transformations. However, the lack of invariance can be cured by replacing the gauge field A_i with a "noncommutative" gauge field $\hat{A}_i$ with the deformed gauge transformation

$$\hat{\delta}\hat{A}_i = \partial_i\lambda + i\lambda \star \hat{A}_i - i\hat{A}_i \star \lambda. \tag{2.3.3.78}$$

Such a transformation is a gauge symmetry in the noncommutative version of $(U\left(1\right))$ Yang-Mills theory. The gauge invariant field strength is

$$\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i - i\hat{A}_i \star \hat{A}_j + i\hat{A}_j \star \hat{A}_i. \tag{2.3.3.79}$$

Indeed, computing the effective action of the open string with the point-splitting method, one finds the noncommutative version of the Dirac-Born-Infeld action (2.3.3.20). We will not go through the details here, but refer the interested reader to[418] and further references to be given in the end of this review.

The effective D-brane action was obtained by setting open string beta functions to zero. Now, we have seen that the outcome can depend on the way we regularize singularities: commutative Dirac-Born-Infeld e.g. for dimensional regularization and noncommutative Dirac-Born-Infeld for point-splitting. From quantum field theory it is known that beta functions which differ by the way of renormalization should be identical up to redefinitions of the couplings. In our example the couplings are A_i in the commutative case, and $\hat{A}_i$ in the noncommutative one. Therefore, there should exist a field redefinition relating commutative gauge theory to noncommutative one. Indeed, such a field redefinition has been found in[418], it is sometimes called the Seiberg-Witten map.

The connection between D-branes and noncommutative field theory has many interesting aspects, which we will, however not further discuss in this review.

2.4 Orientifold fixed planes

In this section we will introduce an extended object which is called orientifold fixed plane. This is nothing but the orbifold plane of section 2.2 whenever the corresponding discrete target space mapping is combined with a worldsheet parity inversion. (Recall that an orbifold fixed plane was defined as an object being invariant under an element of a discrete group acting on the target space.)

At first we will study unoriented closed (type II) strings. These are closed strings which can be emitted or absorbed by an orientifold fixed plane. Afterwards we will investigate how orientifold fixed planes interact via closed strings. We will learn that orientifold fixed planes carry tension and RR charges. In particular, RR charge conservation implies that orientifold fixed planes cannot exist whenever they possess compact transverse dimensions. However, by adding D-branes one can construct models containing orientifold planes with transverse compact dimensions. Such constructions are known as orientifold compactifications. We will present the type I theory and an orientifold analogon of the K3 orbifold discussed in section 2.2.2. (Type I theory is actually not a compactification. Here, the orientifold planes are space filling and do not have transverse dimensions. However, the construction falls into the same category as orientifold compactifications.)

2.4.1 Unoriented closed strings

Recall the mode expansions for type II strings (now with $0 \leq \sigma < 2\pi$). The general solution to the equation of motion for the bosons is

$$X^\mu = X_R^\mu\left(\sigma^-\right) + X_L^\mu\left(\sigma^+\right), \tag{2.4.1.1}$$

with

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}, \tag{2.4.1.2}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}. \tag{2.4.1.3}$$

The mode expansions for the worldsheet fermions are

$$\psi_-^\mu = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-}, \tag{2.4.1.4}$$

$$\psi_+^\mu = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-in\sigma^+}, \tag{2.4.1.5}$$

in the R sectors, and

$$\psi_-^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^-}, \tag{2.4.1.6}$$

$$\psi_+^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-ir\sigma^+} \tag{2.4.1.7}$$

in the NS sectors.

We define an operator Ω which changes the orientation of the worldsheet. For the closed string the action of Ω is

$$\Omega:\ \sigma \leftrightarrow -\sigma. \tag{2.4.1.8}$$

For left handed fermionic modes, we introduce an additional sign such that the product of a left with a right handed fermionic mode is Ω invariant (recall that fermionic modes from the left moving sector anti-commute with fermionic modes from the right moving sector). In formulæ, this means

$$\begin{array}{rr} \Omega\alpha_n^\mu\Omega^{-1} = \tilde{\alpha}_n^\mu, \quad \Omega b_r^\mu \Omega^{-1} = \tilde{b}_r^\mu, & \Omega\tilde{b}_r^\mu\Omega^{-1} = -b_r^\mu, \\ \Omega d_n^\mu\Omega^{-1} = \tilde{d}_n^\mu, & \Omega\tilde{d}_n\Omega^{-1} = -d_n^\mu. \end{array} \tag{2.4.1.9}$$

From this we see that Ω is a symmetry in type IIB theory – the only closed superstring which is left-right symmetric. (Note that the GSO projection operator (2.1.2.59) in the R sector contains an even number of d_0's. Hence, the sign in the transformation (2.4.1.9) cancels out and e.g. P_{GSO}^+ is interchanged with $\tilde{P}_{GSO}^+$.)

Let us study the action of Ω on the massless sector of type IIB excitations. We take the vacuum to be invariant under worldsheet parity reversal. The massless NSNS sector states are (in light cone gauge)

$$b_{-\frac{1}{2}}^i \tilde{b}_{-\frac{1}{2}}^j \left|k\right\rangle \tag{2.4.1.10}$$

The action of Ω on this state interchanges the indices i and j. Thus the states surviving an Ω projection are symmetric in i, j – these are the graviton G_{ij} and the dilaton Φ. Since Ω relates the NSR with the RNS sector only invariant superpositions are kept. Thus we obtain only one gravitino (56 components) and one dilatino (8 components). Half of the target space supersymmetry is broken by the Ω projection. The massless states in the RR sector are obtained from the tensor product of the left with the right moving R vacuum. The R vacua are target space spinor components and Ω interchanges the left with the right moving vacuum. Because spinor components anti-commute the antisymmetrized tensor product survives the Ω projection. This is the 28 dimensional $SO(8)$ representation – the antisymmetric tensor B'_{ij}. We obtain the field content of the heterotic string without the internal fermions λ_+^A. As we stated before, a theory with such a massless spectrum suffers from gravitational anomalies. In the heterotic theory we actually needed 32 worldsheet fermions λ_+^A whose quantization provided exactly the gauge multiplets needed to obtain an anomaly free massless spectrum. Later we will see that one needs to add D-9-branes to the unoriented type IIB theory, for consistency. Before going into that let us study for a while the unoriented closed stringtheory – even though it is not consistent yet.

The theory of unoriented type IIB strings contains orientifold-nine-planes – or short O-9-planes. An O-plane is a set of target space points which is fixed under an element of a discrete group which contains Ω (the element must contain Ω). Because Ω alone does not act on the target space geometry the full target space is fixed under Ω. The fixed set of points is space filling – it is an O-9-plane.

We have seen that when we compactify the type IIB string on a circle and perform a T-duality we obtain type IIA theory compactified on a circle with inverted radius. Let us study what happens to the O-9-planes in this process. Formally, we have the expression (X^9 stands for the bosonic string coordinate)

$$\Omega X^9 \Omega^{-1} \stackrel{\text{T-DUALITY}}{\longrightarrow} T\Omega T^{-1} T X^9 T^{-1} \left(T\Omega T^{-1}\right)^{-1}. \tag{2.4.1.11}$$

We want to know the T-dual of Ω which is denoted by $T\Omega T^{-1}$. This can be computed as follows. We first perform a T-duality, then act with Ω on the T-dual coordinate, and finally T-dualize back. These steps are collected in the following diagram (we use (2.1.5.31) for T-duality)

$$X_L^9 + X_R^9 \xrightarrow{T} X_L^9 - X_R^9 \xrightarrow{\Omega} X_R^9 - X_L^9 \xrightarrow{T^{-1}} -X_R^9 - X_L^9 \tag{2.4.1.12}$$

Thus we see that $T\Omega T^{-1}$ reflects the dimension in which T acts, and also interchanges left with right movers (the second statement can be easily verified by drawing the diagram (2.4.1.12) for the left or right moving piece alone). Thus, for T-duality in X^9 direction we can write

$$T\Omega T^{-1} = R_9 \Omega, \tag{2.4.1.13}$$

where R_9 is the $\mathbb{Z}_2$ element

$$R_9 : X^9 \rightarrow -X^9. \tag{2.4.1.14}$$

The action on the worldsheet fermions can be studied likewise. Now we go to the decompactification limit on the type IIA side. Instead of an O-9-plane we have an O-8-plane, because now only points with $X^9 = 0$ are fixed under the action of ΩR_9. Repeating this argumentations for more than one T-dualized circle we conclude that

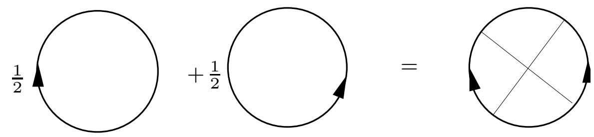

Figure 2.12: The superposition of two strings with opposite orientation can be viewed as a crosscap. The crosscap is a circle with diagonally opposite points being identified.

we have O-p-planes with even (odd) p in type IIA (B) theory. For an O-p-plane with even p, Ω comes combined with a $\mathbb{Z}_2$ operator reflecting an odd number of dimensions. In particular, this combination interchanges e.g. P^+_{GSO} with $\tilde{P}^-_{GSO}$, i.e. it is indeed a symmetry of type IIA strings. The closed string is unoriented only when it is located on an O-plane. A string off the O-plane is oriented. Its counterpart with the opposite orientation is the R_9 image of the string.

2.4.2 O-plane interactions

An O-plane is defined as an object where closed strings become unoriented when they hit it. Topologically this can be depicted by a crosscap as illustrated in figure 2.12.

The opposite process is a crosscap decaying into a pair of strings with different orientations. Only one string out of this pair is physical – the other one is the ΩR image, where R now stands for a target space mapping leaving the O-plane fixed. Thus O-planes can emit or absorb oriented strings. They possibly interact via the exchange of closed oriented strings. This indicates that there is an interaction among O-planes and between D-branes and O-planes. We are going to study these interactions in the following two subsections.

2.4.2.1 O-plane/O-plane interaction, or the Klein bottle

In figure 2.13 we have drawn a process in which two O-planes interact via the exchange of closed strings. We restrict to the special case that the orientifold group element ΩR squares to one. (Combining orbifold compactifications with orientifolds, one can have the more general situation that the orientifold group elements square to a nontrivial orbifold group element. This has to be the same for the two O-planes. Then a twisted sector closed string is exchanged.)

In the following we are going to compute this process. As in the D-brane computation, we take the O-planes to be parallel. The range for the worldsheet coordinates is

$$0 \leq \sigma < 2\pi \ , \ 0 \leq \tau < 2\pi l. \qquad (2.4.2.1)$$

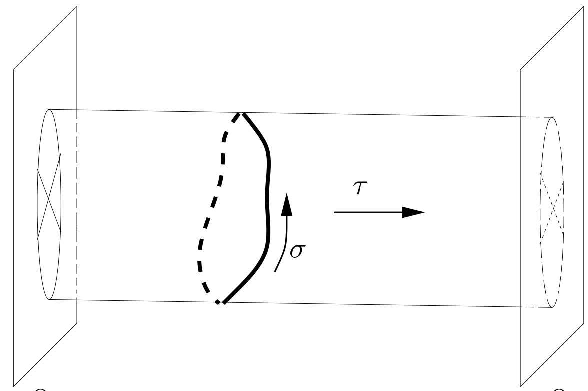

Figure 2.13: O-plane O_i and O-plane O_j talking to each other by exchanging a closed string.

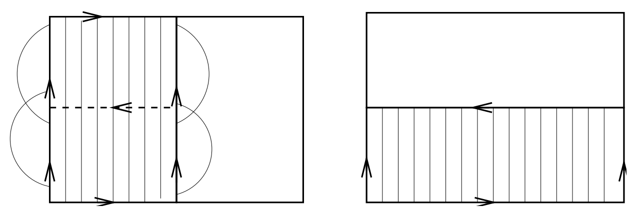

Figure 2.14: The triangulated version of fig. 2.13 on the left. By manipulations preserving the topology this can be mapped onto the triangulated version of a Klein bottle on the right.

Like in section 2.3.2 we want to perform the computation in the scheme where the role of τ and σ are interchanged – i.e. in the worldsheet dual channel. In this dual channel, a virtual pair of closed strings pops out of the vacuum – one of the strings changes its orientation before they rejoin. Therefore, this is called the loop channel. Before performing the transformation to the loop channel, we need to describe the tree channel process fig. 2.13 such that it is periodic in time. The method of doing so differs slightly from the D-brane/D-brane interaction. It is best explained by looking at the triangulated version of picture 2.13 and its double cover which is a torus. We draw this in figure 2.14. In the left picture, the shaded region is the triangulated version of figure 2.13. The half-circles indicate the identifications of the crosscaps. The white region shows our intention to obtain a description which is periodic in τ, with a period $4\pi l$. Now, one cuts the shaded part along the dotted line (with the indicated orientation), and flips the upper rectangle once around its right vertical edge and afterwards shifts it down in the vertical direction. We obtain a process which is indeed periodic in τ, and now $\tau \in [0, 4\pi l)$. (This periodicity appears due to the crosscap identifications indicated by the left half circles. The crosscap identifications for the other O-plane ensure that one can glue the upper rectangle to the lower one after the flip and the shift.) It is difficult to describe this process as a tree channel closed string exchange. Instead we can interchange the roles of σ and τ. Then the interpretation is that a pair of closed strings of length $4\pi l$ pops out of the vacuum, one of the closed strings changes its orientation before they annihilate after a worldsheet time π. This is a vacuum loop amplitude which has the topology of the Klein bottle. The parameter range is

$$0 \leq \sigma < 4\pi l \ , \ 0 \leq \tau < \pi. \tag{2.4.2.2}$$

As in the D-brane case (section 2.3.2) we want to rescale the dualized worldsheet coordinates such that their ranges are the canonical ones, which (now for the closed string) are

$$0 \leq \sigma < 2\pi \ , 0 \leq \tau < 2\pi t. \tag{2.4.2.3}$$

This can be achieved by the redefinitions

$$\tau \rightarrow \tau 2t \ , \sigma \rightarrow \frac{\sigma}{2l}. \tag{2.4.2.4}$$

For the Hamiltonian this induces a rescaling

$$H \rightarrow 2l\,(2t)^2\,H. \tag{2.4.2.5}$$

Analogous to the annulus discussion in section 2.3.2 we require that the action of the rescaling on the time evolution operator is

$$e^{-\pi H} \rightarrow e^{-2\pi t H}. \tag{2.4.2.6}$$

This yields a relation between l and t

$$lt = \frac{1}{4}. \tag{2.4.2.7}$$

Periodic boundary conditions on fermions along the vertical axis of the lhs of figure 2.14 correspond to a $(-)^F = (-)^{\tilde{F}}$ insertion whenever the vertical axis is identified with the worldsheet time on the rhs of figure 2.14. Only closed strings for which the rightmoving $(-)^F$ eigenvalue equals the leftmoving one contribute to the Klein bottle amplitude. (In the R sector an additional sign may occur depending on whether we are looking at type IIA or IIB strings. This does not matter here since the R-sector contributions with a $(-)^F$ insertion vanish anyway.)

Since the connection between tree level periodicities and loop channel insertions is a bit less obvious than in the D-brane/D-brane interaction, let us explain it in some detail here. We take the parameter range (2.4.2.3). We are interested in the behavior of worldsheet fermions under shifts in τ by $4\pi t$. Periodic behavior corresponds to tree level RR exchange whereas anti-periodicity translates to NSNS exchange. Fig. 2.14 tells us how to continue in τ beyond $2\pi t$

$$\psi_{\pm}^{\mu}\left(\tau + 4\pi t, \sigma\right) = \left(-\right)^{F+\tilde{F}} \psi_{\pm}^{\mu}\left(\tau + 2\pi t, 2\pi - \sigma\right), \tag{2.4.2.8}$$

where the $\left(-\right)^{F+\tilde{F}}$ reflects the boundary condition on worldsheet fermions under $2\pi t$ shifts in τ. However, in the Klein bottle amplitude only states with $\left(-\right)^{F} = \left(-\right)^{\tilde{F}}$ contribute because of an Ω insertion in the trace over states. Therefore, the additional factor in (2.4.2.8) is not relevant. Now, the $2\pi t$ shift in τ can be replaced by acting with the trace insertion $\left(-\right)^{F}\Omega$. The $\left(-\right)^{F}$ insertion just cancels a sign included in the definition of the trace over fermions for the right movers. By the same token we have to insert a $\left(-\right)^{\tilde{F}}$ for the left movers. Thus we obtain

$$\psi_{\pm}^{\mu}\left(\tau + 4\pi t, \sigma\right) = \left(-\right)^{\tilde{F}}\Omega\psi_{\pm}^{\mu}\left(\tau, 2\pi - \sigma\right)\Omega^{-1} = \psi_{\pm}^{\mu}\left(\tau, \sigma\right), \tag{2.4.2.9}$$

where in the last step we used our definition for Ω (2.4.1.9) and the 2π periodicity in σ. Thus the $\left(-\right)^{F}$ insertion in the loop channel filters out the RR tree level exchange, indeed. Strictly speaking the above consideration is correct only when the fermions point in directions longitudinal to the O-plane (where the $\mathbb{Z}_2$ reflection R acts as the identity). For the other directions there are two signs canceling each other and leading to the same result. At first, there is an additional minus sign in (2.4.2.8) because the half-circles in fig. 2.14 now contain also the (non-trivial) action of the $\mathbb{Z}_2$ reflection R. This sign is canceled when we replace Ω by ΩR in (2.4.2.9).

We want to filter out the contribution due to RR exchange in the tree channel. Then, the loop channel vacuum amplitude is given by the following expression

$$\begin{aligned}
&\int \frac{dt}{2t}\,\mathrm{Str}\left(\Omega R \frac{\left(-\right)^{F}}{2} e^{-2\pi\alpha' t H}\right) = \\
&\quad \int \frac{dt}{2t}\mathrm{tr}_{\substack{\text{ZERO}\\\text{MODES}}}\left(\Omega R e^{-2\pi\alpha' t H_0}\right)\mathrm{tr}_{\text{BOSONS}}\left(\Omega R e^{-2\pi t\left(H_B + \tilde{H}_B\right)}\right) \\
&\quad \mathrm{tr}_{\substack{\text{NSNS}\\\text{FERMIONS}}}\left(\Omega R \frac{\left(-\right)^{F}}{2} e^{-2\pi t\left(H_{NS} + \tilde{H}_{NS}\right)}\right).
\end{aligned} \tag{2.4.2.10}$$

Here, we split the Hamiltonian into right and left moving parts $H + \tilde{H}$ and these in turn into

$$\alpha' H = \alpha' H_0 + H_B + H_{NS} \tag{2.4.2.11}$$

with[54]

$$H_0 = \frac{p^2}{4} \tag{2.4.2.12}$$

$$H_B = \sum_{i=1}^{8}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n - \frac{1}{24}\right), \tag{2.4.2.13}$$

$$H_{NS} = \sum_{i=1}^{8}\left(\sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r}b^i_r - \frac{1}{48}\right), \tag{2.4.2.14}$$

and the corresponding expressions for the left moving sector. The ΩR insertion projects out contributions of states with zero mode momenta perpendicular to the O-planes, since those states are mapped onto states with the negative momentum in the perpendicular direction by the ΩR insertion. The result for the zero mode contribution reads

$$\mathrm{tr}_{\substack{\text{ZERO}\\ \text{MODES}}} = 2V_{p+1}\int \frac{d^{p+1}k}{(2\pi)^{p+1}}\, e^{-2\pi\alpha' t\frac{k^2}{2}} = 2\cdot 2^{\frac{p+1}{2}} V_{p+1}\left(8\alpha'\pi^2 t\right)^{-\frac{p+1}{2}}. \tag{2.4.2.15}$$

Also here there is an additional factor of two, due to the possible orientations of the closed string. (The trace is taken over oriented strings with an Ω insertion. Another point of view would be that one needs to add to the picture on the rhs of figure 2.14 a picture with reversed orientations on the horizontal edges.) For the traces over excited states we note that the insertion ΩR in the trace means that only states contribute which are eigenstates of ΩR. This means that the left moving excitations have to be identical to the right moving ones. Thus, it is straightforward to modify the calculation presented in section 2.3.2 by just changing the power of the arguments in the functions 2.3.2.20 by two (since the identical left and right moving contributions add). We obtain

$$\mathrm{tr}_{\text{BOSONS}}\left(\Omega R e^{-2\pi t\left(H_B+\tilde{H}_B\right)}\right) = \frac{1}{f_1^8\left(e^{-2\pi t}\right)} \tag{2.4.2.16}$$

for the trace over the bosonic excitations, and

$$\mathrm{tr}_{\substack{\text{NSNS}\\ \text{FERMIONS}}}\left(\Omega R\frac{(-)^F}{2}e^{-2\pi t\left(H_{NS}+\tilde{H}_{NS}\right)}\right) = -\frac{1}{2}f_4^8\left(e^{-2\pi t}\right). \tag{2.4.2.17}$$

Thus we obtain

$$\int\frac{dt}{2t}\mathrm{Str}\left(\Omega R\frac{(-)^F}{2}e^{-2\pi t H}\right) =$$
$$-\frac{1}{2}2^{\frac{p+1}{2}}V_{p+1}\int\frac{dt}{2t}\left(8\alpha'\pi^2 t\right)^{-\frac{p+1}{2}}\frac{f_4^8\left(e^{-2\pi t}\right)}{f_1^8\left(e^{-2\pi t}\right)}. \tag{2.4.2.18}$$

[54]From our treatment in section 2.1.2.2 we would get a factor of $\alpha' 8/2 = 2$ instead of $\alpha' 4/2 = 1$ in the oscillator contributions. Recall, however, that we have changed the σ range from $[0,\pi)$ to $[0,2\pi)$, meanwhile. We have distributed the zero mode contribution symmetrically on H and $\tilde{H}$. Taking into account the effect of rescaling, this gives the factor of 1/4.

Now we undo the worldsheet duality by expressing t in terms of l (2.4.2.7). We use the transformation properties (2.3.2.22) and take the limit $l \to \infty$ in which the contribution of massless closed string excitations dominates. This leads to the expression

$$-\frac{1}{2}\int_{l\to\infty} dl\, 2^{p+1}\, V_{p+1}\left(4\alpha'\pi^2\right)^{-\frac{p+1}{2}} l^{\frac{p-9}{2}}. \tag{2.4.2.19}$$

We see that the result has almost the same structure as the one we obtained for the D-brane/D-brane interaction in table 2.7. (Recall that now, we separated out the RR sector exchange.) The differences are that we do not have the exponential dependence on the distance and that we do have an additional factor of 2^{p+1}. The explanation for the missing exponent is very simple. Since the orientifold planes are all located at a fixed point of the $\mathbb{Z}_2$ action R, they cannot be separated in target space. (However, we could for example compactify the dimensions transverse to the brane. In that case winding modes would play the role of the distance.)

Before we can deduce the ratio of the O-plane RR to the D-brane RR charge, we need to discuss a subtlety appearing because we have modded out reflections in the transverse directions. This has the effect that each transverse direction is "half as long" as in the D-brane computation. The implications of this effect are best seen in a field theory consideration. The field theory result gives a "Coulomb potential" which is of the structure charge-squared times density. (The density appears as the inverse of a second order differential operator.) The charge is obtained as an integral over the transverse space (analogous to $Q = \int d^3x j^0$ in electro-dynamics). In the O-plane case this gives a factor of a half per transverse direction as compared to the D-brane/D-brane interaction. On the other hand the density is multiplied by a factor of two per transverse direction. Hence, the overall net-effect of this transformations is an additional factor of 2^{p-9} which we need to put by hand into the O-plane/O-plane result, before we can compare it with the D-brane/D-brane calculation.[55] Taking this into account, the ratio of the D-p-brane RR charge μ_p to the O-p-plane RR charge μ_p'

$$\mu_p' = \mp 2^{p-4}\mu_p. \tag{2.4.2.20}$$

We cannot fix the sign by the present calculation since the charges enter quadratically the expressions we derived so far. Computing also the contributions without the $(-)^F$ insertions to the Klein bottle, one obtains the square of the O-plane tension. Here, we infer the result by supersymmetry instead. Since the ΩR projection leaves half of the

[55] If we performed a detailed field theory calculation we would find this factor due to the different target spaces (as argued in the text). Later, we will compactify the transverse dimensions. Then this factor will appear "automatically" due to a Poisson resummation of the sum over the winding modes. This must be the case since in the compactified theory D-branes and O-Planes will have the same transverse space.

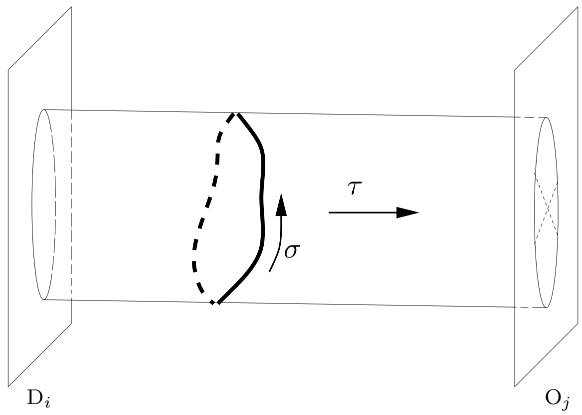

Figure 2.15: D-brane D_i and O-plane O_j talking to each other via the exchange of closed strings.

supersymmetries unbroken, the total one loop amplitude should vanish. This tells us that the ratio of the D-brane tension T_p to the O-plane tension T_p' is

$$T_p' = \mp 2^{p-4} T_p, \tag{2.4.2.21}$$

where at the present stage of the calculation the sign is not known. In order to fix the signs in (2.4.2.20) and (2.4.2.21) we need to study the interaction between D-branes and O-planes. We will do so in the next subsection.

2.4.2.2 D-brane/O-plane interaction, or the Möbius strip

So far, we have seen that D-branes as well as O-planes interact via the exchange of closed type II strings. This suggests that also D-branes interact with O-planes. Such a process is depicted in figure 2.15. We consider the case of parallel D-branes and O-planes. This implies that the D-brane is located in directions where the $\mathbb{Z}_2$ reflection acts with a sign and extended along the other directions.

Again, the range for the worldsheet coordinates is

$$0 \leq \sigma < 2\pi \quad , \quad 0 \leq \tau < 2\pi l. \tag{2.4.2.22}$$

In order to understand how to perform the worldsheet duality transformation it is useful to study the triangulated version of the diagram 2.15. The result of this investigation is drawn in figure 2.16. The right picture is obtained by cutting the left one along the dashed line flipping the upper rectangular around its right edge and afterwards shifting it down. Looking at the left picture with time passing along the vertical axis we see a process in which a pair of open strings pops out of the vacuum. Both ends of the strings are in the worldvolume of the brane D_i. As time goes by one

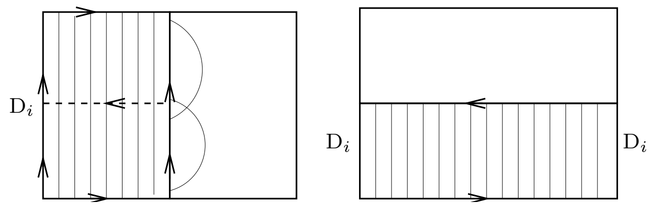

Figure 2.16: The triangulated version of fig. 2.15 on the left. By manipulations preserving the topology this is mapped onto the triangulated version of a Möbius strip on the right.

of the open strings changes its orientation before they finally annihilate. The topology of this diagram is called Möbius strip (or Möbius band).

The range for the worldsheet coordinates after interchanging the role of time and space is

$$0 \leq \sigma < 4\pi l \quad , \quad 0 \leq \tau < \pi, \tag{2.4.2.23}$$

whereas the canonical range for the open string parameters is

$$0 \leq \sigma < \pi \quad , \quad 0 \leq \tau < 2\pi t. \tag{2.4.2.24}$$

Hence, we perform the rescaling

$$\tau \rightarrow \tau 2t \quad , \quad \sigma \rightarrow \frac{\sigma}{4l}. \tag{2.4.2.25}$$

For the Hamiltonian this induces

$$H \rightarrow 4l\,(2t)^2\,H. \tag{2.4.2.26}$$

Finally the time evolution operator should take its canonical form

$$e^{-\pi H} \overset{!}{\rightarrow} e^{-2\pi t H}. \tag{2.4.2.27}$$

This tells us how to relate l and t

$$lt = \frac{1}{8}. \tag{2.4.2.28}$$

Now, we would like to identify which of the loop channel contributions corresponds to an RR exchange in the tree channel. Periodicity under $4\pi t$ shifts in τ translates to RR tree level exchange and anti-periodicity to NSNS tree level exchange. We use fig. 2.15 to identify

$$\psi^\mu_\pm\,(\tau + 4\pi t, \sigma) = (-)^F\,\psi^\mu_\pm\,(\tau + 2\pi t, \pi - \sigma)\,, \tag{2.4.2.29}$$

where the factor of $(-)^F$ appears due to the anti-periodic boundary condition of worldsheet fermions under shifts of $2\pi t$ in τ. Let us study the case where we insert in the loop channel trace just Ω (possibly combined with some target space $\mathbb{Z}_2$ action which we will discuss below). Taking into account the sign when a trace is taken over worldsheet fermions we obtain

$$\begin{aligned} \psi^\mu_\pm(\tau + 4\pi t, \sigma) &= \Omega\psi^\mu_\pm(\tau, \pi - \sigma)\,\Omega^{-1} \\ &= \Omega\psi^\mu_\mp(\tau, \sigma - \pi)\,\Omega^{-1}, \end{aligned} \tag{2.4.2.30}$$

where in the second step we have used the mode expansions (2.3.1.22)–(2.3.1.29). In the open string sector we define the action of Ω as taking $\sigma \to \pi - \sigma$. This finally results in

$$\begin{aligned} \psi^\mu_\pm(\tau + 4\pi t, \sigma) &= \psi^\mu_\mp(\tau, 2\pi - \sigma) \\ &= \psi^\mu_\pm(\tau, \sigma - 2\pi), \end{aligned} \tag{2.4.2.31}$$

where once again the mode expansion has been used. We deduce that open string R sector contributions correspond to closed string RR exchange. (This can also easily be seen in the mode expansions (2.3.1.22)–(2.3.1.29)). The above consideration is correct only in NN directions (in directions in which the D-brane extends). For DD directions there are a couple of signs which cancel each other such that one gets the same result. Since the $\mathbb{Z}_2$ reflection R acts with a sign in those directions, the first line in (2.4.2.30) receives an additional minus sign. Looking at the mode expansion (2.3.1.22)–(2.3.1.29) in DD directions we observe that this sign is canceled when going to the second line in (2.4.2.30). Because now we need to replace Ω by ΩR, the first line in (2.4.2.31) contains an additional minus sign which again is canceled by using the DD mode expansion when going to the second line in (2.4.2.31).

In the above consideration we have only specified how Ω acts on the oscillators, and not how it acts on the vacuum. (In the closed string we tacitly took the vacuum as being invariant under Ω leaving in the NSNS sector the metric invariant and projecting out the B-field.) The computations of the D-brane/D-brane and the O-plane/O-plane interactions provided the absolute values of the corresponding RR charges. The result for the D-brane/O-plane calculation will give the product of the O-plane times the D-brane charge. This should be compatible with our previous result. As we will see in a moment this leaves the two choices that the ΩR eigenvalue of the open string R-vacuum is ± 1. We will choose the minus sign. This corresponds to a D-brane. The action on the NS sector can be inferred by supersymmetry, i.e. it should be such that the complete one loop Möbius strip amplitude vanishes. The result is that the massless states have eigenvalue minus one. (This holds as well for Neumann directions as for

Dirichlet directions, since a sign due to the different mode expansions cancels a sign due to the non-trivial action of R in Dirichlet directions.)

Now, we have collected all the necessary information needed to write down the loop channel amplitude which gives the tree channel RR exchange (recall that a $(-)^F$ insertion leads to a vanishing R sector trace)

$$\begin{aligned} -\int \frac{dt}{2t} \mathrm{tr}_R \Omega R \frac{1}{2} e^{-2\pi\alpha' H} = \\ -\int \frac{dt}{2t} \mathrm{tr}_{\substack{\text{ZERO}\\ \text{MODES}}} \left(\Omega R e^{-2\pi t\alpha' H_0}\right) \mathrm{tr}_{\text{BOSONS}} \left(\Omega R e^{-2\pi t H_B}\right) \\ \mathrm{tr}_{\substack{\text{R}\\ \text{FERMIONS}}} \left(\Omega R \frac{1}{2} e^{-2\pi t H_R}\right). \end{aligned} \tag{2.4.2.32}$$

The expressions for the Hamiltonians can be directly taken from (2.3.2.14)–(2.3.2.18) with the difference that we put $y = 0$ in (2.3.2.15) (because of the ΩR insertion in the trace only D-branes at distance zero from the O-plane contribute). With this difference the trace over the zero modes gives (see (2.3.2.19)

$$\mathrm{tr}_{\substack{\text{ZERO}\\ \text{MODES}}} = 2V_{p+1} \int \frac{d^{p+1}k}{(2\pi)^{p+1}}\, e^{-2\pi t\alpha' k^2} = 2V_{p+1} \left(8\pi^2 \alpha' t\right)^{-\frac{p+1}{2}}, \tag{2.4.2.33}$$

From the mode expansion (2.3.1.20), (2.3.1.21) we learn that

$$\Omega R\, \alpha^\mu_{-n} \left(\Omega R\right)^{-1} = (-1)^n\, \alpha^\mu_{-n}. \tag{2.4.2.34}$$

Modifying the expression (2.3.2.23) accordingly we obtain

$$\mathrm{tr}_{\text{BOSONS}}\, \Omega R e^{-2\pi t H_B} = e^{-i\pi\frac{2}{3}} \frac{1}{f_1^8\left(e^{-\pi(t+\frac{i}{2})}\right)}. \tag{2.4.2.35}$$

The next step is to split the product over integers in the definition of f_1 (2.3.2.20) into a product over even times a product over odd numbers. This gives finally

$$\mathrm{tr}_{\text{BOSONS}}\, \Omega R e^{-2\pi t H_B} = \frac{1}{f_1^8\left(e^{-2\pi t}\right) f_3^8\left(e^{-2\pi t}\right)}. \tag{2.4.2.36}$$

The mode expansion on the fermions (2.3.1.22)–(2.3.1.29) yields

$$\Omega R\, d^\mu_n \left(\Omega R\right)^{-1} = e^{i\pi n} d^\mu_n. \tag{2.4.2.37}$$

Manipulations analogous to the bosonic trace give the result (recall that we have chosen the ΩR eigenvalue of the R vacuum to be minus one)

$$-\mathrm{tr}_{\substack{\text{R}\\ \text{FERMIONS}}} \left(\frac{\Omega R}{2} e^{-2\pi t H_R}\right) = f_2^8\left(e^{-2\pi t}\right) f_4^8\left(e^{-2\pi t}\right), \tag{2.4.2.38}$$

where the 16-fold degeneracy of the R vacuum has been taken into account. We are interested in the contributions due to tree channel RR exchange and have computed now everything we need to obtain the result. However, in order to specify the action of ΩR on the NS vacuum one needs to compute the tree channel NSNS exchange. The requirement that this cancels the RR interaction determines the action of ΩR on the open string NS vacuum. We leave this as an exercise. The result is that the massless vector is odd under ΩR. In the computation of the open string NS sector trace it is useful to apply the identity (2.3.2.21) on the f functions with the shifted arguments and afterwards to proceed as we did above, i.e. to split the product in the definitions of the f's into a product over even and over odd numbers.

So far, we obtained the result

$$-\int \frac{dt}{2l}\Omega R \mathrm{tr}_R \frac{1}{2} e^{-2\pi\alpha' H} = V_{p+1}\int \frac{dt}{2t}\left(8\pi^2\alpha' t\right)^{-\frac{p+1}{2}} \frac{f_2^8\left(e^{-2pit}\right) f_4^8\left(e^{-2\pi t}\right)}{f_1^8\left(e^{-2\pi t}\right) f_3^8\left(e^{-2\pi t}\right)}. \quad (2.4.2.39)$$

Expressing t in terms of l via (2.4.2.28) and using the properties (2.3.2.22) yields finally the tree channel infrared asymptotics

$$2\frac{1}{2}V_{p+1}\int_{l\to\infty} dl \left(4\pi^2\alpha'\right)^{-\frac{p+1}{2}} 2^{p-4} l^{\frac{p-9}{2}}. \quad (2.4.2.40)$$

This expression has to be compared with the second line (RR contribution) of table 2.7 and (2.4.2.19), where (2.4.2.19) has to be multiplied with 2^{p-9} as discussed earlier. (For the Möbius strip we do not need to put such a factor since there is a cancellation between the O-plane charge and the density.) In (2.4.2.40) we have pulled out a factor of two. If we take the D brane distance y to zero in (2.7) we can write down the cumulative infrared asymptotics for a system consisting out of one D-brane and one O-plane (situated at the origin in the transverse space)

$$-V_{p+1}\left(4\pi^2\alpha'\right)^{-\frac{p+1}{2}}\int_{l\to\infty} dl l^{\frac{p-9}{2}}\left(1-2^{p-4}\right)^2. \quad (2.4.2.41)$$

In field theory one obtains a result proportional to $\left(\mu_p+\mu_p'\right)^2$, (recall that μ_p is the D-brane charge and μ_p' the O-plane charge). Thus we obtain finally the ratio between D-brane and O-plane RR charges

$$\mu_p' = -2^{p-4}\mu_p. \quad (2.4.2.42)$$

The Möbius strip computation fixed also the sign of this ratio. However, if we assigned an ΩR eigenvalue of $+1$ to the open string R vacuum we would obtain an additional minus sign in (2.4.2.42). There is an ambiguity here. In the next section we will use our results to construct consistent string theories containing D-branes and O-planes. In this construction this ambiguity cancels out. (In some sense it will turn out that our present choice is the "natural" one.) The ratio of the D-brane tension to the O-plane tension can be inferred by supersymmetry

$$T'_p = -2^{p-4} T_p. \tag{2.4.2.43}$$

2.4.3 Compactifying the transverse dimensions

When we are trying to compactify the transverse directions of a D-brane and/or an O-plane we immediately run into problems. These arise as follows. The equation of motion for the RR field under which the D-p-brane (or the O-p-plane) is charged reads (for otherwise trivial background)

$$d \star F_{p+2} = \star j_{p+1}, \tag{2.4.3.1}$$

where j_p is the external $U(1)$ current indicating the presence of the D-brane (O-plane). Integrating this equation over a compact transverse space gives zero for the left hand side and the D-brane (O-plane) charge on the right hand side. Therefore, the RR charge on the rhs has to vanish. To overcome this problem one may want to add D-branes and O-planes such that the net RR charge is zero. Since one needs more than one D-brane in order to achieve a vanishing net RR charge, one has to specify how ΩR acts on a set of multiple D-branes. For example it could (and actually will) happen that ΩR (anti)symmetrises strings starting and ending on different D-branes. Technically, we define a (projective) representation of the $\mathbb{Z}_2$ (generated by ΩR) on the Chan-Paton labels carried by open string in case of multiple D-branes. The generating element of this representation is denoted by $\gamma_{\Omega R}$. The ΩR action on an open string is

$$\Omega R: \qquad |\psi, ij\rangle \to (\gamma_{\Omega R})_{ii'} \left|\Omega R(\psi), j'i'\right\rangle \left(\gamma^{-1}_{\Omega R}\right)_{j'j}. \tag{2.4.3.2}$$

Here, ψ denotes the oscillator content of the string on which ΩR acts in the same way as discussed previously. In addition, the order of the Chan-Paton labels is altered due to the orientation reversal. Acting twice with ΩR should leave the state invariant. This leads to the condition

$$\gamma_{\Omega R} = \pm \gamma^T_{\Omega R}, \tag{2.4.3.3}$$

i.e. $\gamma_{\Omega R}$ is either symmetric or antisymmetric. By a choice of basis this gives the two possibilities

$$\gamma_{\Omega R} = I \quad \text{or} \quad \gamma_{\Omega R} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}. \tag{2.4.3.4}$$

Let N be the number of D-branes (ΩR images are counted). Then I denotes an $N \times N$ identity matrix for symmetric $\gamma_{\Omega R}$ and an $\frac{N}{2} \times \frac{N}{2}$ identity matrix for antisymmetric $\gamma_{\Omega R}$.

The trace in the open string amplitudes (annulus and cylinder) includes also a trace over the Chan-Paton labels. For the annulus this is simply

$$\sum_{i,j=1}^{N} \langle ij|\, ij\rangle = \sum_{i,j=1}^{N} \delta_{ii}\delta_{jj} = N^2. \tag{2.4.3.5}$$

In the Möbius strip amplitude the trace over the Chan-Paton labels yields the additional factor

$$\sum_{i,j=1}^{N} \langle ij|\, \Omega R\, |ij\rangle = \mathrm{tr}\left(\gamma_{\Omega R}^{-1}\gamma_{\Omega R}^{T}\right) = \pm N, \tag{2.4.3.6}$$

with the (lower) upper sign for (anti-) symmetric $\gamma_{\Omega R}$.

2.4.3.1 Type I/type I′ strings

In the following we are going to investigate the case where the compact space is a torus. The next issue we need to discuss are zero mode contributions due to windings in the compact transverse dimensions. For open strings windings can appear in Dirichlet directions. Since ΩR leaves the winding number of a state invariant these contribute to the annulus, the Klein bottle and the Möbius strip. Including the sum over the winding numbers into the corresponding traces leads to additional factors. The transverse space is a $9-p$–torus:

$$T^{9-p} = \underbrace{S^1 \times \cdots \times S^1}_{9-p \text{ factors}}. \tag{2.4.3.7}$$

For simplicity we take the radii of these S^1s to be identical and denote them by r. It is useful to introduce a dimensionless parameter

$$\rho = \frac{r^2}{\alpha'} \tag{2.4.3.8}$$

for the size of the compact space.

With this ingredients the sum over the winding modes gives the following factors (under the $\frac{dt}{2t}$ integral):

$$\left(\sum_{w=-\infty}^{\infty} e^{-2\pi t\rho w^2}\right)^{9-p} \qquad \text{for the annulus,} \tag{2.4.3.9}$$

$$\left(\sum_{w=-\infty}^{\infty} e^{-\pi t\rho w^2}\right)^{9-p} \qquad \text{for the Klein bottle,} \tag{2.4.3.10}$$

$$\left(\sum_{w=-\infty}^{\infty} e^{-2\pi t\rho w^2}\right)^{9-p} \qquad \text{for the Möbius strip.} \tag{2.4.3.11}$$

In the annulus we have restricted ourselves to the special case that all D-branes are situated at the same point. This configuration gives the correct leading infrared contribution to tree channel amplitude. One can also include distances among the D-branes into the computation. In that case the trace over the Chan-Paton labels cannot be directly taken as in (2.4.3.5) because the zero mode contribution depends on the Chan-Paton label. Taking the infrared limit in the tree channel removes this dependence on the Chan-Paton labels and gives the same result as our slightly simplified computation.[56] Now, we express t in terms of l using (2.3.2.8), (2.4.2.7) and (2.4.2.28) and apply the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} e^{-\pi\frac{(n-b)^2}{a}} = \sqrt{a}\sum_{s=-\infty}^{\infty} e^{-\pi a s^2+2\pi i s b}. \tag{2.4.3.12}$$

We obtain

$$l^{\frac{9-p}{2}}\rho^{\frac{p-9}{2}}\left(\sum_{w=-\infty}^{\infty} e^{-\frac{\pi l w^2}{\rho}}\right)^{9-p} \qquad \text{for the annulus,} \tag{2.4.3.13}$$

$$l^{\frac{9-p}{2}}\rho^{\frac{p-9}{2}}2^{9-p}\left(\sum_{w=-\infty}^{\infty} e^{-\frac{4\pi l w^2}{\rho}}\right)^{9-p} \qquad \text{for the Klein bottle,} \tag{2.4.3.14}$$

$$l^{\frac{9-p}{2}}\rho^{\frac{p-9}{2}}2^{9-p}\left(\sum_{w=-\infty}^{\infty} e^{-\frac{4\pi l w^2}{\rho}}\right)^{9-p} \qquad \text{for the Möbius strip.} \tag{2.4.3.15}$$

In the IR limit $l \to \infty$ the sums become a factor of one. The common ρ dependent factor is a dimensionless quantity representing the volume of the transverse space (sometimes denoted by v_{9-p}). In the Klein bottle as well as in the Möbius strip there is an additional factor of 2^{9-p}. In the Möbius strip this is simply the number of O-planes. (The number of R-fixed points is two per S^1.) Since the Klein bottle amplitude is proportional to the square of the total O-plane charge we would expect another factor of 2^{9-p} here. However, this is "canceled" by the correction factor we put earlier in by hand. As promised in footnote 55 this factor appeared automatically after we compactified the transverse dimensions. This is good because now we would miss the argument for putting it in by hand.

Together with our previous results table 2.7, (2.4.2.19) and (2.4.2.40) we obtain for the infrared limit of the total (tree level RR channel) amplitude

$$-\frac{1}{2}\left(4\alpha'\pi^2\right)^{-\frac{p+1}{2}} V_{p+1}\rho^{\frac{p-9}{2}}\int_{l\to\infty} dl\,\left(N^2+32^2\mp 2N32\right), \tag{2.4.3.16}$$

where the $\mp$ sign is correlated with the $\pm$ sign in (2.4.3.3). We observe that the contributions of the D-brane/D-brane, O-plane/O-plane and D-brane/O-plane interaction

[56] In the limit $l \to \infty$ the distance dependent exponential function in table 2.7 becomes one.

add up to a complete square, proportional to

$$(32 \mp N)^2 . \tag{2.4.3.17}$$

For consistency the total RR charge has to vanish. Thus, we are lead to the conditions

$$\gamma_{\Omega R} = \gamma_{\Omega R}^T \tag{2.4.3.18}$$

and

$$N = 32. \tag{2.4.3.19}$$

Note that the condition (2.4.3.18) is related to our choice that the ΩR eigenvalue of the R vacuum (in the open string sector) is minus one. Since (2.4.3.18) implies that we can choose a basis such that

$$\gamma_{\Omega R} = I, \tag{2.4.3.20}$$

our choice of the ΩR eigenvalue of the R vacuum seems natural. The case $p = 9$ can be obtained from our previous considerations in two ways. The simplest is to set $p = 9$ in (2.4.3.16). Requiring this expression to vanish yields equations (2.4.3.18) and (2.4.3.17) independent of p.[57] An alternative way is to perform T-dualities with respect to all compact directions and to take the decompactification limit in the T-dual model. Both methods lead to the same results. The massless closed string spectrum has been discussed in section 2.4.1. In the NSNS sector one finds the metric G_{ij} and the dilaton Φ. The RR sector provides the antisymmetric tensor B'_{ij}. The Ω invariant combinations of the NSR with the RNS sector massless states yield the space time fermions needed to fill $N = 1$ supermultiplets.

It remains to study the open string sector. The massless NS sector states are

$$\psi^i_{-\frac{1}{2}} \left|k, mn\right\rangle \lambda^{mn}. \tag{2.4.3.21}$$

The ΩR image of these states is

$$-\psi^i_{-\frac{1}{2}} \left|k, mn\right\rangle \left(\lambda^T\right)^{mn} . \tag{2.4.3.22}$$

Hence, the Chan-Paton matrix λ must be antisymmetric

$$\lambda = -\lambda^T. \tag{2.4.3.23}$$

The states (2.4.3.21) are vectors and thus should be interpreted as gauge fields of a certain gauge group. In order to identify the gauge group, we require that the

[57] Recall that a p-dependence cancels in the product of O-plane charges with the number of O-planes.

	# of Q's	# of ψ_μ's	massless bosonic spectrum	
type I $SO(32)$	16	1	NSNS	$G_{\mu\nu}$, Φ
			open string	A_μ^a in adjoint of $SO(32)$
			RR	$B'_{\mu\nu}$

Table 2.8: Consistent string theories in ten dimensions containing open strings.

state (2.4.3.21) transforms in the adjoint and that gauge transformations preserve the condition (2.4.3.23). Thus the commutator of a generator of the gauge group with a 32×32 antisymmetric matrix (λ) should be a 32×32 antisymmetric matrix. This consideration determines the gauge group to be $SO(32)$. The R sector provides fermions in the adjoint of $SO(32)$. NS and R sector together yield an $N = 1$ $SO(32)$ vector multiplet. The list of consistent closed string theories in ten dimensions (table 2.1) is supplemented by the ten dimensional theory containing (unoriented) closed strings and open strings in table 2.8.

2.4.3.2 Orbifold compactification

So far, we have studied the consistency conditions implied by a torus compactification of the transverse dimension. Since type I theory is a ten dimensional $N = 1$ supersymmetric theory, torus compactifications will result in extended supersymmetries in lower dimensions (e.g. $N = 4$ in four dimensions). For phenomenological reasons it is desirable to obtain less supersymmetry. This can be achieved by taking the transverse space to be an orbifold. In the following we will add O-planes and D-branes to the orbifold compactification considered in section 2.2.2. We supplement the $\mathbb{Z}_2$ action (2.2.2.1) with an ΩR action, where R acts on the target space in the same way as given in (2.2.2.1). Hence, our discrete group is generated by R and ΩR. The third non-trivial element is the product of the two generators: Ω. Thus, the theory contains O-5-planes and O-9-planes. We expect that we need to add D-5-branes and D-9-branes in order to preserve RR charge conservation. Before, studying the open strings induced by those D-branes let us discuss the untwisted and twisted closed string sector states. We focus on the massless part of the spectrum. All the information needed to find the untwisted massless states is given in table 2.5. In addition to the $\mathbb{Z}_2$ symmetry we also need to respect the Ω and ΩR symmetry. This is done by symmetrization in the NSNS sector and antisymmetrization in the RR sector. Thus the untwisted NSNS sector contains the metric G_{ij}, the dilaton Φ and ten scalars. In the RR sector one finds a selfdual and an anti-selfdual two form and twelve scalars. The relevant twisted sector states are listed in table 2.6. Taking into account that there are 16 fixed points we obtain 48 massless scalars in the twisted NSNS sector, and 16 massless scalars in the twisted

RR sector. Adding the fermions in the Ω and ΩR invariant combinations of NSR and RNS sector states, one obtains finally a $d = 6$, $N = 1$ supergravity multiplet, one tensor multiplet and 20 hypermultiplets. (We should emphasize again that the present review is not self contained as far as the supergravity representations are concerned. The reader may view the arrangement of the massless states into multiplets as some additional information which is not really employed in the forthcoming discussions. In order to obtain a nice overview about supermultiplets in various dimensions we recommend[399].)

As already mentioned, we need to add D-5- and D-9-branes in order to cancel the O-plane RR charges. Thus the Chan-Paton matrix is built out of the following blocks: $\lambda^{(99)}$ corresponding to strings starting and ending on D-9-branes, $\lambda^{(55)}$ corresponding to strings starting and ending on D-5-branes, $\lambda^{(59)}$ and $\lambda^{(95)}$ corresponding to open strings with ND boundary conditions in the compact dimensions. The action of Ω and ΩR on the Chan-Paton labels is as described in (2.4.3.2). The γ_Ω and $\gamma_{\Omega R}$ posses also a block structure distinguishing between the action on a string end at a D-5- or D-9-brane, e.g. $\gamma_\Omega^{(9)}$ represents the Ω action on a Chan-Paton label corresponding to an open string end on a D-9-brane. Finally, we specify the representation of the $\mathbb{Z}_2$ element R (2.2.2.1) as follows

$$R: \qquad |\psi, ij\rangle \to (\gamma_R)_{ii'} \left|R(\psi), i'j'\right\rangle \left(\gamma_R^{-1}\right)_{j'j} . \tag{2.4.3.24}$$

Also, the γ_R can be split into two blocks: $\gamma_R^{(9)}$ and $\gamma_R^{(5)}$. The requirement that performing twice the same $\mathbb{Z}_2$ action should leave the state invariant leads to the conditions that every gamma-block containing an Ω in the subscript must be either symmetric or anti-symmetric, whereas the gamma-blocks without an Ω in the subscript must square to the identity[58]. At this point, we need to discuss a subtlety of the five-nine sector, i.e. strings with ND boundary conditions along the compact dimensions. In that sector the Fock space state (without the Chan-Paton label) has an Ω^2 and an $(\Omega R)^2$ eigenvalue of minus one. Unfortunately, we did not develop the techniques needed to show this, in this review. An argument employing an isomorphism between the algebra of vertex operators and Fock space states can be found in[204]. Since Ω^2 and $(\Omega R)^2$ should leave states invariant, this minus sign needs to be canceled by an appropriate action on the Chan-Paton labels. For example a symmetric $\gamma_\Omega^{(9)}$ implies an anti-symmetric $\gamma_\Omega^{(5)}$, and a symmetric $\gamma_{\Omega R}^{(5)}$ implies an anti-symmetric $\gamma_{\Omega R}^{(9)}$.

Let us now study the amplitudes in the loop channel. For strings starting and ending on D-9-branes there is a tower of Kaluza-Klein momentum modes but no winding modes. The D-9-branes are space filling and thus must lie on top of each other. Open

[58] We fix a possible phase to one.

strings with both ends on a D-5-brane can have winding modes and can be separated in the compact directions. (In our previous computation we did not consider this separation, because it is not relevant in the tree channel infrared limit. In order to see explicitly that the dependence on the distance among D-5-branes drops out, we will take it into account, here. We call the component of the position of the ith brane c_i^a.) Further, all amplitudes obtain an additional insertion

$$\frac{1+R}{2}, \tag{2.4.3.25}$$

ensuring that we trace only over states which are invariant under the orbifold group. For terms containing an R insertion the D-5-brane must be located at R fixed points, since otherwise the states in the (55) sector are not eigenstates under R. For the same reason the winding or Kaluza-Klein momentum modes have to vanish in the presence of an R insertion. Further, there will be additional signs for oscillators pointing into a compact dimension. This modifies the oscillator contribution to the trace in a straightforward way. (We leave the details as an exercise.) Taking into account all these effects and the discussion in section 2.3.2 one finds for the annulus amplitude $\mathcal{A}$

$$\begin{aligned}\mathcal{A} \;=\; & -\frac{V_6}{4}\int\frac{dt}{t^4}\left(8\pi^2\alpha'\right)^{-3}\left[\frac{f_4^8\left(e^{-\pi t}\right)}{f_1^8\left(e^{-\pi t}\right)}\left\{\left(\mathrm{tr}\,\gamma_1^{(9)}\right)^2\left(\sum_{n=-\infty}^{\infty}e^{-\frac{2\pi t n^2}{\rho}}\right)^4\right.\right.\\ & \left.+\sum_{i,j\in 5}\left(\gamma_1^{(5)}\right)_{ii}\left(\gamma_1^{(5)}\right)_{jj}\prod_{a=6}^{9}\sum_{w=-\infty}^{\infty}e^{-t\frac{\left(2\pi w\sqrt{\rho\alpha'}+c_i^a-c_j^a\right)^2}{2\pi\alpha'}}\right\}\\ & -2\frac{f_2^4\left(e^{-\pi t}\right)f_4^4\left(e^{-\pi t}\right)}{f_1^4\left(e^{-\pi t}\right)f_3^4\left(e^{-\pi t}\right)}\left(\sum_{I=1}^{16}\mathrm{tr}\,\gamma_{R,I}^{(5)}\right)\left(\mathrm{tr}\,\gamma_R^{(9)}\right)\\ & \left.+4\frac{f_3^4\left(e^{-\pi t}\right)f_4^4\left(e^{-\pi t}\right)}{f_1^4\left(e^{-\pi t}\right)f_2^4\left(e^{-\pi t}\right)}\left\{\left(\mathrm{tr}\,\gamma_R^{(9)}\right)^2+\sum_{I=1}^{16}\left(\mathrm{tr}\,\gamma_{R,I}^{(5)}\right)^2\right\}\right],\end{aligned} \tag{2.4.3.26}$$

where we have formally assigned a gamma with subscript 1 to the action of the identity element of the orbifold group on the Chan-Paton labels. The sum over $i, j \in 5$ means that we sum over all Chan-Paton labels belonging to an open string end on a D-5-brane. The index $I = 1, \ldots, 16$ labels the fixed 5-planes, and a corresponding subscript at a $\gamma^{(5)}$ indicates that the D-5-brane is located on the Ith fixed plane.

Next, we want to compute the Klein bottle amplitude $\mathcal{K}$. It contains the insertions Ω and ΩR. In principle, we have to take the trace over untwisted and twisted sector states (with the $(-)^F$ insertion). Because half of the RR sector states have the opposite $(-)^F$ eigenvalue than the other half, RR sector states do not contribute to the trace with a $(-)^F$ insertion. The same applies to RR and NSNS twisted sector states. Eigenstates of Ω have zero winding numbers whereas for eigenstates of ΩR the Kaluza-

Klein momenta are zero. With this ingredients we find

$$\mathcal{K} = -8\frac{V_6}{4}\int \frac{dt}{t^4}\left(8\pi^2\alpha'\right)^{-3}$$
$$\frac{f_4^8\left(e^{-2\pi t}\right)}{f_1^8\left(e^{-2\pi t}\right)}\left\{\left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi t n^2}{\rho}}\right)^4 + \left(\sum_{w=-\infty}^{\infty} e^{-\pi t\rho w^2}\right)^4\right\}. \qquad (2.4.3.27)$$

Finally, for the Möbius strip amplitude $\mathcal{M}$ we need to trace over R sector states with an $\Omega + \Omega R$ insertion. Eigenstates correspond to open strings starting and ending on the same brane. According to our earlier assignments, the ΩR eigenvalue of the R vacuum corresponding to a string ending on a D-5-brane is minus one, and so is the Ω eigenvalue of the R vacuum corresponding to a string ending on D-9-branes. To determine the remaining eigenvalues one has to act with R on the Ramond vacuum. R can be viewed as a $180°$ rotation and the R vacua as target space spinors. Hence, half of the Ramond vacua have R eigenvalue minus one and the other half plus one. For this reason, only D-9-branes contribute to the term with the Ω insertion whereas only D-5-branes give a non-vanishing result for the trace containing an ΩR insertion. The result for the Möbius strip is

$$\begin{aligned}\mathcal{M} &= \frac{V_6}{4}\int \frac{dt}{t^4}\left(8\pi^2\alpha'\right)^{-3}\frac{f_2^8\left(e^{-2\pi t}\right)f_4^8\left(e^{-2\pi t}\right)}{f_1^8\left(e^{-2\pi t}\right)f_3^8\left(e^{-2\pi t}\right)}\\ &\quad \left\{\mathrm{tr}\left(\left(\gamma_\Omega^{(9)}\right)^{-1}\left(\gamma_\Omega^{(9)}\right)^T\right)\left(\sum_{n=-\infty}^{\infty} e^{-\frac{2\pi t n^2}{\rho}}\right)^4\right.\\ &\quad \left.+\mathrm{tr}\left(\left(\gamma_{\Omega R}^{(5)}\right)^{-1}\left(\gamma_{\Omega R}^{(5)}\right)^T\right)\left(\sum_{w=-\infty}^{\infty} e^{-2\pi t\rho w^2}\right)^4\right\}. \end{aligned} \qquad (2.4.3.28)$$

With the next steps necessary to compute the total RR charge of the system we are familiar by now. We replace $t = \frac{1}{2l}$ in the annulus, $t = \frac{1}{4l}$ in the Klein bottle and $t = \frac{1}{8l}$ in the Möbius strip. In order to be able to read off the infrared (large l) asymptotics we use formulæ (2.3.2.22) and (2.4.3.12). The final result is

$$\begin{aligned}\mathcal{A}+\mathcal{K}+\mathcal{M} &\longrightarrow -\frac{V_6}{4}\int_{l\to\infty} dl\ \left(4\pi^2\alpha'\right)^3\\ &\quad \left[\rho^2\left\{\left(\mathrm{tr}\gamma_1^{(9)}\right)^2 - 32\mathrm{tr}\left(\left(\gamma_\Omega^{(9)}\right)^{-1}\left(\gamma_\Omega\right)^T\right) + 32^2\right\}\right.\\ &\quad +\frac{1}{\rho^2}\left\{\left(\mathrm{tr}\gamma_1^{(5)}\right)^2 - 32\mathrm{tr}\left(\left(\gamma_{\Omega R}^{(5)}\right)^{-1}\left(\gamma_\Omega\right)^T\right) + 32^2\right\}\\ &\quad \left.+\frac{1}{4}\sum_{I=1}^{16}\left(\mathrm{tr}\gamma_R^{(9)} + 4\mathrm{tr}\gamma_{R,I}^{(5)}\right)^2\right]. \end{aligned} \qquad (2.4.3.29)$$

The setup respects RR charge conservation if (2.4.3.29) vanishes. Thus, we need 32 D-9-branes and 32 D-5-branes. (A gamma representing the identity is of course the identity matrix.) Further, we take

$$\gamma_{\Omega R}^{(5)} = \left(\gamma_{\Omega R}^{(5)}\right)^T \quad , \quad \gamma_{\Omega}^{(9)} = \left(\gamma_{\Omega}^{(9)}\right)^T . \tag{2.4.3.30}$$

Our previous discussion of the (59) sector implies

$$\gamma_{\Omega}^{(5)} = -\left(\gamma_{\Omega}^{(5)}\right)^T \quad , \quad \gamma_{\Omega R}^{(9)} = -\left(\gamma_{\Omega R}^{(9)}\right)^T . \tag{2.4.3.31}$$

The remaining representation matrices can be found by imposing that the gammas should form a projective[59] representation of the orientifold group ($\mathbb{Z}_2 \times \mathbb{Z}_2$). We simply choose

$$\gamma_R^{(5)} = \gamma_{\Omega R}^{(5)}\gamma_{\Omega}^{(5)} \tag{2.4.3.32}$$

$$\gamma_R^{(9)} = \gamma_{\Omega R}^{(9)}\gamma_{\Omega}^{(9)} . \tag{2.4.3.33}$$

By fixing a basis in the Chan-Paton labels we obtain

$$\gamma_{\Omega R}^{(5)} = \gamma_{\Omega}^{(9)} = I, \tag{2.4.3.34}$$

where the rank of the identity matrix is 32. The antisymmetric form is

$$\gamma_{\Omega}^{(5)} = \gamma_{\Omega R}^{(9)} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \tag{2.4.3.35}$$

with I being a 16×16 identity matrix, here. Note that our choice is consistent with the requirement that $\gamma_R^{(\cdot)}$ squares to the identity. So far, we did not take into account that the last term in (2.4.3.29) has to vanish. With $\gamma_R^{(\cdot)}$ being traceless this is ensured.

We have now all the ingredients needed to determine the open string spectrum. Let us first study strings starting and ending on the D-9-branes, or in short the (99) sector. We keep states which are invariant under each element of the orientifold group. (The D-9-branes are fixed under each element of the orientifold group.) In the NS sector we find massless vectors with the Chan-Paton matrix

$$\lambda_{\text{vector}}^{(99)} = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}, \tag{2.4.3.36}$$

where A denotes a real antisymmetric and otherwise arbitrary 16×16 matrix and S stands for a real 16×16 symmetric matrix. For the scalars in the NS sector one finds

$$\lambda_{\text{scalars}}^{(99)} = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}, \tag{2.4.3.37}$$

[59] "Projective" means up to phase factors, which drop out since the gamma acts in combination with its inverse on the Chan-Paton label.

where the A_i are 16×16 antisymmetric matrices. Let us ignore the D-5-branes for a moment and determine the gauge group and its action on the scalars in the (99) sector. Since the vectors are in the adjoint of the gauge group, the gauge group should be 16^2 dimensional. $U(16)$ is a good candidate. Further, we know that the vector should transform in the adjoint under global gauge transformations under which it should not change the form specified by (2.4.3.36). Thus, we define an element of the gauge group as

$$g^{(9)} = \exp \begin{pmatrix} A_g & S_g \\ -S_g & A_g \end{pmatrix}, \tag{2.4.3.38}$$

where S_g (A_g) are real anti-(symmetric) matrices with infinitesimal entries. A gauge transformation acts on the Chan-Paton matrix as

$$\lambda^{(99)} \to g^{(9)} \lambda^{(99)} \left(g^{(9)} \right)^{-1} = \left[\begin{pmatrix} A_g & S_g \\ -S_g & A_g \end{pmatrix}, \lambda \right]. \tag{2.4.3.39}$$

We observe that the vectors transform in the adjoint and the form of the Chan-Paton matrix is preserved. Note also that $g^{(9)}$ is unitary and has 16^2 parameters. It is a $U(16)$ element. The $U(1)$ subgroup corresponds to $A_g = 0$ and S_g proportional to the identity. From our assignment that the Chan-Paton matrix transforms in the adjoint of $U(16)$ it is also clear that the Chan-Paton label i and j transform in the $\mathbf{16}$ and $\overline{\mathbf{16}}$ of $U(16)$ Thus, the scalars can be decomposed into the antisymmetric $\mathbf{120} + \overline{\mathbf{120}}$. One may also explicitly check that the form of the Chan-Paton matrix for the scalars is not altered by a gauge transformation. We leave the discussion of the fermions in the R sector as an exercise. The result is that that half of them carry the Chan-Paton matrix (2.4.3.36) and the other half the matrix (2.4.3.39). Altogether the (99) sector provides a vector multiplet in the adjoint of $U(16)$ and a hypermultiplet in the $\mathbf{120}$ + $\overline{\mathbf{120}}$.

Now, we include the D-5-branes. Here, we have to distinguish between the case that a D-5-brane is situated at a fixed plane or not. In the first case the (55) strings have to respect the ΩR and R symmetry, whereas in the second case these orientifold group elements just fix the fields on the image brane. Suppose we have $2m_I$ D-5-branes at the Ith fixed plane. (The number of the D-5-branes per fixed plane must be even, since otherwise they cannot form a representation of the orientifold group.) The NS sector leads again to a massless vector and massless scalars with almost the same Chan-Paton matrices as in the (99) sector ((2.4.3.36) and (2.4.3.39)). The only difference is that the antisymmetric and symmetric matrices are now $m_I \times m_I$ instead of 16×16. Hence, we obtain a vector multiplet in the adjoint of $U(m_I)$ and a hypermultiplet in the $\frac{\mathbf{m}_I}{\mathbf{2}}(\mathbf{m}_I\textbf{-1}) + \overline{\frac{\mathbf{m}_I}{\mathbf{2}}(\mathbf{m}_I\textbf{-1})}$.

Let $2n_j$ D-5-branes be situated away from the fixed plane (but on top of each other). For the (55) sector belonging to those D-5-branes we impose invariance under Ω, only. The solution for $\gamma_\Omega^{(5)}$ is given in (2.4.3.35). Together with the minus eigenvalue on the massless NS sector Fock space state, this leads to the result that the vector in the (55) sector is an element of the $USp\left(2n_j\right)$ Lie algebra in the adjoint representation. Taking into account (part of) the R sector this is promoted to a $USp\left(2n_j\right)$ vector multiplet. The scalars together with the remaining R sector states form a hypermultiplet in the antisymmetric $\mathbf{n}_j\left(\mathbf{2n}_j\text{-}\mathbf{1}\right)$ representation.

It remains to study the (95) sector. (Here, one has to take into account that along the compact directions NS sector fermions are integer modded whereas R sector fermions are half integer modded. This is quite similar to the twisted sector closed string. In particular, the (95) NS sector ground state is already massless. Hence, the NS sector does not give rise to massless vectors. We do not impose Ω or ΩR invariance on (95) strings since they are mapped onto (59) strings by the worldsheet parity inversion. If the considered D-5-branes are situated at one of the fixed planes we impose R invariance. In this case, one finds in the NS sector two scalars with the Chan-Paton matrix

$$\lambda^{(95)}=\left(\begin{array}{cc} X_1 & X_2 \\ -X_2 & X_1 \end{array}\right), \tag{2.4.3.40}$$

where the X_i are general $m_i \times 16$ matrices. Together with the R sector this leads to a hypermultiplet in the $\left(\mathbf{16}\ ,\ \mathbf{m}_I\right)$ of $U\left(16\right)\times U\left(m_I\right)$, (the hypermultiplet is neutral under the gauge group living on D-5-branes not situated at the Ith fixed plane). For D-5-branes which are not a fixed plane the (95) sector provides a hypermultiplet in the $\left(\mathbf{16},\ \mathbf{2n}_J\right)$ of $U\left(16\right)\times USp\left(2n_j\right)$.

Altogether we find the gauge group is

$$U\left(16\right)\times\prod_{I=1}^{16}U\left(m_I\right)\times\prod_J USp\left(2n_j\right), \tag{2.4.3.41}$$

where j labels the D-5-brane packs away from fixed planes. In addition the total number of D-5-branes has to be 32 (images are counted), i.e.

$$\sum_{I=1}^{16}2m_I+2\sum_j 2n_j=32. \tag{2.4.3.42}$$

There are hypermultiplets in the representation

$$\begin{aligned} &2\left(\mathbf{120}\ ,\ \mathbf{1},\ \mathbf{1}\right)+\sum_{I=1}^{16}\left\{2\left(\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}}\mathbf{m}_I\left(\mathbf{m}_I\text{-}\mathbf{1}\right),\mathbf{1}\right)_I+\left(\mathbf{16},\mathbf{m}_I,\mathbf{1}\right)_I\right\} \\ &+\sum_j\left\{\left(\mathbf{1},\mathbf{1},\mathbf{n}_j\left(\mathbf{2n}_j\text{-}\mathbf{1}\right)\text{-}\mathbf{1}\right)_j+\left(\mathbf{1},\mathbf{1},\mathbf{1}\right)+\left(\mathbf{16},\mathbf{1},\mathbf{2n}_j\right)_j\right\}, \end{aligned} \tag{2.4.3.43}$$

where we have split the anti-symmetric representation of $USp\,(2n_j)$ into its irreducible parts and an index I, j refers to the gauge group on the D-5-brane pack at a fixed plane and off a fixed plane, respectively. It can be checked that the effective six dimensional theory is free of anomalies. The models belonging to different distributions of the D-5-branes on and off fixed planes can be continuously transformed into each other. In the field theory description this corresponds to the Higgs mechanism.

We have seen that adding to the orbifold compactification of section 2.2 O-planes and D-branes gives a very interesting picture. Apart from the closed string twisted sector states located at the orbifold fixed planes we obtain various fields from open strings ending on D-branes. These D-branes can be moved within the compact directions while keeping the geometry fixed. The techniques described in this section can be also applied to phenomenologically more interesting setups leading to four dimensional theories. A description of such models is beyond the scope of the present review.

Chapter 3

Non-Perturbative description of branes

3.1 Preliminaries

In the previous sections we gave a perturbative description of various extended objects: the fundamental string, orbifold planes, D-branes and Orientifold planes. The string plays an outstanding role in the sense that field theories on the worldvolumes of the other extended objects are effective string theories. The quantization of the fundamental string is performed in a trivial target space (i.e. the target space metric is the Minkowski metric and all other string excitations are constant or zero). Further, the worldsheet topology is specified to the spherical (for closed strings) or disc (for open strings) topology (after Wick rotating to Euclidean worldsheet signature). Our treatment leads to a perturbative expansion in the genus of the worldsheet (see section 2.1.4). The perturbative expansion is governed by the string coupling

$$g_s = e^{\langle\Phi\rangle}, \tag{3.1.0.1}$$

which needs to be small. Perturbative closed string theory has an effective field theory description which contains supergravity. How does one obtain insight into regions where g_s is large? Clearly, the perturbation theory breaks down in this case, and indeed this region is rather difficult to study. There are, however, a few results one can obtain also for strong couplings. Let us recall how non perturbative effects in Yang-Mills theory can be studied. Apart from the trivial vacuum, (Euclidean) Yang-Mills theory contains several other stable vacua, *viz.* the instantons. Studying fluctuations around an instanton vacuum, one finds an additional weight factor in the path integral which comes from the background value of the action and is of the form

$$e^{-\frac{n}{g^2}},$$

where n is the instanton number and g is the Yang-Mills gauge coupling. As long as g is small, the fluctuations around an instanton vacuum are heavily suppressed. However, as soon as g becomes large, the suppression factor becomes large. Thus, knowing about the instanton solutions in Yang-Mills theory gives a handle on non perturbative effects. But how can one know, that one does not have to include strong coupling effects into the theory before deriving the instanton solutions? The answer is that instantons are stable, they are characterized by a topological number which cannot be changed in a continuous way when taking g from small to large. Therefore, instantons can give information about strongly coupled Yang-Mills theory even though they are found as solutions to the perturbative formulation of Yang-Mills theories. States (vacua) with such a feature are called BPS states.[1][2]

Therefore, our aim will be to find BPS states in string theory. In the low energy limit, the various superstring theories are described by supergravities. Insights into non-perturbative effects in string theory can be gained by finding the BPS states of perturbative string theory. As a guiding principle, we will look for solutions to the effective equations of motion that preserve part of the supersymmetry (i.e. are invariant under a subset of the supersymmetry transformations). Roughly speaking, it is then the number of preserved supersymmetries which cannot be changed continuously when taking the string coupling from weak to strong. We will see that such solutions can be viewed as branes. The number of branes takes the role of the instanton number in the Yang-Mills example discussed above. We will be very brief in our analysis and essentially only summarize some of the important results. The classical review on branes as supergravity solutions is[152] and we will give more references in the end of this review.

3.2 Universal Branes

From section 2.1.4 we recall that all the closed superstring theory effective actions contain a piece

$$S_{univ} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\left(\partial\Phi\right)^2 - \frac{1}{12}H^2\right). \tag{3.2.0.1}$$

In the present section we will truncate all closed string effective field theories to (the supersymmetric extension of) (3.2.0.1). This is consistent because we will restrict on backgrounds where the discarded part of the action vanishes and the corresponding equations of motion are satisfied trivially. By adding appropriate terms including

[1]There are also stable non BPS states (for reviews see e.g. [424, 319, 195]). We will not discuss these.

[2]BPS stands for the names Bogomolny, Prasad and Sommerfield, and refers to the papers[76, 379].

fermions (2.1.4.5) can be promoted to an $N = 1$ supersymmetric theory. (For type II theories this is a sub-symmetry of the $N = 2$ supersymmetry.) The supersymmetric extension is usually given in the Einstein frame. The action (3.2.0.1) is written in the string frame were the string tension is a constant and independent of the dilaton. The Einstein frame is obtained by the metric redefinition

$$g_{\mu\nu} = e^{-\frac{\Phi}{2}} G_{\mu\nu}, \tag{3.2.0.2}$$

where $G_{\mu\nu}$ is the string frame metric and $g_{\mu\nu}$ is the Einstein frame metric. The action (3.2.0.1) takes the form

$$S_{E,univ} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} \left(\partial\Phi\right)^2 - \frac{1}{12} e^{-\Phi} H^2 \right). \tag{3.2.0.3}$$

We observe that (3.2.0.3) starts with the familiar Einstein Hilbert term (therefore the name "Einstein frame"). Further, the kinetic term of the dilaton has the "correct" sign now, and the coupling of the B field is Φ dependent. In the supersymmetric extension, a gravitino and a dilatino are added. We do not give the supersymmetric action explicitly. For us, it will suffice to know the supersymmetry transformations of the gravitino and the dilatino. These are

$$\delta\psi_\mu = D_\mu \epsilon + \frac{1}{96} e^{-\frac{\Phi}{2}} \left(\Gamma_\mu{}^{\nu\rho\kappa} - 9 \delta_\mu^\nu \Gamma^{\rho\kappa} \right) H_{\nu\rho\kappa} \epsilon, \tag{3.2.0.4}$$

$$\delta\lambda = -\frac{1}{2\sqrt{2}} \Gamma^\mu \partial_\mu \Phi \epsilon + \frac{1}{24\sqrt{2}} e^{-\frac{\Phi}{2}} \Gamma^{\mu\nu\rho} H_{\mu\nu\rho} \epsilon, \tag{3.2.0.5}$$

where ψ_μ denotes the gravitino and λ the dilatino. The Gamma matrices with curved indices are obtained from ordinary Gamma matrices (16×16 matrices satisfying the usual Clifford algebra in ten dimensional Minkowski space) by transforming the flat index with a vielbein to a curved one. A Gamma with multiple indices denotes the anti-symmetrized product of Gamma matrices. The spinor ϵ is the supersymmetry transformation parameter.

Sometimes it is useful to formulate the theory in a slightly different way. To this end, one adds to the action (3.2.0.3) a Lagrange multiplier term providing the constraint of a fulfilled Bianchi identity. Calling the Lagrange multipliers $A_{\mu_1 \ldots \mu_6}$, such a term looks like

$$\int d^{10}x \, \epsilon^{\mu_1 \ldots \mu_{10}} A_{\mu_1 \ldots \mu_6} \partial_{\mu_7} H_{\mu_8 \mu_9 \mu_{10}}. \tag{3.2.0.6}$$

The $A_{\mu_1 \ldots \mu_6}$ equation of motion yields the Bianchi identity of the B field strength $H_{\mu\nu\rho}$. However, one can alternatively solve the B field equation of motion with the result

$$H_{\mu\nu\rho} \sim e^{\Phi} \epsilon_{\mu\nu\rho\mu_1 \ldots \mu_7} K^{\mu_1 \ldots \mu_7}, \tag{3.2.0.7}$$

with

$$K_{\mu_1 \ldots \mu_7} = \partial_{[\mu_1} A_{\mu_2 \ldots \mu_7]}. \tag{3.2.0.8}$$

This means that we can trade the antisymmetric tensor B for a six form potential A. Choosing an appropriate normalization for the Lagrange multiplier terms (3.2.0.6), the effective action (3.2.0.3) in terms of the six form potential A reads

$$\tilde{S}_{E,univ} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} \left(\partial\Phi\right)^2 - \frac{1}{2 \cdot 7!} e^{\Phi} K^2 \right). \tag{3.2.0.9}$$

Also in this form the action can be supersymmetrized. In terms of the six form potential A the gravitino and dilatino supersymmetry transformations read

$$\delta\psi_\mu = D_\mu \epsilon + \frac{1}{2 \cdot 8!} e^{\frac{\Phi}{2}} \left(3\Gamma_\mu{}^{\nu_1 \ldots \nu_7} - 7\delta_\mu^{\nu_1} \Gamma^{\nu_2 \ldots \nu_7} \right) K_{\nu_1 \ldots \nu_7} \epsilon, \tag{3.2.0.10}$$

$$\delta\lambda = -\frac{1}{2\sqrt{2}} \Gamma^\mu \partial_\mu \Phi \epsilon - \frac{1}{2 \cdot 2\sqrt{2} \cdot 7!} e^{\frac{\Phi}{2}} \Gamma^{\mu_1 \ldots \mu_7} K_{\mu_1 \ldots \mu_7} \epsilon. \tag{3.2.0.11}$$

In the following two subsections, we will present two solutions preserving half of the supersymmetry.

3.2.1 The fundamental string

The solutions we are going to discuss in the present and subsequent sections are generalizations of extreme Reissner–Nordström black holes. Reissner–Nordström black holes are solutions of Einstein–Hilbert gravity coupled to an electro magnetic field. They carry mass and electric or magnetic charge. Extreme Reissner–Nordström black holes satisfy a certain relation between the charge and the mass. (In our case such a relation will be dictated by the requirement of partially preserved supersymmetry.) Replacing the electro magnetic field strength F by its dual $\star F$ interchanges electric with magnetic charge. (For a more detailed discussion of Reissner–Nordström black holes see e.g.[442].)

The action (3.2.0.3) bears some analogy to four dimensional Einstein gravity coupled to an electro magnetic field. The difference is that the theory is ten dimensional instead of four dimensional and the electro magnetic field strength is replaced by the three form H. In addition, there is the scalar Φ. Since the gauge potential is now a two form which naturally couples to the worldvolume of a string, we look for "extreme Reissner–Nordström black strings" instead of black holes. The corresponding ansatz for the metric is

$$ds^2 = e^{2A} \eta_{ij} dx^i dx^j + e^{2B} \delta_{ab} dy^a dy^b, \tag{3.2.1.1}$$

with $i, j = 0, 1$ and $a, b = 2, \ldots, 9$. Further, A and B are functions of

$$r = \sqrt{\delta_{ab} y^a y^b}, \tag{3.2.1.2}$$

only. Here, we have taken the second step before the first one in the sense that first we should have thought about what kind of isometries we would like to obtain and only afterwards we should have written down a general ansatz respecting the isometries. Therefore, let us perform the first step now and discuss the isometries of the ansatz. Clearly, there is an $SO(1,1)$ isometry acting on the x^i. This means, that up to Lorentz boosts, x^i spans the worldvolume of a straight static string. There is no further dependence on x^i in the ansatz because we do not wish to distinguish a point on the worldvolume of the string. The other isometry acts as an $SO(8)$ on the y^a. This is the natural extension of the $SO(3)$ isometry associated with non-rotating four dimensional black holes. It is $SO(8)$ now because the space transverse to the string is eight dimensional (whereas in 4d black holes the space transverse to the hole is three dimensional). The r dependence respects the $SO(8)$ isometry. Distinguishing between different values of r means specifying the position of the string, i.e. r measures the radial distance from the string.

In order not to spoil the above symmetries, we choose for the remaining fields the ansatz

$$B_{01} = -e^{C} \quad , \quad \Phi = \Phi(r), \tag{3.2.1.3}$$

where C is also a function of r only. All other components of B are zero. Viewed as a two form, B is proportional to the invariant volume form of the string worldvolume. The factor e^C may depend on r.

The ansatz for the target space spinors is that they all vanish. As mentioned earlier we are interested in situations where the solution preserves part of the supersymmetry because this ensures that we can continuously take the string coupling from weak to strong. In particular, the unbroken supersymmetry is parameterized by spinors ϵ for which the gravitino and dilatino values of zero do not change under supersymmetry transformations, i.e. those ϵ for which the rhs of (3.2.0.4) and (3.2.0.5) vanish. In order to find such solutions for our ansatz it is convenient to represent the ten dimensional Gamma matrices $A, B = 0, \ldots, 9$,

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \tag{3.2.1.4}$$

as a tensor product of 2×2 matrices γ_i in $1+1$ and 8×8 matrices Σ_a in 8 dimensions[3]

$$\Gamma_A = (\gamma_i \otimes \mathbf{I}, \gamma_3 \otimes \Sigma_a), \tag{3.2.1.5}$$

[3]The corresponding algebras are $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$ and $\{\Sigma_a, \Sigma_b\} = 2\delta_{ab}$.

where $\mathbf{I}$ is the 8×8 identity matrix and

$$\gamma_3 = \gamma_0 \gamma_1 \tag{3.2.1.6}$$

squares to the 2×2 identity matrix. Further, we have to take into account that the ten dimensional $N = 1$ supersymmetry parameter ϵ is subject to the constraint

$$\Gamma_{11} \epsilon = \epsilon. \tag{3.2.1.7}$$

Under certain conditions to be specified below the variations of the gravitino and the dilatino vanish for

$$\epsilon = e^{\frac{3\Phi}{8}} \varepsilon_0 \otimes \eta_0, \tag{3.2.1.8}$$

where ε_0 and η_0 are $SO(1,1)$ and $SO(8)$ constant spinors, respectively, which satisfy the lower dimensional chirality conditions

$$(1 - \gamma_3)\, \varepsilon_0 = 0 \ , \quad \left(1 - \prod_{a=2}^{9} \Sigma_a\right) \eta_0 = 0. \tag{3.2.1.9}$$

This breaks the supersymmetry to half the amount of the perturbative (trivial) vacuum. (The condition (3.2.1.7) could be also satisfied by choosing simultaneously the opposite chiralities in the two equations (3.2.1.9).)

We already mentioned that only under certain conditions we can find unbroken supersymmetries at all. Requiring that asymptotically ($r \to \infty$) we obtain the perturbative vacuum, these conditions read

$$A = \frac{3}{4}\left(\Phi - \Phi_0\right), \tag{3.2.1.10}$$

$$B = -\frac{1}{4}\left(\Phi - \Phi_0\right), \tag{3.2.1.11}$$

$$C = 2\Phi - \frac{3}{2}\Phi_0 \tag{3.2.1.12}$$

where Φ_0 is the asymptotic value of Φ. Hence supersymmetry leaves only one function out of our ansatz undetermined. This function can be taken to be the dilaton whose equation of motion boils down to

$$\delta^{ab} \partial_a \partial_b e^{-2\Phi(r)} = 0, \tag{3.2.1.13}$$

i.e. the "flat" Laplacian of the transverse space (spanned by the y^a) acting on $e^{-2\Phi}$ has to vanish. As in the case of four dimensional black holes, we solve this equation everywhere but at the origin $r = 0$, where there are additional contributions due to a source string. (We do not add the source string explicitly here, but will infer its properties (tension and charge) in an indirect way below. For the explicit inclusion of the source term see e.g.[152].) The solution to (3.2.1.13) reads

$$e^{-2\Phi} = e^{-2\Phi_0}\left(1 + \frac{k}{r^6}\right), \tag{3.2.1.14}$$

where k is an integration constant which will be related to the string tension below. Plugging this back into (3.2.1.10) – (3.2.1.12) and in the ansatz gives the final solution.

Next, we would like to deduce the tension of the string source from our solution. This is done by studying the Newtonian limit of general relativity. In particular, by comparing the Einstein equation with the geodesic equation of a point particle (which has constant mass in the Einstein frame) one finds that the Newton potential of the string-source is encoded in the subleading term in a large r expansion of g_{00}. Therefore, we first observe that for large r

$$g_{00} = -1 + \frac{3k}{4r^6} + \dots. \tag{3.2.1.15}$$

The relation between g_{00} and the Newton potential of a string is explicitly such that[4]

$$\frac{1}{r^7}\partial_r\left(r^7\partial_r g_{00}\right) = -\frac{3}{2}\kappa^2 T_E \frac{\delta(r)}{\Omega_7 r^7} \tag{3.2.1.16}$$

holds, with the understanding that terms denoted by $\dots$ in (3.2.1.15) are neglected. The string tension is denoted by T_E. Further, the unit volume of a seven–sphere Ω_7 enters the expression. Hence, we obtain

$$T_E = \frac{3k}{\kappa^2}\Omega_7. \tag{3.2.1.17}$$

We put the index E at the tension in order to indicate that it is measured in the Einstein frame. What we are actually interested in, is the tension in the string frame. This is readily obtained by noticing that transforming back to the string frame (asymptotically) implies

$$\kappa^2 \to e^{2\Phi_0}\kappa^2 \ , \ \ \Omega_7 \to e^{\frac{7\Phi_0}{2}}\Omega_7. \tag{3.2.1.18}$$

Thus in the string frame the tension is[5]

$$T = \frac{3k}{\kappa^2}e^{\frac{3\Phi_0}{2}}\Omega_7. \tag{3.2.1.19}$$

[4] Equation (3.2.1.16) has the same form as the equation satisfied by the Newton potential. The numerical factor of $\frac{3}{2} = \frac{7-p}{4}$ on the rhs of (3.2.1.16) is a matter of convention, which fixes the relation between κ, the speed of light ($c = 1$), and Newton's constant (see e.g.[432]). We choose our convention in agreement with[152], where explicit source terms (containing the tension) are added.

[5] On dimensional grounds, one would expect a different scaling of T. In order to obtain this, one has to take into account that k is a dimensionful quantity. In (3.2.1.19) k is given Einstein frame units.

Recalling that our "elementary particle" is a string of tension $\frac{1}{2\pi\alpha'}$ and requiring that any string like object must consist out of an integer number N of elementary strings we finally determine the integration constant k to be[6]

$$k = N\frac{\kappa^2}{6\pi\alpha'\Omega_7}e^{-\frac{3\Phi_0}{2}}. \tag{3.2.1.20}$$

It remains to compute the $U(1)$ charge carried by the vacuum. This is basically done by integrating the B equation of motion over the transverse space. The result is

$$\mu = \frac{1}{\sqrt{2}\kappa}\int_{S^7} e^{-\Phi} \star H, \tag{3.2.1.21}$$

where the integration is over an asymptotic seven-sphere enclosing the string source. The $U(1)$ charge is denoted by μ. Expressing the result in terms of the string tension one obtains

$$\mu = \sqrt{2}\kappa\frac{N}{2\pi\alpha'}. \tag{3.2.1.22}$$

This equality is related to partially unbroken supersymmetry. If the configuration was not stable the tension of the bound state would be larger than the sum of the elementary tensions. Hence, the rhs of (3.2.1.22) is larger for general (non BPS) states. The BPS state saturates a general inequality. Since the BPS state is stable, there can be no state with less tension and the same charge since otherwise the BPS state would decay into such a state. The lower bound on the tension set by the BPS state is called the Bogomolnyi bound.

3.2.2 The NS five brane

In this subsection we repeat the analysis of the previous section, however, with the action (3.2.0.9) instead of (3.2.0.3). Thus, we will obtain the magnetic dual of the previously discussed string solution. This is called the NS five brane. Its properties (tension, charge) will be fixed in terms of the string properties via the Dirac quantization condition. (For generalizations of the Dirac quantization condition to extended objects see[353, 438].) As the derivation of the NS five brane solution goes along the same lines as the one given in the previous subsection, we will be even more sketchy here. Instead of the two form potential, we have now the six form potential A. Since an object which extends along five spatial dimensions naturally couples to a six form potential, we choose the following ansatz for the metric

$$ds^2 = e^{2A}\eta_{ij}dx^i dx^j + e^{2B}\delta_{ab}dy^a dy^b, \tag{3.2.2.1}$$

[6]Here we use the fact that for a superposition of BPS states there is no binding energy, i.e. the total tension is obtained by simply summing the tensions of the individual BPS states.

where now $i, j = 0, \ldots, 5$ and $a, b = 6, \ldots, 9$. The five brane worldvolume extends along the x^i directions and the functions A and B are allowed to depend on the radial distance from the five brane r with

$$r = \sqrt{\delta_{ab} y^a y^b}. \tag{3.2.2.2}$$

The ansatz for the six form potential is

$$A_{012345} = -e^C, \tag{3.2.2.3}$$

where C is a function of r. The components of A which cannot be obtained by permuting the indices in (3.2.2.3) are zero. The final input is that also the dilaton depends only on r,

$$\Phi = \Phi\left(r\right). \tag{3.2.2.4}$$

All fermionic fields are again set to zero. There is an unbroken supersymmetry if we can find a spinor such that the gravitino and dilatino transformations (3.2.0.10) and (3.2.0.11) vanish. It turns out that half of the supersymmetry is preserved if the following relations hold

$$A = -\frac{1}{4}\left(\Phi - \Phi_0\right), \tag{3.2.2.5}$$

$$B = \frac{3}{4}\left(\Phi - \Phi_0\right), \tag{3.2.2.6}$$

$$C = -2\Phi + \frac{3}{2}\Phi_0, \tag{3.2.2.7}$$

where Φ_0 denotes again the asymptotic $r \to \infty$ value of the dilaton. (In addition to partially unbroken supersymmetry we have once again imposed that for large r the vacuum should approach the perturbative vacuum.) Under these conditions the equations of motion boil down to

$$\delta^{ab}\partial_a\partial_b e^{2\Phi} = 0. \tag{3.2.2.8}$$

We solve this equation everywhere but at $r = 0$ where we allow for additional contributions due to source terms. One finds

$$e^{2(\Phi - \Phi_0)} = 1 + \frac{\tilde{k}}{r^2}. \tag{3.2.2.9}$$

The integration constant $\tilde{k}$ can be fixed by exploiting the Dirac quantization condition. To this end we compute the charge carried by the vacuum

$$\tilde{\mu} = \frac{1}{\sqrt{2}\kappa}\int_{S^3} e^{\Phi} \star K = \frac{\sqrt{2}\Omega_3 \tilde{k}}{\kappa} e^{\frac{\Phi_0}{2}}, \tag{3.2.2.10}$$

where the integral is over an asymptotic three–sphere surrounding the five brane and Ω_3 denotes the volume of a unit three–sphere. Now, the Dirac quantization condition reads

$$\tilde{\mu}\mu = 2\pi\tilde{N}N, \tag{3.2.2.11}$$

where μ is the charge of N elementary strings (3.2.1.22). The number of five branes (number of magnetic charges) is $\tilde{N}$. This fixes the integration constant,

$$\tilde{k} = \frac{\pi\tilde{N}}{T_S\Omega_3}e^{-\frac{\Phi_0}{2}}, \tag{3.2.2.12}$$

where T_S is the elementary string tension ($T_S = \frac{1}{2\pi\alpha'}$ in the string frame). By computing the gravitational potential in the Newtonian limit, one finds the tension of the five brane in the Einstein frame (with T_S and κ also in Einstein frame units)

$$\tilde{T}_E = \frac{\pi\tilde{N}}{T_S\kappa^2}e^{-\frac{\Phi_0}{2}}. \tag{3.2.2.13}$$

The mass dimension of $\tilde{T}_E$ is six. Hence, we obtain

$$\tilde{T} = e^{-2\Phi_0}\frac{2\pi\alpha'\pi\tilde{N}}{\kappa^2} \tag{3.2.2.14}$$

in the string frame. We observe that the five brane tension behaves as $1/g_s^2$. In the perturbative region the NS five brane is very heavy whereas it becomes lighter when the string coupling increases.

The NS five brane is an extended object for which we did not give a perturbative description. Indeed, such a description is not known. One could try to quantize strings in the NS five brane background. This is possible only in certain spatial regions. Firstly, for large r the background becomes flat, and we know how to quantize strings there. But also in the background at $r \to 0$ (the near horizon limit) one can find a quantized string theory. In that limit the string frame metric reads

$$ds_s^2 = \eta_{ij}dx^i dx^j + \tilde{k}\left(d\log r\right)^2 + \tilde{k}d\Omega_3^2, \tag{3.2.2.15}$$

and the dilaton is linear in $\rho = \log r$. With $d\Omega_3^2$ we denote the metric of a unit three–sphere. The NSNS field strength H is a constant times the volume element $d\Omega_3$. The geometry factorises into a $5+1$ dimensional Minkowski space times the direction on which the dilaton depends linearly times an S^3. Since S^3 is an $SU\left(2\right)$ group manifold, string theory can be quantized in such a background. For more details see[97] or the review[96].

3.3 Type II branes

Like in the previous sections, we are interested in setups where only a truncated version of the effective actions (see section 2.1.4) is relevant. The bosonic part of the truncated type II action reads

$$S = \frac{1}{2\kappa^2}\int d^{10}x\left(e^{-2\Phi}\left(R + 4\left(\partial\Phi\right)^2\right) - \frac{1}{2\left(p+2\right)!}F_{p+2}^2\right), \tag{3.3.0.1}$$

where F_{p+2} denotes the field strength of an RR $p+1$ form potential. For type IIA p is even whereas it is odd for type IIB theory. For $p = 3$ the action has to be supplemented by the constraint that the field strength is selfdual. The ($p+2$ form) field strengths are not all independent but related by Hodge duality to the field strength corresponding to $6-p$ (an $8-p$ form field strength). We will restrict the discussion to the cases $0 \le p < 7$. For $p = 7$ the solution presented in[225] is relevant. The 8-brane appears as a solution of massive type IIA supergravity[56]. How this is related to string theory (or rather M-theory) is discussed in the recent paper[236] (see also references therein). We will consider only a single relevant p at a time. The field redefinition (3.2.0.2) takes us to the action in the Einstein frame

$$S_E = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\left(\partial\Phi\right)^2 - \frac{1}{2\left(p+2\right)!}e^{\frac{3-p}{2}\Phi}F_{p+2}^2\right). \tag{3.3.0.2}$$

The p-brane ansatz reads

$$ds^2 = e^{2A}\eta_{ij}dx^i dx^j + e^{2B}\delta_{ab}dy^a dy^b, \tag{3.3.0.3}$$

with $i, j = 0, \ldots, p$ and $a, b = p+1, \ldots, 9$. A and B are functions of

$$r = \sqrt{\delta_{ab}y^a y^b} \tag{3.3.0.4}$$

The dilaton is also taken to be a function of r. Let us first exclude the case $p = 3$ from the discussion. For the $p+1$ form gauge field we choose

$$A_{0,\ldots,p} = -e^C, \tag{3.3.0.5}$$

where C is a function of r, and all other components of A (which cannot be obtained by permuting the indices in (3.3.0.5)) are zero. All the other fields (NSNS B-field, the remaining RR forms, and the fermions) are zero. The BPS condition leaves one out of the four functions A, B, C and Φ undetermined. Choosing for convenience C to be the undetermined function, these conditions read

$$A = \frac{7-p}{16}\left(C - C_0\right), \tag{3.3.0.6}$$

$$B = -\frac{p+1}{16}\left(C - C_0\right), \tag{3.3.0.7}$$

$$\Phi = \frac{p-3}{4}\left(C - C_0\right) + \frac{4C_0}{p-3}, \tag{3.3.0.8}$$

where again the boundary condition that for $r \to \infty$ the background should be trivial has been imposed. C_0 denotes the asymptotic value of C which is related to the asymptotic dilaton value

$$C_0 = \frac{p-3}{4}\Phi_0. \tag{3.3.0.9}$$

The equations of motion reduce to

$$\delta^{ab}\partial_a\partial_b e^{-C} = 0. \tag{3.3.0.10}$$

We solve this by

$$e^{-C} = e^{-C_0} + \frac{k_p}{r^{7-p}}. \tag{3.3.0.11}$$

The RR charge of the vacuum is

$$\mu_p = \frac{1}{\sqrt{2}\kappa}\int_{S^{8-p}} e^{\frac{3-p}{2}\Phi} \star F_{p+2} = \frac{7-p}{\sqrt{2}\kappa}\Omega_{8-p}k_p, \tag{3.3.0.12}$$

where the integration is over an asymptotic $(8-p)$–sphere surrounding the p brane, and Ω_{8-p} is the volume of the unit sphere. Now we try whether we can identify the type II p-branes with the D-branes discussed in section 2.3. This trial is motivated by the observation that the D-branes considered in section 2.3 are also extended objects carrying RR charge. In section 2.3 we computed the charge of a single D-brane to be (see (2.3.2.31) and (2.3.2.29))

$$\mu_p^{\text{single brane}} = \sqrt{2\pi}\left(4\pi^2\alpha'\right)^{\frac{3-p}{2}}. \tag{3.3.0.13}$$

Assuming that the vacuum considered in the present section is composed out of an integer number N_p of single D-branes, we identify (T_S denotes the frame dependent tension of a single fundamental string)

$$k_p = N_p\frac{2\kappa\left(4\pi^2/T_S\right)^{\frac{3-p}{2}}\sqrt{\pi}}{(7-p)\,\Omega_{8-p}}, \tag{3.3.0.14}$$

such that

$$\mu_p = N_p\mu_p^{\text{single brane}}. \tag{3.3.0.15}$$

A first consistency check is to observe that the Dirac quantization condition

$$\mu_p\mu_{6-p} = 2\pi N_p N_{6-p} \tag{3.3.0.16}$$

is satisfied. After we have fixed the integration constant k_p the tension of the brane solution is determined. Because the vacuum considered here and the D-branes considered in section 2.3 are BPS objects the tension is related to the charge and we expect that our tension should be in agreement with (2.3.2.29). Let us nevertheless compute it explicitly. To this end, we write down the asymptotic expansion of g_{00}

$$g_{00} = -1 + \frac{7-p}{8}\frac{k_p}{r^{7-p}}e^{\frac{p-3}{4}\Phi_0} + \dots . \tag{3.3.0.17}$$

The tension is given via (see also (3.2.1.16))

$$\frac{1}{r^{8-p}}\partial_r\left(r^{8-p}\partial_r g_{00}\right) = -\frac{7-p}{4}\kappa^2 T_{p,E}\frac{\delta(r)}{\Omega_{8-p}r^{8-p}}, \tag{3.3.0.18}$$

where $T_{p,E}$ denotes the tension in Einstein frame units and it is understood that terms denoted by dots in (3.3.0.17) are neglected. This yields

$$T_{p,E} = \frac{7-p}{2\kappa^2}\Omega_{8-p}k_p e^{\frac{p-3}{4}\Phi_0}. \tag{3.3.0.19}$$

Since $T_{p,E}$ has mass dimension $p+1$, it receives a factor of $e^{-\frac{p+1}{4}\Phi}$ under the transformation to the string frame. Taking this and (3.3.0.14) into account, we find for the string frame tension

$$T_p = N_p e^{-\Phi_0}\kappa\sqrt{\pi}\left(4\pi^2\alpha'\right)^{\frac{3-p}{2}}\frac{1}{\kappa}, \tag{3.3.0.20}$$

in agreement with (2.3.2.29). Thus, we found that the p-brane vacuum can be viewed as consisting out of an integer number of "elementary" (or magnetic) D-branes considered in section 2.3.

So far, we have derived this result only in the case $p \neq 3$. In the case $p = 3$, the condition (3.3.0.8) is changed into

$$\Phi = \Phi_0 \quad , \quad C_0 = 0. \tag{3.3.0.21}$$

The selfduality condition can be imposed by replacing F_5 from our ansatz with $F_5 + \star F_5$,

$$F_5 \to F_5 + \star F_5. \tag{3.3.0.22}$$

The solution for C is

$$e^{-C} = 1 + \frac{k_3}{r^4}. \tag{3.3.0.23}$$

The "electric" charge is

$$\mu_3 = \frac{1}{\sqrt{2}\kappa}\int_{S^5}\star dA_{0123} = \frac{4}{\sqrt{2}\kappa}\Omega_5 k_3. \tag{3.3.0.24}$$

The replacement (3.3.0.22) implies that the solution carries also a magnetic charge $\tilde{\mu}_p = \mu_p$. Thus, the Dirac quantization condition yields (N_3 is the number of D3-branes)

$$k_3 = N_3\sqrt{\pi}\frac{\kappa}{2\Omega_5}, \tag{3.3.0.25}$$

such that

$$\mu_3 = \sqrt{2\pi} N_3. \tag{3.3.0.26}$$

For the tension, one obtains in the string frame

$$T_3 = N_3 e^{-\Phi_0} \frac{\sqrt{\pi}}{\kappa}. \tag{3.3.0.27}$$

Thus, also for $p = 3$ we can consistently assume that the vacuum solution is made out of an integer number of the D3-branes introduced in section 2.3. We will return to the solution for $p = 3$ in section 4.3.

We stop our discussion on the appearance of branes as vacua of the effective actions at this point. We should, however, mention that there are many more configurations which can be constructed. For example, one can find vacua where branes lie within the worldvolume of other higher dimensional branes. Such studies confirm the result of section 2.3.1.3 that supersymmetry is completely broken unless the number of ND directions is an integer multiple of four.

A final remark about the BPS vacua of type I theory is in order. Although we called this section "type II branes", the discussion applies to type I theory as well. In the closed string sector of type I theory the NSNS B field is projected out, and thus there is neither a fundamental string nor an NS five brane vacuum in the effective type I theory. On the other hand, the RR two form potential survives the projection. Hence, type I theory possesses the D1 and the D5 brane vacua.

Chapter 4

Applications

In this chapter, we are going to present some applications of the branes discussed so far. In the following, we will show that branes are a useful tool in supporting duality conjectures involving an interchange between strong and weak couplings. As a first example we consider dualities among different string theories. Thereafter, field theory dualities will be translated into manipulations within certain brane setups. Next, we want to present the AdS/CFT correspondence – a duality between closed and open string theory, or in first approximation between gravity and gauge theory. Finally, we argue that branes allow constructions in which the string scale is about a TeV. Such setups have the prospect of being discovered in the near future. There are many more applications of branes in theoretical physics. Some of them we will list in chapter 6 containing suggestions for further reading.

4.1 String dualities

There are many excellent reviews on string dualities and we do not plan to provide an introduction into this subject here. We just want to summarize how branes are mapped among each other under duality transformations. We start by drawing the M-theory star in figure 4.1. The idea behind this picture is that the theories written at the tips of the "star" are different descriptions of one underlying theory called M-theory. This underlying theory is not known. It is assumed to possess a moduli space which looks like figure 4.1. The picture is supported by evidences for the conjecture that all the other theories appearing in figure 4.1 are related by motions in their moduli spaces.

Let us briefly summarize how these theories are connected. We start at the top of the star (11D SUGRA) and work our way down to the bottom (type I), first counter clockwise. Compactifying eleven dimensional supergravity on an interval ($S^1/\mathbb{Z}_2$) yields the effective field theory of the heterotic $E_8 \times E_8$ string. The dilaton is re-

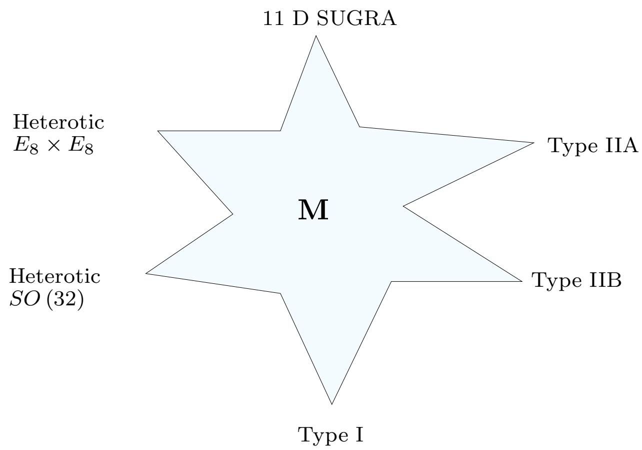

Figure 4.1: M-theory star

lated to the length of the interval such that the string coupling is small when the interval is short. The $E_8 \times E_8$ fields live as twisted sector fields at the ends of the interval (the orbifold nine planes). If we take the string coupling of the $E_8 \times E_8$ heterotic string to be strong, 11D supergravity on an interval provides the more suitable description. The connection between the $E_8 \times E_8$ and the $SO(32)$ string was already discussed in section 2.1.5.4. It does not relate strong with weak coupling but small with large compactification radii in nine dimensions. The heterotic $SO(32)$ string is connected to type I strings by a strong/weak coupling duality. Now, let us go back to the top of the star and go down clockwise. Type IIA supergravity can be obtained by compactifying 11 dimensional supergravity on a circle. The radius of the circle determines the vev of the dilaton. For small string coupling the circle is small, and for strong coupling it is large. The connection between type IIA and type IIB strings is seen by compactifying further down to nine dimensions and inverting the radius, as argued in section 2.1.5.4. Type I theory is obtained by gauging worldsheet parity of type IIB strings and adding the D-branes needed to ensure RR-charge conservation (in a sense these can be viewed as twisted sector states).

Because the branes we have discussed are stable under deformations in the moduli space, they should be mapped in a one-to-one way onto each other by string dualities. Since eleven dimensional supergravity did not appear until now in our discussion (it does not correspond to an effective weakly coupled string theory), we have to list the relevant BPS branes of 11 dimensional supergravity. 11 dimensional supergravity contains a three form gauge potential which can be Hodge dualized to a six form gauge potential. Analogously to the solutions found in the previous chapter, one finds thus a membrane (2 brane) and a five brane.

Let us now walk once around the star in figure 4.1 in a clockwise direction and follow the branes along this journey. Upon compactifying one of the eleven dimensions the momentum into this direction becomes quantized. The off diagonal metric components containing one 11 label become a Kaluza Klein gauge field – a one form potential, which can be Hodge dualized (with respect to the non compact directions) to a seven form potential. The associated BPS states are zero and six branes. These become the D0 and the D6 branes in the type IIA picture. For the branes which exist already in the uncompactified theory, there are two options within the compactification. The compact dimension can be transverse or longitudinal. Hence, the membrane will be either described by a fundamental string or by a D2 brane in weakly coupled type IIA theory, and the five brane yields the D4 brane and the NS five brane of type IIA theory.

Compactifying further down to nine dimensions and taking the decompactification limit after a T-duality transformation, type IIA theory goes over into type IIB theory. The D-branes gain or lose one spatial direction due to the T-duality, and hence we obtain all the D-branes of type IIB theory. Type IIB theory possesses a symmetry which is not depicted in figure 4.1. This is an $SL\left(2,\mathbb{Z}\right)$ symmetry which we do not want to discuss in detail. For later use we state that the $SL\left(2,\mathbb{Z}\right)$ symmetry contains a transformation called S duality. S duality interchanges strong with weak coupling, the D1 brane with a fundamental string and the D5 brane with the NS five brane. The D3 brane stays a D3 brane under S-duality.[1]

Type I strings are obtained by projecting out worldsheet parity in type IIB strings. This removes the fundamental string, the NS five brane, and the D3 brane from the spectrum of BPS states. The remaining states are the D1 and the D5 brane. Under the strong/weak coupling duality mapping of type I theory to the $SO\left(32\right)$ heterotic theory, these become the fundamental string and the NS five brane of the heterotic string. The BPS spectrum is not affected when going over to the $E_8 \times E_8$ heterotic string via T-duality. The $E_8 \times E_8$ theory is supposed to be dual to 11 dimensional supergravity on $S^1/\mathbb{Z}_2$. Therefore, let us discuss which of the branes of 11 dimensional supergravity survive the $\mathbb{Z}_2$ projection. First of all, the zero and the six branes are projected out since the Kaluza-Klein gauge field is odd under changing the sign of the eleventh coordinate. In order to deduce the $\mathbb{Z}_2$ action on the three form potential C, we note that the action of 11 dimensional supergravity contains a Chern Simons term

$$\int C \wedge dC \wedge dC.$$

This term is symmetric under the $\mathbb{Z}_2$ if C receives an additional sign, i.e. a C component

[1] We do not include the D7 brane and its counterpart, the D instanton (related by Hodge duality), into the discussion.

containing an 11 label is even under the $\mathbb{Z}_2$. Conversely, the $\mathbb{Z}_2$ even components of the dual six form potential do not contain an 11 label. From the ten dimensional perspective, a one brane and a five brane survive the $\mathbb{Z}_2$ projection. These are the fundamental string and the NS five brane in the heterotic description.

Hence, we have seen that continuous changes of M theory moduli preserve the spectrum of BPS branes. We have identified dual descriptions of branes. Note also that not all tips of the star in figure 4.1 are connected by continuous changes of moduli. For example, 11 dimensional supergravity on S^1 is not continuously connected to 11 dimensional supergravity on $S^1/\mathbb{Z}_2$. Therefore, the BPS branes of the circle compactified 11 dimensional supergravity have a one-to-one description in type IIA theory, but the type IIA BPS branes cannot all be given a heterotic description, and so on.

4.2 Dualities in Field Theory

Another area where supersymmetry allows insight into strongly coupled regions of perturbatively formulated theories are supersymmetric field theories. In this section we will focus on four dimensional $N = 1$ gauge theories with matter in the fundamental representation (supersymmetric QCD). For the various other examples we refer to the literature (see chapter 6). In supersymmetric theories, non-renormalization theorems allow to study the moduli space in strongly coupled regions. In $N = 1$ theories, the superpotential must be holomorphic in the fields. This often restricts its form, and the moduli space is found by searching for flat directions in the superpotential. A thorough analysis of $N = 1$ $SU(N_c)$ gauge theory with N_f chiral multiplets in the fundamental representation led Seiberg to the conjecture that perturbatively completely different looking theories are connected in moduli space. Analyzing results on beta functions in such theories, one finds that for $\frac{3}{2}N_c < N_f < 3N_c$ the beta function becomes zero at a certain (strong) coupling. Hence, such gauge theories flow to a conformal fixed point in the infrared (they are asymptotically free). The amazing result of Seiberg's analysis is that an $SU(N_f - N_c)$ theory with N_f chiral multiplets in the fundamental representation of $SU(N_f - N_c)$ and N_f^2 gauge singlets flows to the same infrared fixed point as the above mentioned $SU(N_c)$ theory. Thus, the moduli spaces of the two theories are connected in the strong coupling region. The field theory analysis involves first finding a duality map between conformal primaries at the infrared fixed point and to test whether the picture is consistent under continuous deformations. Another quite non trivial consistency check is that the 't Hooft anomaly matching conditions are satisfied. In the present section we will sketch how the moduli spaces of the two theories mentioned above can be connected by simply playing around with

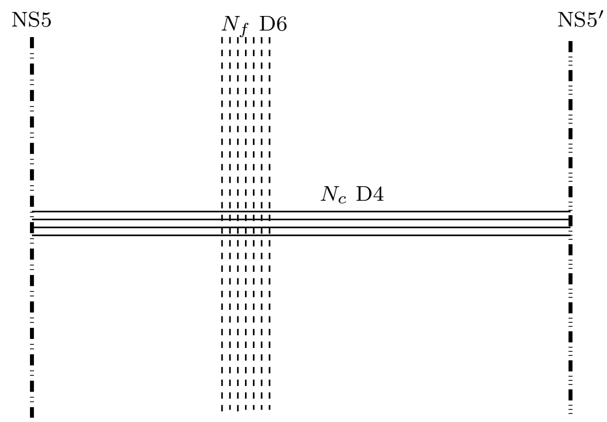

Figure 4.2: Brane setup for supersymmetric QCD. It has to be looked at in combination with table 4.1.

	0	1	2	3	4	5	6	7	8	9
NS5	–	–	–	–	–	–	·	·	·	·
NS5′	–	–	–	–	·	·	·	·	–	–
D4	–	–	–	–	·	·	–	·	·	·
D6	–	–	–	–	·	·	·	–	–	–

Table 4.1: Brane setup for supersymmetric QCD. The numbers in the first line label the dimensions. Hyphens denote longitudinal and dots transverse dimensions.

branes. Throughout this section we will neglect the back reaction of branes on the target space geometry, i.e. we take a limit where gravity decouples.

Our first task is to translate $N = 1$ $SU(N_c)$ supersymmetric gauge theory with N_f chiral multiplets in the fundamental representation into a brane setup. A setup yielding the desired theory is drawn in figure 4.2. Since it is difficult to draw pictures in ten dimensions we supplement the figure by table 4.1 where hyphens stand for longitudinal and dots for transverse dimensions. The D4-brane stretches in the sixth direction between the two NS5 branes. Hence, its extension along the sixth dimension is given by the finite distance of the NS5 and the NS5′ brane. If this distance is shorter than the experimental resolution, the theory on the D4-branes is effectively 3+1 dimensional. The positions of the NS5, NS5′ and the D4 in the seventh dimension must coincide (simply for geometrical reasons). We take N_c of such D4-branes in order to obtain $SU(N_c)$ gauge theory. The position of the D4-branes in the transverse directions is fixed by the condition that it stretches between the NS5 and NS5′ brane. The scalar fields in the adjoint of the gauge group correspond to collective coordinates for those positions. They are projected out by the boundary condition. Therefore,

	0	1	2	3	4	5	6	7	8	9
D5	–	–	–	–	–	–	·	·	·	·
D5′	–	–	–	–	·	·	·	·	–	–
F1	–	·	·	·	·	·	–	·	·	·
D3	–	·	·	·	·	·	·	–	–	–

Table 4.2: Dual brane setup for supersymmetric QCD. This can easily be checked to be consistent.

the theory on the D4-branes can admit at most $N = 1$ supersymmetry (viewed from a $3 + 1$ dimensional perspective). We will argue in a moment that there is partially unbroken supersymmetry in the above setup. Before, let us comment on the role of the D6-branes. A string starting on a D6 and ending on a D4-brane transforms in the fundamental representation of $SU(N_c)$. If we take N_f D6-branes we obtain the N_f desired multiplets in the fundamental representation of the gauge group. The $SU(N_f)$ gauge theory becomes the flavour symmetry of our supersymmetric QCD. (Indeed, the effective four dimensional coupling is obtained by integrating over the extra dimensions. It is inversely proportional to the volume of the extra dimensions. Since the D6-brane worldvolume contains non-compact extra dimensions, the $SU(N_f)$ dynamics decouples and we are left with a global symmetry.)

After we successfully constructed a brane setup for the gauge theory we are interested in, we should check whether this brane setup is consistent. One could, for example, couple it to gravity and look for explicit solutions describing such a setup. This is, however, rather complicated. What we will do instead, is to take a setup from which we know that it is consistent and connect it to the above setup through a chain of string dualities. A setup of which we know that it is consistent is given in table 4.2. Here, a fundamental string (F1) stretches between two D5-branes (D5 and D5′). This is consistent by the definition of a D-brane. Above, we have argued that at most $N = 1$ supersymmetry survives on the (interval compactified) D4-brane. An argument that this is exactly the case can be given by counting the ND directions of the setup in table 4.2 (see also section 2.3.1.3). The D5 and the D3-brane provide eight ND directions (all but the zeroth and sixth dimension). An open string starting on a D5′-brane and ending on a D3-brane has four ND directions (1237). Finally, the string stretched between the D5 and D5′-brane has mixed (ND) boundary conditions along the four dimensions: 4589. Hence, the number of ND directions is always an integer multiple of four, indicating that the spectrum possesses some supersymmetry. (There are also more precise methods to investigate the number of preserved supersymmetries. One can study the conditions for vanishing gravitino and dilatino variations in the rigid

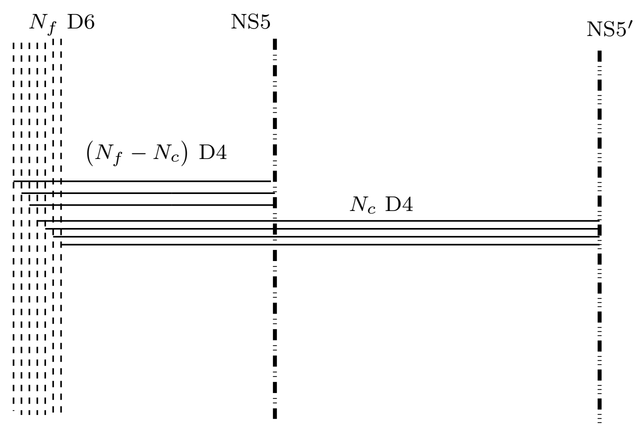

Figure 4.3: Brane setup of figure 4.2 after pushing the D6-branes past the NS5 brane.

limit, see e.g. [158] for such an analysis within the present context.) It remains to see that the setup in table 4.2 is connected to the one which we are interested in (table 4.1). Table 4.2 contains branes of type IIB theory. Therefore, we can apply an S-duality (shortly described in the previous section) on this system. This takes the D5 and D5′-brane to the NS5 and NS5′ brane of table 4.1. The fundamental string (F1) turns into a D1 string and the D3-brane remains invariant under S-duality. Performing a T duality along the first, second and third dimension (replacing type IIB by IIA) yields the configuration of table 4.1.

In the following we will describe a path in the moduli space of the setup in figure 4.2 taking us to the dual theory found by Seiberg. We will do so by essentially interchanging the position of the NS5 with the NS5′ brane. This involves however some subtleties which we will mention but not elaborate on. For more details we ask the interested reader to consult[158] or literature to be given in chapter 6. Our first step is to move the D6-branes to the left of the NS5 brane. When the D6-branes cross the NS5 brane, N_f additional D4-branes stretching between the D6-branes and the NS5 brane are created[238]. After the D6-branes have been moved to the left of the NS5 brane, there is a point in moduli space where there are no D4-branes stretching between the NS5 and the NS5′ brane. This can be achieved by connecting the N_c D4-branes stretching between NS5′ and NS5 branes with N_c out of the N_f D4-branes which stretch between D6-branes and NS5′ branes. The result of performing this first step in moduli space is drawn in figure 4.3.

Now, the boundary conditions are such that we can displace the NS5 brane in the seventh dimension. After doing so, it can be moved to the right of the NS5′ brane

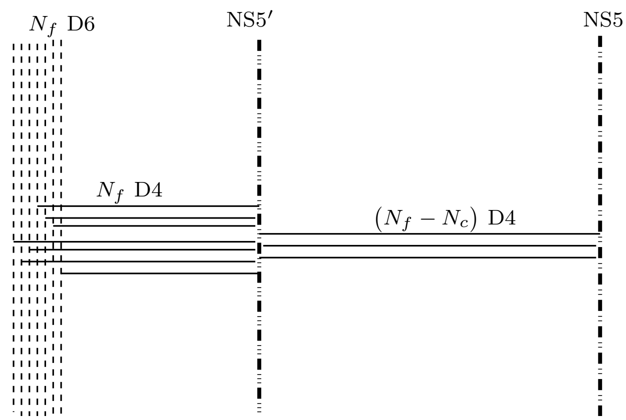

Figure 4.4: Brane setup for the dual gauge theory.

along the sixth dimension.[2] As soon as the NS5 brane is situated to the right of the NS5′ brane, we realign it in the seventh dimension with the positions of the NS5′ and the D4-branes. There are now $(N_f - N_c)$ D4-branes starting at the D6-branes passing through the NS5′ branes and ending on the NS5 brane. These we break on the NS5′ brane. The picture drawn in figure 4.4 emerges.

Finally, we need to read off the perturbative formulation of the field theory corresponding to figure 4.4. The gauge group of the theory living on the D4-branes stretching between the NS5 and the NS5′ brane is $SU(N_f - N_c)$. There are N_f chiral multiplets in the fundamental of the gauge group coming from strings stretching between the N_f D4-branes on the left and the $N_f - N_c$ D4-branes in the middle. The D4-branes to the left can move (fluctuate) in the eighth and ninth dimension. This gives rise to N_f^2 chiral multiplets which are singlets under the gauge group.

In this section we have seen that branes can be useful tools in deriving (or at least illustrating) quite nontrivial connections between gauge field theories. Our purpose was to provide the rough ideas on how this works within an example. The reader who found this interesting is strongly advised to check the literature (chapter 6) for more details and subtleties.

4.3 AdS/CFT correspondence

In this section we will describe a duality between gravity and field theory, or from a stringy perspective between closed string excitations and open string excitations. We will focus on the most prominent example where the field theory is $\mathcal{N} = 4$ supercon-

[2]Here, it is important that the NS branes do not meet when passing each other.

formal $SU(N)$ Yang-Mills theory[3] (the theory of open string excitations ending on D3 branes) and gravity lives on an $AdS_5 \times S^5$ space (the near horizon geometry of D3 branes). In the next subsection we will state the duality conjecture and mention the most obvious consistency checks. Instead of elaborating on the various more involved consistency checks which have been performed in the literature, we will discuss an application of the duality. We will use the gravity side of the conjecture (the theory of closed string excitations) to compute a Wilson loop in field theory. This will be done in a semiclassical approximation. We will also discuss next to leading order corrections. In order to avoid disappointment, we should mention here that we will not give a quantitative result for the next to leading order corrections.

4.3.1 The conjecture

From section 3.3 we recall that in the case of the D3 branes the truncated action in the Einstein frame and in the string frame look almost the same. We will work in the string frame and absorb the constant dilaton into the definition of the gravitational coupling κ. Choosing in addition a convenient[4] numerical relation between α' and κ, we can write (see (3.3.0.23))

$$e^{-C} = 1 + \frac{4\pi g_s N \alpha'^2}{r^4}, \tag{4.3.1.1}$$

where g_s denotes the string coupling, and $N = N_3$ is the number of D3 branes. Recall also that the metric is (3.3.0.3) (use (3.3.0.6) and (3.3.0.7) and $i, j = 0, \ldots, 4$, and we parameterize the transverse space by polar coordinates, i.e. $d\Omega_5^2$ is the metric of a unit five sphere)

$$ds^2 = e^{\frac{C}{2}} \eta_{ij} dx^i dx^j + e^{-\frac{C}{2}} \left(dr^2 + r^2 d\Omega_5^2 \right). \tag{4.3.1.2}$$

Now we take the near horizon limit following the prescription

$$\alpha' \to 0 \quad \text{and} \quad U \equiv \frac{r}{\alpha'} \quad \text{fixed.} \tag{4.3.1.3}$$

The first limit ensures that the field theory on the brane decouples from gravity living in the bulk. The second limit implies that we zoom in on the near horizon region. It is taken such that the mass of an open string stretching between the N D3 branes and

[3] In order to distinguish between the number of branes and the number of supersymmetries we use $\mathcal{N}$ for the number of supersymmetries in the present section.

[4] The precise relation is (in Einstein frame units) $\kappa^2 = 16\pi^7 \alpha'^4$, where $\Omega_5 = \pi^3$ has been used (see (3.3.0.25)). Plugging this into (3.2.1.22), one finds that for this choice the NSNS charge of the fundamental string equals the RR charge of the D1 string (3.3.0.13). This implies that the numerics involved in the S duality of type IIB simplifies to $g_s \to \frac{1}{g_s}$.

some probe brane at a finite distance is constant. Performing the limit (4.3.1.3) the metric (4.3.1.2) becomes[5]

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} \eta_{ij} dx^i dx^j + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]. \qquad (4.3.1.4)$$

This describes an $AdS_5 \times S^5$ geometry. Before taking a short detour on the description of AdS_5 spaces as hypersurfaces of a six dimensional space let us check the validity region of (4.3.1.4) by focusing on the S^5 part. The radius of the S^5 is

$$R^2 = \alpha' \sqrt{4\pi g_s N}. \qquad (4.3.1.5)$$

In order to avoid high curvature (where higher derivative corrections become important, or even the effective gravity description may break down) one should take this radius to be large, i.e.

$$g_s N \gg 1. \qquad (4.3.1.6)$$

In addition we should keep the string coupling small which implies that the number of D3 branes we look at has to be large. Now, we recall that the field theory living on the D3 branes is $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory. The gauge coupling is $g_{YM}^2 = 2\pi g_s$ (see (2.3.3.22)). So, at first sight it seems that the gauge coupling is small whenever the string coupling is small. However, we should also impose the condition (4.3.1.6), in particular the large N limit. For large N $SU(N)$ gauge theories ‘t Hooft developed a perturbative expansion in the parameter (the ‘t Hooft coupling) $g_{YM}^2 N$[435]. The condition (4.3.1.6) implies that the ‘t Hooft coupling is large whenever the effective gravity description is reliable. We will argue below that gravity (or closed type IIB strings) in the space (4.3.1.4) is dual to the gauge theory living on the D3 branes. Because of (4.3.1.6) this is a strong/weak coupling duality. One of the first things one should check before publishing a conjecture on dual pairs is that the global symmetries of the dual descriptions should match. (Global symmetries are observable.) Therefore, let us take a short detour and describe the AdS_5 space as a hypersurface in a six dimensional space. This will enable us to see the isometries of AdS_5 in much the same way as one sees the $SO(6)$ isometry of S^5 when viewing it as a hypersurface in six dimensional space.

The space in which we will find an AdS_{p+2} space as a hypersurface is a $2 + p + 1$ dimensional space with the metric

$$ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{\alpha=1}^{p+1} dX_\alpha^2. \qquad (4.3.1.7)$$

[5] One can obtain this metric directly when dropping the requirement of an asymptotically trivial background in the search for BPS branes in section 3.3.

Analogously to a sphere, the AdS_{p+2} space is defined as the set of points satisfying the condition

$$-X_{-1}^2 - X_0^2 + \sum_{\alpha=1}^{p+1} X_\alpha^2 = -R^2, \tag{4.3.1.8}$$

where R is called the radius of the AdS_{p+2} space. We solve this equation by

$$\begin{aligned} X_{-1} + X_{p+1} &= U, && (4.3.1.9)\\ \text{for } i = 0, \ldots, p: \quad X_i &= x^i \frac{U}{R}, && (4.3.1.10)\\ X_{-1} - X_{p+1} &= \frac{x^2 U}{R^2} + \frac{R^2}{U}, && (4.3.1.11) \end{aligned}$$

where $x^2 = \eta_{ij} x^i x^j$ and U and x^i parameterize the hypersurface (4.3.1.8). Plugging (4.3.1.9) – (4.3.1.11) into (4.3.1.7), we obtain the AdS_{p+2} metric

$$ds^2 = \frac{U^2}{R^2} \eta_{ij} dx^i dx^j + R^2 \frac{dU^2}{U^2}. \tag{4.3.1.12}$$

Comparison with (4.3.1.4) shows us that the limit (4.3.1.3) took us to an $AdS_5 \times S^5$ space where the radii of the AdS_5 and the S^5 coincide and are given by (4.3.1.5).

After this detour we can easily read of the isometries of (4.3.1.4). The isometry is $SO(4,2) \times SO(6)$. These isometries show up in the field theory on the D3 branes as follows. The $SO(6)$ $(= SU(4))$ is the R symmetry of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The beta function of the gauge theory vanishes exactly, i.e. the gauge theory is a conformal field theory. The $SO(4,2)$ part of the isometry corresponds to the conformal group which is a symmetry in the gauge theory. Taking into account the preserved supersymmetries[6], one observes that the isometry group $SO(4,2) \times SO(6)$ can be extended to the superconformal group acting in the field theory. Thus the global symmetries of the two descriptions match. In the asymptotic region $U \to \infty$ the AdS part of the metric (4.3.1.4) becomes (up to a conformal factor) the 3+1 dimensional Minkowski space. This is the boundary of the AdS space. The $SO(4,2)$ isometry acts as the group of conformal transformations on the Minkowski space. In this sense, one can identify the boundary of the AdS space with the location of the D3 branes, although one should not think of the two descriptions simultaneously, because whenever the parameters are such that the gravity description is reliable, the perturbative description of the gauge theory breaks down and vice versa.

Moreover, one can identify the $SL(2, \mathbb{Z})$ duality of the type IIB string with the Montonen Olive duality[346, 463, 360, 457] of $\mathcal{N} = 4$ super Yang Mills theory.

[6] These are enhanced in the near horizon limit.

Thus, we have seen some evidence that the AdS/CFT correspondence conjecture holds. More checks have been performed, but we will not discuss those here. In the following section we want to illustrate the duality by computing Wilson loops in gauge theory using type IIB superstrings. Before doing so, let us summarize the AdS/CFT correspondence (duality) conjecture.

- Type IIB superstrings living in an $AdS_5 \times S^5$ background are dual to open superstrings ending on a stack of D3 branes.
- The AdS_5 and the S^5 have the same radius whose value (in units of α') is related to the 't Hooft coupling of the gauge theory via equation (4.3.1.5), and $g_{YM}^2 = 2\pi g_s$.
- The type IIB string theory is in its perturbative regime if g_s is small, and higher curvature effects are not dangerous as long as (4.3.1.6) holds. In this region, the gauge theory is in the large N limit and strongly coupled.

In a somewhat weaker statement, one should replace "type IIB superstrings" by "type IIB supergravity" and "open strings ending on D3 branes" by $\mathcal{N} = 4$ $SU(N)$ gauge theory. We will take the duality conjecture as stated in the items.

4.3.2 Wilson loop computation

4.3.2.1 Classical approximation

A Wilson loop is the (normalized) partition function of gauge theory in the presence of an external quark anti-quark pair. A perturbative description of this situation in a D3 brane setup for static quarks is drawn in figure 4.5. In order to employ the AdS/CFT duality conjecture, we need to translate figure 4.5 to type IIB strings living on $AdS_5 \times S^5$. The prescription is that the open strings in figure 4.5 translate into a background string of type IIB theory on $AdS_5 \times S^5$. In the previous section we have argued that the position of the N D3 branes is translated to the boundary of the AdS_5 space. (We should point out again that the emphasis is on "translated" since the gauge theory description breaks down whenever the AdS prescription is reliable.) Therefore, the background string should fulfill the boundary condition that its ends on the AdS_5 boundary are separated by a distance L. Classically, the background string is then uniquely determined by the requirement of minimal worldsheet area. As we will see in a moment, the picture drawn in figure 4.6 arises. The fact that the string with minimal area goes down from the boundary into the AdS space and up again to satisfy the boundary condition is a result of the non trivial metric. The corresponding

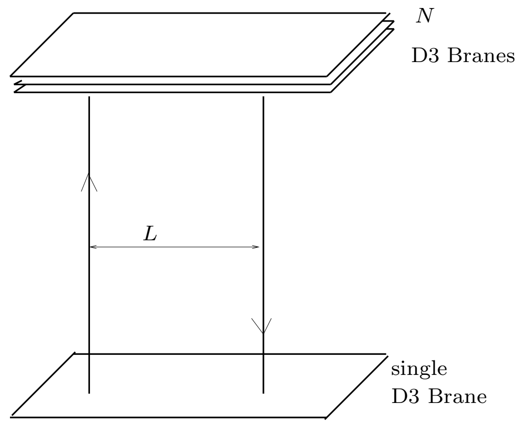

Figure 4.5: The perturbative Wilson loop setup. The quark anti-quark pair corresponds to the ends of open strings on the N D3 branes. The open strings have opposite orientation. The quark anti-quark pair is chosen to be static. The dynamics of the quarks decouples as long as the single D3 brane is very far away from the N D3 branes. The distance between the quark and the anti-quark is L.

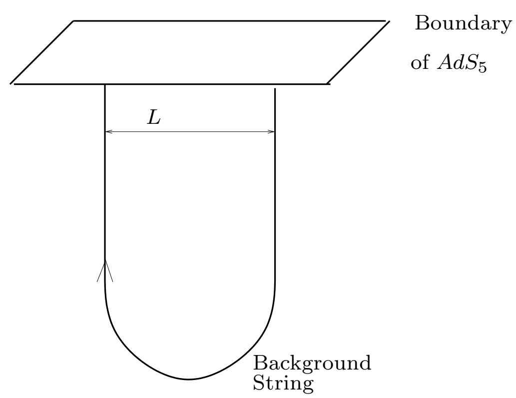

Figure 4.6: The non perturbative Wilson loop setup. The quark anti-quark pair corresponds to a background string ending on the AdS_5 boundary.

calculation can be carried out explicitly. For the sake of a minor simplification, we redefine the coordinate $U = \frac{R^2}{\alpha'}u$ such that the metric (4.3.1.4) reads

$$ds^2 = R^2\left[u^2\eta_{ij}dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2\right]. \tag{4.3.2.1}$$

The worldsheet area of the background string is

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\det g} \tag{4.3.2.2}$$

where $g_{\alpha\beta}$ is the induced metric (2.1.1.2). As an ansatz for the background string we take

$$X^0 = \tau \quad , \quad X^1 = \sigma \quad , \quad X^4 = U(\sigma)\,, \tag{4.3.2.3}$$

and the rest of the string positions is constant in σ and τ. The indices are assigned in the order in which coordinates appear in (4.3.2.1), and $X^4 = U$ (the capital U denotes the string position in the space with the metric (4.3.2.1) and should not be confused with the capital U in (4.3.1.3)). The first two equations in (4.3.2.3) represent the static gauge and the sigma dependence of U allows for the string to describe the curve of figure 4.6. This is the simplest consistent ansatz for the given boundary conditions.

The induced metric is

$$\frac{ds^2_{ind}}{R^2} = -U^2 d\tau^2 + \left(U^2 + \frac{(\partial_\sigma U)^2}{U^2}\right)d\sigma^2. \tag{4.3.2.4}$$

and thus the Nambu-Goto action (4.3.2.2) reads

$$S = \frac{TR^2}{2\pi}\int d\sigma\sqrt{(\partial_\sigma U)^2 + U^4}, \tag{4.3.2.5}$$

where T denotes the time interval we are considering and we have set α' to one (R^2 is then a dimensionless quantity giving the AdS radius in units of α'). The action (4.3.2.5) (and also the Lagrange density $\mathcal{L}$ obtained by dividing S by T) does not depend explicitly on σ. This implies

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial\sigma} \\ &= -(\partial_\sigma U)\frac{\partial\mathcal{L}}{\partial U} - \left(\partial_\sigma^2 U\right)\frac{\partial\mathcal{L}}{\partial(\partial_\sigma U)} + \frac{d\mathcal{L}}{d\sigma} \\ &= -\frac{d}{d\sigma}\left((\partial_\sigma U)\frac{\partial\mathcal{L}}{\partial(\partial_\sigma U)} - \mathcal{L}\right), \end{aligned} \tag{4.3.2.6}$$

where the Euler-Lagrange equations have been used in the last step. For our system we obtain

$$\partial_\sigma U = \pm\frac{U^2}{U_0^2}\sqrt{U^4 - U_0^4}, \tag{4.3.2.7}$$

where U_0 is a constant related to the integration constant of the last equation in (4.3.2.6). U_0 is the lower bound on the curve in figure (4.6) at $\sigma = 0$. Thus, we can solve for σ as a function of U

$$\sigma = \pm \int_{U_0}^{U} d\tilde{U} \frac{U_0^2}{\tilde{U}^2 \sqrt{\tilde{U}^4 - U_0^4}}, \tag{4.3.2.8}$$

where the plus-minus sign appears due to the two branches of the curve in figure 4.6 (σ is a horizontal coordinate and U a vertical one in this figure). At the boundary ($U \to \infty$) the difference between the two values of σ should be L. Some straightforward manipulations with the integral in (4.3.2.8) yield

$$\frac{L}{2} = \frac{1}{4U_0} B\left(\frac{3}{4}, \frac{1}{2}\right), \tag{4.3.2.9}$$

where

$$B(\alpha, \beta) = \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

denotes Euler's Beta function. Using the identities

$$x\Gamma(x) = \Gamma(x+1) \quad , \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

one finds for the integration constant

$$U_0 = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma\left(\frac{1}{4}\right)^2 L}. \tag{4.3.2.10}$$

This shows our earlier statement that the background string is uniquely determined by the boundary condition. The Wilson loop $W[C]$ is the partition function for the background string. For the classical approximation we find

$$W[C] = e^{-TE}, \tag{4.3.2.11}$$

with

$$E = \frac{R^2}{2\pi} \int d\sigma \sqrt{(\partial_\sigma U)^2 + U^4}. \tag{4.3.2.12}$$

Plugging in the classical solution (4.3.2.7) (and taking into account a factor of two due to the two branches) yields

$$E = \frac{R^2}{\pi} \int_{U_0}^{\infty} dU \frac{U^2}{\sqrt{U^4 - U_0^4}}. \tag{4.3.2.13}$$

Now we split this integral into two pieces (the motivation for this will become clear below)

$$E = E_c + E_s, \tag{4.3.2.14}$$

with

$$E_s = \frac{R^2}{\pi}\int_{U_0}^{\infty} dU \frac{U^4+U_0^4}{U^2\sqrt{U^4-U_0^4}}, \tag{4.3.2.15}$$

and

$$E_c = -\frac{R^2}{\pi}\int_{U_0}^{\infty} dU \frac{U_0^4}{U^2\sqrt{U^4-U_0^4}}. \tag{4.3.2.16}$$

Let us first discuss the integral E_s. This integral is divergent due to the upper integration bound and we regularize it by a cutoff U_{max}. The asymptotic expansion for large U_{max} is

$$\begin{aligned} E_s &= \frac{R^2 U_0}{\pi}\int_1^{\frac{U_{max}}{U_0}} dy \frac{y^4+1}{y^2\sqrt{y^4-1}} \\ &= \frac{R^2 U_0}{\pi}\left[\frac{\sqrt{y^4-1}}{y}\right]_1^{\frac{U_{max}}{U_0}} = \frac{R^2}{\pi}U_{max} + \ldots\,, \end{aligned} \tag{4.3.2.17}$$

where the dots stand for terms going to zero as U_{max} is taken to infinity. Thus, E_s corresponds to the self energy of the two strings in figure 4.5. It does not depend on the distance L and diverges as the length of the string is taken to infinity. Here, we observe one interesting feature of the AdS/CFT correspondence. From the AdS perspective U_{max} is a large distance, i.e. an IR cutoff. On the field theory side this appears as a cutoff for high energies, i.e. a UV cutoff. This interchange between infrared and ultraviolet cutoffs is a general characteristics of the correspondence[434]. Let us give a technical remark in connection with the integral E_s. Plugging in our classical solution (4.3.2.7) into the induced metric (4.3.2.4), we find the classical value of the induced metric (for later use we call this $R^2 h_{\alpha\beta}$)

$$ds^2_{class} \equiv R^2 h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = R^2\left(-U^2 d\tau^2 + \frac{U^6}{U_0^4}d\sigma^2\right). \tag{4.3.2.18}$$

The scalar curvature computed from $h_{\alpha\beta}$ reads

$$R^{(2)} = 2\frac{U^4+U_0^4}{U^4}. \tag{4.3.2.19}$$

With this information it is easy to verify that the structure of the self energy integral is

$$E_s = \frac{R^2}{4\pi T}\int d^2\sigma \sqrt{-h}R^{(2)}. \tag{4.3.2.20}$$

(This does not contradict the Gauss–Bonnet theorem because the worldsheet of the background string is not compact.)

Now, let us come to the second contribution in (4.3.2.14). This will turn out to be the more interesting one. Its computation is quite similar to the computation of U_0 in terms of L. Therefore, let us just give the result

$$E_c = -\frac{4\pi^2\sqrt{2g_{YM}^2 N}}{\Gamma\left(\frac{1}{4}\right)^4 L}, \tag{4.3.2.21}$$

where (4.3.1.5) and $g_{YM}^2 = 2\pi g_s$ has been used. This is the part of the quark anti-quark potential which arises due to gluon exchange among the two quarks. It is a Coulomb potential. Since L is the only scale appearing in the setup and $\mathcal{N} = 4$ supersymmetric Yang–Mills theory has a conformal symmetry, there can be only a Coulomb potential. Anything else would need another scale to produce an energy, but this cannot appear due to conformal invariance.

In that respect models with less or none supersymmetry are more interesting because one can observe confinement in those models. The corresponding literature is listed in chapter 6. The case we are considering here, is the one where the AdS/CFT correspondence is perhaps best understood. We will study a question which is interesting from a more theoretical perspective, namely whether there are corrections to the result (4.3.2.21).

4.3.2.2 Stringy corrections

Before discussing corrections to (4.3.2.21) we should envisage the possibility that (4.3.2.21) is an exact result. There are some results which may lead to this conclusion. By analyzing the structure of possible corrections to the $AdS_5 \times S^5$ geometry, physicists[43, 340, 276] found that this geometry is exact. Still, there is a very simple argument destroying the hope that (4.3.2.21) might be exact. Namely, the above Wilson loop computation can also be performed in the perturbative regime, where the ‘t Hooft coupling is small. Then one finds, of course, also a Coulomb law but the dependence on the ‘t Hooft coupling is linear instead of a square root dependence (which actually cannot be obtained in a perturbative calculation). This does not contradict the result (4.3.2.21) but tells us that taking the ‘t Hooft coupling smaller should result in corrections such that finally for very small ‘t Hooft coupling the square root like dependence goes over into a linear one.

After we have excluded corrections to the $AdS_5 \times S^5$ geometry, we will study fluctuations of the IIB string around the background string in figure 4.6. That is, we consider the Wilson loop as the quantum mechanical partition function

$$W[C] = \int [\mathcal{D}\delta X]\,[\mathcal{D}\delta\theta]\, e^{-S_{IIB}(X+\delta X,\delta\theta)}, \tag{4.3.2.22}$$

where δX denote bosonic fluctuations and $\delta\theta$ fermionic ones (the fermionic background of the string is trivial). Before going into the details of the computation, let us describe the expansion we are going to perform. From (4.3.2.2) and (4.3.2.1) we see that the square root of the 't Hooft coupling appears as an overall constant in front of the metric. (This is also true for terms containing fermions.) Therefore, the expansion parameter $\hbar$ (or α' in section 2.1.3) is identified with the inverse square root of the 't Hooft coupling. The expression (4.3.2.22) can be computed as a power series in this parameter. In particular, the next to leading order correction to (4.3.2.21) will not depend on the 't Hooft coupling. It is this correction we will discuss in some detail in the following. In order to be able to use (4.3.2.22) for explicit calculations we need to know the type IIB string action in an $AdS_5 \times S^5$ background. Fortunately, this has been constructed in the literature[340]. These authors gave a type IIB action in the Green Schwarz formalism, which is appropriate in the presence of non-trivial RR backgrounds. The construction is similar to the one discussed in section 2.1.1.3. One uses target space supersymmetry and kappa symmetry as a guide. The technicalities are rather involved and we will not discuss them here. Because we will restrict ourselves to terms second order in fluctuations we only need a truncated version of the action of type IIB strings on $AdS_5 \times S^5$. The complete action does not contain terms with an odd number of target space fermions, in particular no terms linear in target space fermions. Since the fermionic background is trivial, contributions quadratic in fluctuations can either have two fermionic fluctuations and no bosonic fluctuation or only bosonic fluctuations. The part of the action which is quadratic in the fluctuations consists of a sum of terms with only bosonic fluctuations and terms with only fermionic fluctuations.

Let us discuss the bosonic fluctuations first. The type IIB action is a kappa symmetric extension of (4.3.2.2). For the bosonic fluctuations only the contribution (4.3.2.2) is relevant (for the lowest non trivial contribution). As in section 2.1.3 we parameterize the fluctuations by tangent vectors to geodesics connecting the background with the actual value, i.e. we perform a normal coordinate expansion. The quantum fields are

$$\xi^a = E^a_\mu \xi^\mu, \tag{4.3.2.23}$$

where E^a_μ are the vielbein components obtained by taking the square root of the diagonal metric components. A number as a label on a ξ will stand for a flat index (a), unless stated explicitly otherwise. The computation of the term second order in the ξ^a is a bit lengthy but straightforward with the information given in section 2.1.3. (The only difference to section 2.1.3 is that we expand now a Nambu-Goto action instead of a Polyakov action.) Before giving the result it is useful to perform a local

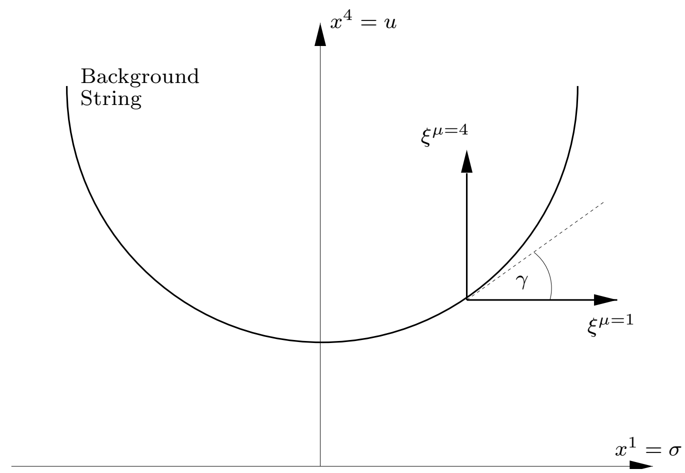

Figure 4.7: Perpendicular and longitudinal fluctuations in the one–four plane.

Lorentz rotation in the space spanned by the ξ^a. The rotation is[7]

$$\begin{pmatrix} \xi^{\|} \\ \xi^{\perp} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^4 \end{pmatrix}, \tag{4.3.2.24}$$

with

$$\cos\alpha = \frac{U_0^2}{U^2} \quad , \quad \sin\alpha = \frac{\sqrt{U^4 - U_0^4}}{U^2}. \tag{4.3.2.25}$$

Note that the determinant of the matrix appearing in (4.3.2.24) is one. The fields $\xi^{\|}$ and $\xi^{\perp}$ describe fluctuations parallel and perpendicular to the worldsheet, respectively. This is illustrated in figure 4.7. Fluctuations drawn into figure 4.7 carry Einstein indices which we indicated explicitly. The angle γ is given by the slope of the background string

$$\tan\gamma = \partial_\sigma U. \tag{4.3.2.26}$$

The combination

$$\xi^{\mu=4} - \tan\gamma\, \xi^{\mu=1} \tag{4.3.2.27}$$

vanishes for

$$\tan\gamma = \frac{\xi^{\mu=4}}{\xi^{\mu=1}}, \tag{4.3.2.28}$$

[7] We employ the symmetry of the background string and take $0 \leq \sigma \leq \frac{L}{2}$ and the plus sign in (4.3.2.7).

i.e. if $\xi^{\mu=1} + \xi^{\mu=4}$ is tangent to the background string. Thus, the combination in (4.3.2.27) is normal to the background string. Transforming the indices to flat ones (with the vielbein) and ortho-normalization yields $\xi^{\parallel}$ and $\xi^{\perp}$ with the given interpretation. When writing down the Lagrangian second order in the fluctuations, we can set $R = 1$ since we know the general R dependence (*viz.* none) from the argument given above. The Lagrangian for the bosonic fluctuations comes out to be

$$\mathcal{L}^{(2)}_{bosons} = \mathcal{L}^{(2)}_{AdS_5} + \mathcal{L}^{(2)}_{S^5} \tag{4.3.2.29}$$

with

$$\begin{aligned} \mathcal{L}^{(2)}_{AdS_5} &= \frac{1}{2}\sqrt{-h}\Bigg[\sum_{a=2,3,\perp} \xi^a \Delta \xi^a - 2\left(\xi^2\right)^2 - 2\left(\xi^3\right)^2 \\ &\quad + \left(R^{(2)} - 4\right)\left(\xi^{\perp}\right)^2\Bigg], \end{aligned} \tag{4.3.2.30}$$

$$\mathcal{L}^{(2)}_{S^5} = \frac{1}{2}\sqrt{-h}\sum_{a'=5}^{9} \xi^{a'} \Delta \xi^{a'}, \tag{4.3.2.31}$$

where total derivative terms have been dropped (the fluctuations should satisfy Dirichlet boundary conditions in order not to change the classical boundary conditions). Further, Δ denotes the two dimensional Laplacian with respect to the metric $h_{\alpha\beta}$ (4.3.2.18) and $R^{(2)}$ is the corresponding scalar curvature (4.3.2.19). We observe that the longitudinal fluctuations ξ^0 and $\xi^{\parallel}$ drop out of the action. Hence, we can fix the worldsheet diffeomorphisms via

$$\xi^0 = \xi^{\parallel} = 0. \tag{4.3.2.32}$$

If the normalization of the functional integral in (4.3.2.22) contains a division by the volume of the worldsheet diffeomorphisms, we cancel the ξ^0 and $\xi^{\parallel}$ integration against this term in the normalization. This may be problematic, and we will comment on this issue later.

It remains to study the fermionic fluctuations. Since the fermionic background is trivial we just need to copy the Lagrangian from[340] (truncated to quadratic terms) and to plug in our background. The result of the copying task is

$$\begin{aligned} \mathcal{L}_F &= -\frac{1}{2}\sqrt{-h}h^{\alpha\beta}\left(E^{\hat{a}}_{\alpha} - i\bar{\theta}^I\hat{\gamma}^{\hat{a}}\left(D_\alpha\theta\right)^I\right)\left(E^{\hat{a}}_{\beta} - i\bar{\theta}^J\hat{\gamma}^{\hat{a}}\left(D_\beta\theta\right)^J\right) \\ &\quad - i\epsilon^{\alpha\beta}E^{\hat{a}}_{\alpha}\left(\bar{\theta}^1\hat{\gamma}^{\hat{a}}\left(D_\beta\theta\right)^1 - \bar{\theta}^2\hat{\gamma}^{\hat{a}}\left(D_\beta\right)^2\right). \end{aligned} \tag{4.3.2.33}$$

First, we need to explain some of the notation. The index $\hat{a} = 0, \ldots, 9$ labels the tangent space coordinates of $AdS_5 \times S^5$. Later, we will use an index $a = 0, \ldots, 4$ for the tangent space of AdS_5 and $a' = 5, \ldots, 9$ for the tangent space of S^5. The vielbein with a worldsheet index is

$$E_\alpha^{\hat{a}} = E_\mu^{\hat{a}} \partial_\alpha X^\mu, \tag{4.3.2.34}$$

where X^μ is the position of the background string. The indices $I, J = 1, 2$ label the two target space supersymmetries. The derivative D_α is defined as

$$\begin{aligned} (D_\alpha \theta)^I &= \left[\delta^{IJ} \left(\partial_\alpha + \frac{1}{4} (\partial_\alpha X^\mu) \, \omega_\mu^{ab} \gamma^{ab} \right) - \frac{i}{2} \epsilon^{IJ} (\partial_\alpha X^\mu) \, E_\mu^a \gamma^a \right] \theta^J \\ &\equiv \nabla_\alpha \theta^I - \frac{i}{2} \epsilon^{IJ} (\partial_\alpha X^\mu) \, E_\mu^a \gamma^a \theta^J. \end{aligned} \tag{4.3.2.35}$$

Here, ω_μ^{ab} denotes the target space spin connection and we have used the fact that our background is trivial in S^5 directions. The gamma matrices $\hat{\gamma}^{\hat{a}} = \left(\gamma^a, i\gamma^{a'} \right)$ satisfy $SO(4,1)$ and $SO(5)$ Clifford algebras, respectively. The θ^I are sixteen component spinors each. They are conveniently labeled by a double spinor index $\theta^{\alpha\alpha'}$ where α is a spinor index in the tangent space of the AdS_5, and α' a spinor index in the tangent space of S^5. The γ^a and $\gamma^{a'}$ are four times four matrices tensored with four times four identity matrices. (In the following we will suppress target space spinor indices in order to avoid confusion with worldsheet indices which are also labeled by small Greeks.)

We do not intend to give a derivation of (4.3.2.33) but let us have a brief look at its structure before proceeding. Expression (4.3.2.35) is a tensor (density) with indices $\alpha\beta$ contracted either with $h^{\alpha\beta}$ or $\epsilon^{\alpha\beta}$. The terms with $h^{\alpha\beta}$ can be thought of as arising from the replacement (2.1.1.36)[8] whereas the $\epsilon^{\alpha\beta}$ contracted terms come from the Wess Zumino term (2.1.1.38) needed for kappa symmetry. The details differ from the discussion in section 2.1.1.3 due to the different target space geometry and the RR four form flux.

For our background, the Lagrangian (4.3.2.33) can be written in a compact way

$$\mathcal{L}_F = -\sqrt{-h} \left(\bar{\theta}^1, \bar{\theta}^2 \right) \begin{pmatrix} 2iE_\mu^a (\partial_\alpha X^\mu) \, \gamma^a \mathcal{P}_-^{\alpha\beta} \nabla_\beta & 1 - \mathcal{B} \\ -1 - \mathcal{B} & 2iE_\mu^a (\partial_\alpha X^\mu) \, \gamma^a \mathcal{P}_+^{\alpha\beta} \nabla_\beta \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}, \tag{4.3.2.36}$$

where X^μ stands for the background position of the string and

$$\mathcal{P}_\pm^{\alpha\beta} = \frac{1}{2} \left(h^{\alpha\beta} \pm \frac{\epsilon^{\alpha\beta}}{\sqrt{-h}} \right), \tag{4.3.2.37}$$

$$\mathcal{B} = \frac{1}{2\sqrt{-h}} \epsilon^{\alpha\beta} E_\mu^a E_\nu^b (\partial_\alpha X^\mu) (\partial_\beta X^\nu) \, \gamma^{ab}. \tag{4.3.2.38}$$

[8] In general, the combination in (2.1.1.36) contains also terms higher than quadratic order in the fermions. This is because the superalgebra is altered in the $AdS_5 \times S^5$ case as compared to a flat target space. This should be clear by noting that the isometries form a subgroup of the supersymmetry transformations.

As usual, a gamma with a multiple index is the antisymmetrised product of gamma matrices.

It is useful to perform the rotation (4.3.2.24) also on the spinors (with α as given in (4.3.2.25))

$$\theta^I = \left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\gamma^{14}\right)\psi^I. \tag{4.3.2.39}$$

In order to compute the partition function, we should fix the kappa symmetry. This is conveniently done in terms of the nilpotent matrices

$$\gamma^{\pm} = \frac{1}{2}\left(\gamma^0 \pm \gamma^1\right). \tag{4.3.2.40}$$

In analogy to section 2.1.1.3, we choose the kappa fixing conditions

$$\gamma^-\psi^1 = 0 \quad , \quad \gamma^+\psi^2 = 0. \tag{4.3.2.41}$$

We assume that the integration over spinors not satisfying (4.3.2.41) cancels the volume of kappa transformations appearing as a normalization factor in the functional integral. This may be problematic, and we will comment on this issue later. Spinors satisfying the kappa fixing condition are then governed by the Lagrangian

$$\mathcal{L}_F = -\sqrt{-h}\,(\bar\psi^1,\,\bar\psi^2)\begin{pmatrix} i\gamma^+\nabla_+ & 2 \\ -2 & i\gamma^-\nabla_- \end{pmatrix}\begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix}, \tag{4.3.2.42}$$

where $\nabla_\pm$ are tangent space derivatives defined as follows

$$\nabla_\pm = e_0^\tau\nabla_\tau \pm e_1^\sigma\nabla_\sigma = \frac{1}{U}\nabla_\tau \pm \frac{U_0^2}{U^3}\nabla_\sigma, \tag{4.3.2.43}$$

where e_0^τ and e_1^σ are two dimensional (inverse) vielbein components obtained from the square roots of the diagonal elements of $h^{\alpha\beta}$. Note also that the covariant derivative simplifies when acting on spinors satisfying (4.3.2.41). Defining partial tangent space derivatives analogously to (4.3.2.43) one finds

$$\nabla_\pm\psi^I = \left(\partial_\pm \pm \frac{\omega_\pm}{2}\right)\psi^I, \tag{4.3.2.44}$$

where

$$\omega_\pm = e_0^\tau\omega_\tau^{01} \pm e_1^\sigma\omega_\sigma^{01} \tag{4.3.2.45}$$

are tangent space components of the two dimensional spin connection ω_α^{01} computed from the zweibeinen defined in (4.3.2.43). Let us further define the matrices

$$\rho^+ = \begin{pmatrix} 0 & 0 \\ \gamma^0 & 0 \end{pmatrix} \quad , \quad \rho^- = \begin{pmatrix} 0 & -\gamma^0 \\ 0 & 0 \end{pmatrix}. \tag{4.3.2.46}$$

These are the same matrices as in (2.1.1.17) with i replaced by $\gamma^0/2$. Finally, we rewrite (4.3.2.42) in a suggestive way as follows

$$\begin{aligned}
\mathcal{L}_F &= -\sqrt{-h}\,(\bar\psi_2,\,-\bar\psi_1)\begin{pmatrix} -2 & i\gamma^-\nabla_- \\ -i\gamma^+\nabla_+ & -2 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \\
&= -\sqrt{h}\,(\bar\psi_2,\,-\bar\psi_1)\begin{pmatrix} -2 & 2i\gamma^0\nabla_- \\ -2i\gamma^0\nabla_+ & -2 \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \\
&= 2\sqrt{-h}\,(\bar\psi_2,\,-\bar\psi_1)\,(i\rho^m\nabla_m+1)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad (4.3.2.47)
\end{aligned}$$

where in the second line (4.3.2.41) has been used and a repeated index m stands for the sum over the labels $+$ and $-$. Comparison with the expressions in section 2.1.1.2 shows that the part of the action containing target space spinor fluctuations 'metamorphosed' into an action for worldsheet spinors after imposing the kappa fixing condition (4.3.2.41). The difference is that the derivative contains the spin connection due to the non trivial worldsheet metric $h_{\alpha\beta}$ (4.3.2.18), and the mass terms appearing due to the constant non vanishing curvature of the AdS space.

Now we have collected all the information needed to express the second order fluctuation contribution to (4.3.2.22) in terms of determinants of two dimensional differential operators. (For Dirac operators one uses the formal identity $\det A = \sqrt{\det A^2}$.)

Integration over the fluctuations leads to determinants of operators which can be read off from (4.3.2.30), (4.3.2.31) and (4.3.2.47). The corrected expression for the Wilson loop reads

$$W[C] = e^{-TE_{class}}\frac{\det\left(-\Delta_F - \frac{1}{4}R^{(2)} + 1\right)}{\det\left(-\Delta+2\right)\det^{\frac{1}{2}}\left(-\Delta+4-R^{(2)}\right)\det^{\frac{5}{2}}\left(-\Delta\right)}. \qquad (4.3.2.48)$$

The exponential is the classical contribution with E_{class} given by (4.3.2.14), (4.3.2.20) and (4.3.2.21). Note also that the operator appearing in the numerator of (4.3.2.48) is a four times four matrix. The Laplacian acting on worldsheet fermions Δ_F is $\eta^{mn}\nabla_n\nabla_m$. Unfortunately it is not known how to evaluate the determinants in (4.3.2.48) exactly. What is known exactly are the divergent contributions. These are given in (2.1.3.33). They are of the form

$$E_{div} \sim \int d^2\sigma\sqrt{-h}R^{(2)}. \qquad (4.3.2.49)$$

Comparing with (4.3.2.20), we find that this divergence renormalizes the self-energy which is infinite anyway. A correction to the Coulomb charge of the quarks will be finite. Unfortunately, we cannot give it in a more explicit way (as a number).

In addition, there is also a conceptual puzzle with the divergent contribution. Although for our problem it is not relevant, it should not be there. The argument that something might have gone wrong goes as follows. The string action is equivalent to a Polyakov type action, at least at a classical level (see section (2.1.1.1)). The Polyakov action is conformally invariant, and in a consistent string background the conformal invariance should not be broken by quantum effects. Therefore, divergences which introduce a cutoff (or renormalization group scale) cannot occur. Indeed, it was argued in[149] that a treatment analogous to ours but with a Polyakov instead of the Nambu-Goto action leads to a finite result. This treatment is a bit more complicated since the worldsheet metric appears as an independent field which also fluctuates. The advantage is however that subtle contributions due to the occurring integral measures are well understood. Such contributions are typically of the structure (4.3.2.49)[16, 193, 328]. (Note, however, that if the worldsheet metric is identified with the induced metric, the term in (4.3.2.49) is not really distinguishable from $(\partial X)^2$ terms (see e.g.[155])). In our derivation, we have mentioned already two places where nontrivial measure contributions could arise. This could happen when we cancel the integration over the longitudinal fluctuations against the volume of the worldsheet diffeomorphisms and in the kappa fixing procedure. Unfortunately, the Nambu-Goto case is less understood than the Polyakov formulation. (For a recent attempt to fix the functional measures in the bosonic part see[347].) Fortunately, the result of the better understood calculation in the Polyakov approach is identical to the one given here (up to the irrelevant divergence)[149].

With these open questions we close our discussion on the AdS/CFT correspondence. The reader who wants to know more will find some references in chapter 6.

4.4 Strings at a TeV

So far we have not determined the numerical value of the string scale (set by α') in terms of a number. We restricted our discussions mostly to the massless excitations of the string. This was motivated by the belief that the string scale (in energies) is large compared to observed energy scales. Often it is comparable to the Planck scale. This identification is motivated by studies of heterotic weakly coupled strings which provided for a long time the most promising starting point in constructing phenomenologically interesting models. Such models are obtained by compactifying the ten dimensional heterotic (mostly $E_8 \times E_8$) string down to four dimensions on a Calabi–Yau manifold. Let us give a rough estimate for the resulting four dimensional couplings. The effective four dimensional heterotic action is of the form

$$S_{het} = \int d^4x \frac{V}{g_h^2}\left(l_h^{-8}R^{(4)} - l_h^{-6}\mathrm{tr}F^2 + \dots\right), \qquad (4.4.0.1)$$

where we drop details which are not relevant for the present estimate on scales. In (4.4.0.1) l_h is the heterotic string scale (set by α'), g_h the heterotic string coupling (fixed by the dilaton vev), and V is the volume of the compact space. The quantities in which four dimensional physics is usually described are the four dimensional Planck mass M_p and the gauge coupling g_{YM}. These are related to the input data (g_h, l_h and V) as follows

$$M_p^2 = \frac{V}{g_h^2 l_h^8} \quad , \quad \frac{1}{g_{YM}^2} = \frac{V}{g_h^2 l_h^6}. \tag{4.4.0.2}$$

Expressing g_h in the first equation in terms of the second equation and further defining the string mass scale as $M_h = 1/l_h$ the above equation can be rewritten as

$$M_h = g_{YM} M_p \quad , \quad g_h = g_{YM} \frac{\sqrt{V}}{l_h^3}. \tag{4.4.0.3}$$

Now we assume that a $g_{YM} \sim 0.2$ is a realistic value. (This is the gauge coupling of the minimal supersymmetric standard model at the GUT scale.) Plugging $g_{YM} = 0.2$ into the first equation in (4.4.0.3) we find that the heterotic string scale is

$$M_h \sim 10^{18}\text{GeV}, \tag{4.4.0.4}$$

i.e. of the order of the Planck scale. The second equation in (4.4.0.3) implies that the compact space is also of the Planck size if we want to stay within the region where the string coupling is small.

Now let us investigate how the above estimates on scales are altered in a theory containing branes. Phenomenologically interesting models arise also as orientifold compactifications of type II theories. As we have seen in section 2.4, these contain typically D-branes on which the gauge interactions are localized whereas the gravitational sector corresponds to closed string excitations which propagate in all dimensions. Assuming that the gauge sector (and charged matter) is confined to live on Dp-branes the effective action for the orientifold compactification will be of the form

$$S_{ori} = \int d^{10}x \frac{1}{g_{II}^2 l_{II}^8} R - \int d^{p+1}x \frac{1}{g_{II} l_{II}^{p-3}} \mathrm{tr} F^2, \tag{4.4.0.5}$$

where l_{II} and g_{II} are the string scale and coupling of the underlying type II theory, respectively. Assuming further that our orientifold construction is such that the compact space has dimensions which are transverse to all relevant D-branes we denote by $V_\perp$ the volume of the compact space transverse to the branes and by $V_\|$ the volume of the compact space longitudinal to the branes (such that the overall compact volume is $V = V_\perp V_\|$). With this notation the four dimensional action reads

$$S_{ori} = \int d^4x \frac{V_\| V_\perp}{g_{II}^2 l_{II}^8} R - \int d^4x \frac{V_\|}{g_{II} l_{II}^{p-3}} \mathrm{tr} F^2, \tag{4.4.0.6}$$

from which we obtain the four dimensional Planck length l_p and gauge coupling g_{YM}

$$\frac{1}{l_p^2} = \frac{V_\parallel V_\perp}{g_{II}^2 l_{II}^8} \quad , \quad \frac{1}{g_{YM}^2} = \frac{V_\parallel}{g_{II} l_{II}^{p-3}}. \tag{4.4.0.7}$$

Hence, the four dimensional Planck mass ($M_p = 1/l_p$) and the string coupling g_{II} are

$$M_p^2 = \frac{v_\perp l_{II}^{-2}}{v_\parallel \left(g_{YM}\right)^4} \quad , \quad g_{II} = g_{YM}^2 v_\parallel, \tag{4.4.0.8}$$

where

$$v_\parallel = V_\parallel \, l_{II}^{3-p} \quad , \quad v_\perp = V_\perp \, l_{II}^{p-9} \tag{4.4.0.9}$$

are dimensionless numbers describing the size of the compact space in string scale units. The relations (4.4.0.8) allow to take the string length l_{II} larger than the four dimensional Planck length l_p. This can be achieved by taking $v_\perp$ large. The size of the parallel volume is taken to be of the "string size", i.e. $v_\parallel \sim 1$. If the parallel volume is smaller than the string size, we T-dualize with respect to the smaller dimension. This dimension will then contribute to the perpendicular volume since the string changes boundary conditions. Hence, the $v_\parallel < 1$ case is T dual to the considered case of large $v_\perp$. On the other hand if $v_\parallel > 1$, the second equation in (4.4.0.8) tells us that in this case the string coupling becomes strong, and our description breaks down. (Moreover, it is problematic for gauge interactions to be compactified on large volumes because the corrections to the four dimensional gauge interactions are usually ruled out by experimental accuracy.)

Let us analyse in some detail what happens if we choose a TeV for the string scale. This is about the lowest value which is just in agreement with experiments. (For a lower value massive string excitations should have shown up in collider experiments.) With $M_p \sim 10^{16}$TeV we find

$$v_\perp \sim 10^{28} \;\rightarrow\; V_\perp = 10^{28} \frac{1}{(\mathrm{TeV})^{9-p}}. \tag{4.4.0.10}$$

The Planck length is about 10^{-33}cm and hence in our units one TeV corresponds to $1/\left(10^{-18}\mathrm{m}\right)$. Thus we obtain

$$V_\perp \sim 10^{28-(9-p)18} \, (\mathrm{m})^{9-p} \,. \tag{4.4.0.11}$$

For the case $p = 8$ (one extra large dimension) we obtain that the perpendicular dimension is compactified on a circle of the size

$$p = 8 \;\rightarrow\; R_\perp \sim 10^{10}\mathrm{km}. \tag{4.4.0.12}$$

Such a value is certainly excluded by observations. (In the next subsection we will compute corrections to Newton's law due to Kaluza-Klein massive gravitons and see that the size of the compact space should be less than a mm.) For $p = 7$ we obtain (distributing the perpendicular volume equally on the two (extra large) dimensions)

$$p = 7 \rightarrow R_\perp \sim 0.1\text{mm}. \tag{4.4.0.13}$$

This value is just at the edge of being experimentally excluded. The situation improves the more extra large dimensions there are. For example in the case $p = 3$ (and again a uniform distribution of the perpendicular volume on the six dimensions ($V_\perp = R_\perp^6$)) we obtain

$$p = 3 \ \rightarrow \ R_\perp \sim 10^{-10}\text{m}, \tag{4.4.0.14}$$

which is in good agreement with the experimental value ($R_\perp^{exp} = 0...0.1$mm).

We have seen that D branes allow the construction of models where the string scale is as low as a TeV. (Note also, that in the above discussion we can perform T-dualities along the string sized parallel dimensions. This changes p but leaves the large extra dimensions unchanged. Actually, it might be preferable to have $p = 3$ in order to avoid Kaluza–Klein gauge bosons of a TeV mass.) This gives the exciting perspective that string theory might be at the horizon of experimental discovery. In near future collider experiments, massive string modes would be visible. In addition, the extra large dimensions could be also discovered soon. This can happen either by the production of Kaluza-Klein gravitons in particle collisions or by short distance Cavendish like experiments. However, it might as well be the case that models with less "near future discovery potential" are realized in nature.

Apart from the prospect of being observed soon, strings at a TeV scale are interesting for another reason. If the string scale is at a TeV, we would call this a fundamental scale. Thus the hierarchy problem would be rephrased. With the fundamental scale at a TeV we should wonder why the (four dimensional) Planck scale is so much higher, or why gravitational interactions are so much weaker than the other known interactions. This hierarchy is now attributed to the size of the extra large dimensions. Supersymmetry may not be necessary to explain the hierarchy between the Planck scale and the weak scale. Therefore, in the above models supersymmetry could be broken already by the compactification. In such models the question of stability is typically a problematic issue.

The above considerations are also interesting if one does not insist on a direct connection to string theory. If one just starts 'by hand' with a higher dimensional setup containing branes, one would also obtain the first equation in (4.4.0.8). In this case, one calls l_{II} the higher dimensional Planck length, which in turn can be chosen to be 1/TeV.

4.4.1 Corrections to Newton's law

In the previous section we stated that observations provide experimental bounds on the size of extra dimensions. In the brane setup in which we found the possibility of large (as compared to the Planck length) extra dimensions, these extra dimensions are typically tested only by gravitational interactions. Therefore, let us describe the influence of additional dimensions on the gravitational interaction in some more detail. We will be interested in the Newtonian limit of gravity. For simplicity, we assume that the space is of the structure $M_4 \times T^n$, where M_4 is the $3+1$ dimensional compact space and T^n is an n dimensional torus of large volume. (There might be an additional compact space of Planck size. This does not enter the computation carried out below.)

The analysis we will carry out here is similar to the discussion of the massless scalar in section 2.1.5.1, where the role of the scalar is taken over by the Newton potential. Let us arrange the spatial coordinates into a vector $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ corresponds to the M^4 and $\mathbf{y}$ to the T^n. For simplicity we assume that the torus is described by a quadratic lattice and the uniform length of a cycle is $2\pi R$, i.e.

$$\mathbf{y} \equiv \mathbf{y} + 2\pi R. \tag{4.4.1.15}$$

The $n+4$ dimensional Newton potential V_{n+4} of a point particle with mass μ located at the origin is given by the equation

$$\Delta_{n+3} V_{n+4} = (n+1)\, \Omega_{n+2} G_{n+4}\, \mu\, \delta^{(n+3)}\left(\mathbf{x}, \mathbf{y}\right), \tag{4.4.1.16}$$

where Δ_{n+3} is the three dimensional flat Laplacian and Ω_{n+2} is the volume of a unit $n+2$ sphere. Any solution to (4.4.1.16) should be periodic under (4.4.1.15). This can be ensured by expanding the potential in terms of eigenfunctions $\psi_{\mathbf{k}}\left(\mathbf{y}\right)$ of a Laplace operator. The eigenvalue equation is

$$\Delta_n \psi_{\mathbf{k}}\left(\mathbf{y}\right) = -m_k^2 \psi_{\mathbf{k}}\left(\mathbf{y}\right). \tag{4.4.1.17}$$

Thus an orthonormal set of eigenfunctions is

$$\psi_{\mathbf{k}} = \frac{1}{(2\pi R)^{\frac{n}{2}}} e^{i \frac{\mathbf{k}}{R} \mathbf{y}}, \tag{4.4.1.18}$$

where $\mathbf{k}$ is an n dimensional vector with integer entries. We expand the higher dimensional Newton potential into a series of the eigenfunctions with $r = |\mathbf{x}|$ dependent coefficients

$$V_{n+4} = \sum_{\mathbf{k}} \phi_{\mathbf{k}}\left(r\right) \psi_{\mathbf{k}}\left(\mathbf{y}\right). \tag{4.4.1.19}$$

Plugging this ansatz into equation (4.4.1.16) determines the Fourier coefficients[9]

$$\phi_{\mathbf{k}}\left(r\right) = -\frac{\Omega_n G_{n+4}\,\mu\,\psi^\star_{\mathbf{k}}\left(0\right)}{2}\frac{1}{r}e^{-\frac{|\mathbf{k}|}{R}}. \tag{4.4.1.20}$$

Now, we consider the case that all particles with which we can test the gravitational potential are localized at $\mathbf{y} = 0$. (This is natural from the brane picture since we can test gravity only with matter which is confined to live on the brane. Recall that we neglected the effects of the Planck sized longitudinal compact dimensions.) We are interested in the Newton potential at $\mathbf{y} = 0$. This comes out to be

$$V_4 \equiv V_{n+4} = -\frac{G_4\mu}{r}\sum_{\mathbf{k}} e^{-r\frac{|\mathbf{k}|}{R}}, \tag{4.4.1.21}$$

where the four dimensional and the higher dimensional Newton constant are related via

$$G_4 = \frac{\Omega_n G_{n+4}}{2\left(2\pi R\right)^n}. \tag{4.4.1.22}$$

For $\mathbf{k} = \mathbf{0}$ we obtain the usual four dimensional Newton potential. The other terms are additive Yukawa potentials. They arise due to the exchange of massive Kaluza Klein gravitons.

Experimentalists usually parameterize deviations from Newton's law via the expression [323]

$$V_4\left(r\right) = -\frac{G_4\mu}{r}\left(1 + \alpha e^{-\frac{r}{\lambda}}\right). \tag{4.4.1.23}$$

In the paper[323] the experimental values are discussed. These maybe outdated by now but for us only the order of magnitude is important (and the fact that so far no deviation from Newton's law has been observed). Depending on the size of α an upper bound on λ varying from the μm range to the cm range has been measured. This tells us that a scenario with two extra large dimensions is almost excluded whereas setups with more than two extra large dimensions are in agreement with the experimental tests of Newton's law.

[9] Here, one uses the completeness relation satisfied by the $\psi_{\mathbf{k}}$.

Chapter 5

Brane world setups

In the last section of the previous chapter we have argued that branes allow for scenarios with large extra dimensions transverse to the brane. This is because those extra large dimensions can be tested only via gravitational interactions which are (due to their weakness) measured only at scales down to about 0.1 mm. We obtained such models via investigations of string theory. One could, however, just postulate the existence of branes (on which charged interactions are located). In this last chapter we will take this latter point of view and not worry whether the setups we are going to discuss have a stringy origin. Because in the presence of branes we can attribute the hierarchy between the Planck and the weak scale to the size of the transverse dimensions, we do not need supersymmetry in such setups. Without supersymmetry, quantum effects usually create vacuum energies. A non vanishing vacuum energy on a brane will back react on the geometry of the space in which the brane lives. Taking into account such back reactions leads to so called warped compactifications. This means that the higher dimensional geometry is sensitive to the position of a brane. The most prominent example of such warped compactifications are the Randall Sundrum models which we will discuss next.

5.1 The Randall Sundrum models

5.1.1 The RS1 model with two branes

In the model we are going to describe in this section there is one extra dimension which will be denoted by ϕ. The five dimensional space is a foliation with four dimensional Minkowski slices. The fifth dimension is compactified on an orbifold $S^1/\mathbb{Z}_2$. 3-branes are located at the orbifold fixed planes (at $\phi = 0$ and $\phi = \pi$) . Hence the action is of the form

$$S = S_{bulk} + S_{b1} + S_{b2}, \tag{5.1.1.1}$$

where S_{b1} and S_{b2} denote the actions on the branes. For the bulk action we take five dimensional gravity with a bulk cosmological constant,

$$S_{bulk} = \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{-G}\left(2M^3 R - \Lambda\right), \tag{5.1.1.2}$$

where M denotes the five dimensional Planck mass and G_{MN} is the five dimensional metric. The branes are located in ϕ and we identify the brane coordinates with the remaining 5d coordinates x^μ, $\mu = 0, \ldots, 3$. Then the induced metrics on the branes are simply

$$g^{b1}_{\mu\nu} = G_{\mu\nu|\phi=0} \quad , \quad g^{b2}_{\mu\nu} = G_{\mu\nu|\phi=\pi} \,. \tag{5.1.1.3}$$

We assume that fields being localized on the brane are in the trivial vacuum and take into account only nonzero vacuum energies on the branes. Calling those vacuum energies T_1 and T_2, the brane actions read

$$S_{b1} + S_{b2} = -\int d^4x \left(T_1\sqrt{-g^{b1}} + T_2\sqrt{-g^{b2}}\right), \tag{5.1.1.4}$$

where the first (second) term on the lhs is attributed to the first (second) term on the rhs. Instead of working out the solutions to the system on an interval $S^1/\mathbb{Z}_2$ it is technically easier to construct a solution in a non compact space, such that the solution is periodic in

$$\phi \equiv \phi + 2\pi, \tag{5.1.1.5}$$

and even under

$$\phi \to -\phi. \tag{5.1.1.6}$$

A vacuum with this property yields then automatically a compact interval in ϕ. (The equivalent[1] and more complicated alternative is to define the theory on an interval from the very beginning and take into account surface terms when deriving the equations of motion as well as Gibbons Hawking[200] boundary terms (for a discussion in the context of brane worlds see also[132, 133]).) With these remarks the Einstein equations of motion read (capital indices run over all dimensions $M, N = 0, \ldots, 4$)

$$\sqrt{-G}\left(R_{MN} - \frac{1}{2}G_{MN}\right) = \\ -\frac{1}{4M^3}\left[\Lambda\sqrt{-G}G_{MN} + \sum_{i=1}^{2} T_i\sqrt{-g^{bi}}g^{bi}_{\mu\nu}\delta^\mu_M\delta^\nu_N\delta\left(\phi - \phi_i\right)\right], \tag{5.1.1.7}$$

[1] I acknowledge discussions with Radosław Matyszkiewicz on this topic.

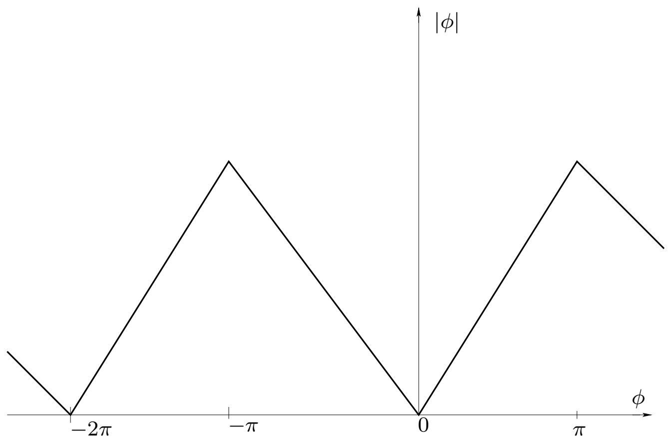

Figure 5.1: The periodic modulus function.

with $\phi_1 = 0$ and $\phi_2 = \pi$. The delta functions appearing on the rhs of (5.1.1.7) are defined on a real line. The most general metric ansatz possessing a four dimensional Poincaré transformation as isometry is

$$ds^2 = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2. \tag{5.1.1.8}$$

We could rescale ϕ such that the r_c dependence drops out, but that would change the periodicity condition (5.1.1.5). Plugging this ansatz into the equations of motion (5.1.1.7) yields (a prime denotes differentiation with respect to ϕ)

$$\frac{6\sigma'^2}{r_c^2} = -\frac{\Lambda}{4M^3}, \tag{5.1.1.9}$$

$$\frac{3\sigma''}{r_c^2} = \frac{T_1}{4M^3 r_c}\delta(\phi) + \frac{T_2}{4M^3 r_c}\delta(\phi - \pi). \tag{5.1.1.10}$$

The solution to (5.1.1.9) is

$$\sigma = r_c |\phi| \sqrt{\frac{-\Lambda}{24M^3}}, \tag{5.1.1.11}$$

where the modulus function is defined as usual in the interval $-\pi < \phi < \pi$,

$$|\phi| = \begin{cases} -\phi & , -\pi < \phi < 0 \\ \phi & , 0 < \phi < \pi \end{cases} . \tag{5.1.1.12}$$

This ensures that the solution is even under $\phi \to -\phi$. In order to incorporate (5.1.1.5), we define the modulus function on the real line by the periodic continuation of (5.1.1.12). The resulting function is drawn in figure 5.1. Away from the points at $\phi = 0$ and integer multiples of π, the second derivative of σ vanishes and

(5.1.1.10) is fulfilled in those regions. In order to take into account the delta function sources in (5.1.1.10), one integrates this equation over an infinitesimal neighborhood around the location of the brane sources. This gives rise to the constraints

$$T_1 = -T_2 = 24M^3k \quad ,\text{with} \quad k^2 = -\frac{\Lambda}{24M^3} \tag{5.1.1.13}$$

on the parameters of the model. These constraints can be thought of as fine tuning conditions for a vanishing effective cosmological constant in four dimensions. We will come back to this point in section 5.2.3. Our final solution is

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu} + r_c^2 d\phi^2, \tag{5.1.1.14}$$

where k^2 is defined in (5.1.1.13), and we take k to be positive (for a negative k just redefine $\phi \to \pi - \phi$).

We observe that by taking into account the back reaction of the branes onto the geometry, we obtain a metric which depends on the position in the compact direction. For the particular model we consider this dependence is exponential. That opens up an interesting alternative explanation for the large hierarchy between the Planck scale and the weak scale. We take all the input scales (M, Λ , r_c) to be of the order of the Planck scale. First, we should check whether this provides the correct four dimensional Planck mass. To this end, we expand a general 4d metric around the classical solution

$$ds^2 = e^{-2kr_c}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu + r_c^2 d\phi^2. \tag{5.1.1.15}$$

In principle we should also allow the four-four component of the metric r_c^2 to fluctuate. Since r_c is an integration constant, such fluctuations will be seen as massless scalars in the effective four dimensional theory. This is a common problem known as moduli stabilization problem. We will assume here that some unknown mechanism gives a mass to the fluctuations of G_{44} and take it to be frozen at the classical value r_c^2. The Kaluza-Klein gauge fields $G_{\mu 4}$ are projected out by the $\mathbb{Z}_2$. Plugging (5.1.1.15) into the action and integrating over ϕ yields the effective action for four dimensional gravity

$$S_{eff} = M_p^2 \int d^4x\sqrt{-g}R^{(4)}(g)\,, \tag{5.1.1.16}$$

where $R^{(4)}(g)$ denotes the four dimensional scalar curvature computed from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and the four dimensional Planck mass M_p is given by

$$M_p^2 = M^3 r_c \int_{-\pi}^{\pi} d\phi e^{-2kr_c|\phi|} = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right). \tag{5.1.1.17}$$

This tells us that choosing five dimensional scales of the order of the Planck scale gives the correct order of magnitude for the four dimensional Planck scale.

Now, let us consider matter living on the branes. On the first brane located at $\phi = 0$, the induced metric is just the Minkowski metric and Lagrangians for matter living on that brane will just have their usual form. On the other hand, matter living on the second brane (located at $\phi = \pi$) feels the ϕ dependence of the bulk metric. Let us focus on a Higgs field being located at the second brane. Its action will be of the form

$$S^{b2}_{Higgs} = \int d^4x e^{-4kr_c\pi} \left\{ e^{2kr_c\pi} \eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda \left(|H|^2 - v_o^2 \right)^2 \right\}, \tag{5.1.1.18}$$

where the overall exponential factor originates from the determinant of the induced metric. Rescaling the Higgs field H such that the kinetic term in (5.1.1.18) takes its canonical form induces the rescaling

$$v_0 \to v_{eff} = e^{-kr_c\pi} v_0. \tag{5.1.1.19}$$

This means that a symmetry breaking scale which is written as v_0 into the model effectively is multiplied by a factor of $e^{-kr_c\pi}$. Repeating the above argument for any massive field, one finds that any mass receives such a factor

$$m_0 \to m_{eff} = e^{-kr_c\pi} m_0, \tag{5.1.1.20}$$

when going to an effective description in which kinetic terms are canonically normalized. Choosing $kr_c \approx 10$ (which is roughly a number of order one), one can achieve that the exponential in (5.1.1.20) takes Planck sized input masses to effective masses of the order of a TeV. Hence, in the above model we can obtain the TeV scale from the Planck scale without introducing large numbers, provided we live on the second brane.

5.1.1.1 A proposal for radion stabilization

In the previous section, we have already mentioned that the internal metric component G_{44} gives rise to a massless field in an effective description. This means that its vev r_c is very sensitive against any perturbation and rather unstable. For the discussion of the hierarchy problem it is important that the distance of the branes r_c is of the order of the Planck length. Therefore, it is desirable to stabilize this distance, i.e. to give a mass to G_{44} in the effective description. In the present section we briefly present a proposal of Goldberger and Wise how a stabilization might be achieved via an additional scalar living in the bulk. We will neglect the back reaction of the scalar field on the geometry. This means that we just consider a scalar field in the RS1 background constructed in the previous section. The action consists out of three parts

$$S = S_{bulk} + S_{b1} + S_{b2}, \tag{5.1.1.21}$$

where S_{bulk} defines the five dimensional dynamics of the field and S_{b1} and S_{b2} its coupling to the respective branes. We choose

$$S_{bulk} = \frac{1}{2}\int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{-G}\left(G^{MN}\partial_M \Phi \partial_N \Phi - m^2\Phi^2\right), \tag{5.1.1.22}$$

where Φ is the scalar field and G_{MN} is given in (5.1.1.14). The coupling to the branes is taken to be

$$\begin{aligned} S_{b1} &= -\int d^4x \sqrt{-g^{b1}}\lambda_1 \left(\Phi^2 - v_1^2\right)^2, && (5.1.1.23)\\ S_{b2} &= -\int d^4x \sqrt{-g^{b2}}\lambda_2 \left(\Phi^2 - v_2^2\right)^2, && (5.1.1.24)\end{aligned}$$

where v_i and λ_i are dimensionfull parameters whose values will be discussed below. With the ansatz that Φ does not depend on the x^μ for $\mu = 0, \ldots, 3$ the equation of motion for the scalar is

$$\begin{aligned} e^{-4kr_c|\phi|}\Bigg(&-\frac{e^{4kr_c|\phi|}}{r_c^2}\partial_\phi\left(e^{-4kr_c|\phi|}\partial_\phi\Phi\right) + m^2\Phi \\ &+4\lambda_1\Phi\left(\Phi^2 - v_1^2\right)\frac{\delta(\phi)}{r_c} + 4\lambda_2\Phi\left(\Phi^2 - v_2^2\right)\frac{\delta(\phi-\pi)}{r_c}\Bigg) = 0. \end{aligned} \tag{5.1.1.25}$$

With $\nu = \sqrt{4 + \frac{m^2}{k^2}}$ the solution inside the bulk $0 < \phi < \pi$ is written as

$$\Phi = e^{2kr_c|\phi|}\left(Ae^{kr_c\nu|\phi|} + Be^{-kr_c\nu|\phi|}\right), \tag{5.1.1.26}$$

where the integration constants A and B will be fixed below. Plugging this solution back into the Lagrangian yields an r_c dependent constant, i.e. a potential for the distance of the two branes,

$$\begin{aligned} V(r_c) \;=\; & k(\nu+2)A^2\left(e^{2\nu kr_c\pi} - 1\right) + k(\nu-2)B^2\left(1 - e^{-2\nu kr_c\pi}\right) \\ &+\lambda_1\left(\Phi(0)^2 - v_1^2\right)^2 + \lambda_2 e^{-4kr_c\pi}\left(\Phi(\pi)^2 - v_2^2\right)^2. \end{aligned} \tag{5.1.1.27}$$

Because of the dependence of Φ on the modulus function (see figure 5.1) the second derivative in the first term in (5.1.1.25) will lead to delta functions whose argument vanishes at the position of the branes. Matching this with the delta function source terms in (5.1.1.25) yields equations for the integration constants A and B. Instead of writing down and solving those equations explicitly we suppose that λ_1 and λ_2 are large enough for the approximation

$$\Phi(0) = v_1 \quad , \quad \Phi(\pi) = v_2 \tag{5.1.1.28}$$

to be sufficiently accurate. In this approximation one obtains

$$A = v_2 e^{-(2+\nu)kr_c\pi} - v_1 e^{-2\nu kr_c\pi}, \tag{5.1.1.29}$$
$$B = v_1\left(1+e^{-2\nu kr_c\pi}\right) - v_2 e^{-(2+\nu)kr_c\pi}. \tag{5.1.1.30}$$

The next approximation lies in the assumption that

$$\epsilon = \frac{m^2}{4k} \ll 1. \tag{5.1.1.31}$$

In evaluating the potential $V(r_c)$ (5.1.1.27), we neglect terms of order ϵ^2 but do not treat ϵkr_c as a small number. This yields

$$\begin{aligned} V(r_c) &= k\epsilon v_1^2 + 4ke^{-4kr_c\pi}\left(v_2 - v_1 e^{-\epsilon kr_c\pi}\right)^2\left(1+\frac{\epsilon}{4}\right) \\ &\quad - k\epsilon v_1 e^{-(4+\epsilon)kr_c\pi}\left(2v_2 - v_1 e^{-\epsilon kr_c\pi}\right). \end{aligned} \tag{5.1.1.32}$$

Up to orders of ϵ, this potential has a minimum at

$$kr_c = \frac{4k^2}{\pi m^2}\log\left(\frac{v_1}{v_2}\right). \tag{5.1.1.33}$$

In figure 5.2, we have drawn the potential in a neighborhood of the minimum (using Maple). (What is actually drawn is $V - k\epsilon v_1^2$.) With the appropriate choice for the scales, the minimum of the potential is clearly visible. One should note, however, the exponentially suppressed height of the right wall of the potential. If we had chosen a larger scale for the drawing, figure 5.2 would just show a runaway potential which rapidly reaches its asymptotic value. This might be a drawback of the stabilization mechanism.

The expression for the stable distance between the branes (5.1.1.33) shows that no extreme fine tuning is needed in order to obtain the wanted value of about ten for kr_c. It remains to investigate whether the various approximations (including the neglection of the back reaction) are sensible. This investigation has been carried out in[215] by estimating the size of next to leading order corrections. The result is that the approximations are fine.

To close this section, we should mention that the described stabilization method is often called "Goldberger Wise mechanism" in the literature. We preferred to use the term "proposal" because we are not certain that this mechanism is the commonly established method for solving the problem of moduli stabilization. We decided to present a brief description of the method because it is one of the most prominent lines of thought in the context of the Randall Sundrum model. In general, the problem of moduli stabilization is not very well understood.

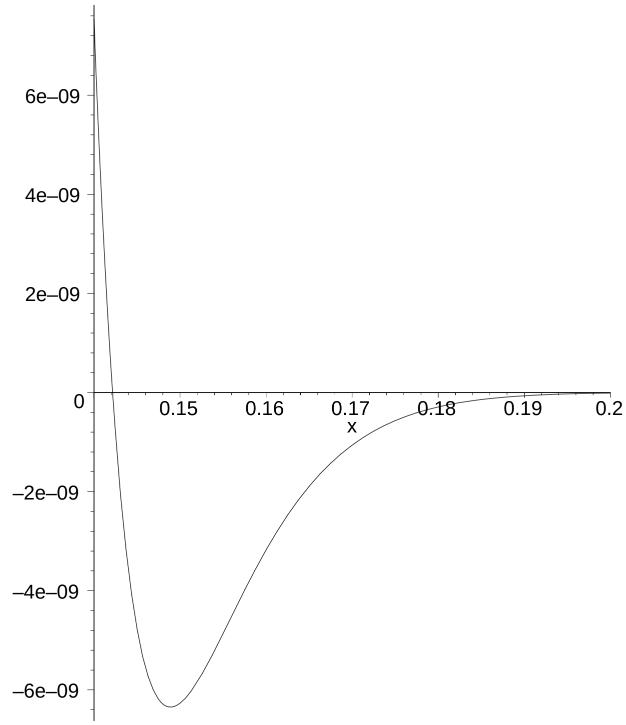

Figure 5.2: The Goldberger Wise potential for $k = 10$, $m = 9$, $v_2 = 1$, $v_1 = 3$. The vertical axis shows $V - k\epsilon v_1^2$ whereas on the horizontal axis r_c is drawn.

5.1.2 The RS2 model with one brane

In this section we are going to consider a variant of the model presented in section 5.1.1 where the second brane is removed. Since for the solution of the hierarchy it was essential that the observers live on this second brane, we now give up the goal of solving the hierarchy problem (at least temporarily). The construction of the single brane solution is very simple. The extra dimension is not compact anymore and therefore we use the coordinate y instead of ϕ. We do not impose the periodicity condition (5.1.1.5) but still require a $\mathbb{Z}_2$ symmetry under

$$y \to -y. \tag{5.1.2.34}$$

Further, we remove S_{b2} from the action (5.1.1.1). Since the extra dimension is not compact, we can perform rescalings of y in order to remove the r_c dependence of the ansatz (5.1.1.8). Without loss of generality we take $r_c = 1$. Thus, in the single brane case, the solution for the metric is

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{5.1.2.35}$$

With a non compact extra dimension, one may worry that gravity is five dimensional now. However, taking the $r_c \to \infty$ limit of (5.1.1.17), one finds that the effective four dimensional Planck mass is finite. This means that the graviton zero mode is normalizable and yields a four dimensional Newton law. Apart from the zero modes, there will be also massive gravitons who lead to corrections of Newton's law. In the following subsection we will investigate these corrections.

5.1.2.1 Corrections to Newton's law

The Newton potential is obtained by studying fluctuations around the background (5.1.2.35), for example

$$G_{00} = -e^{-2k|y|} - V(x, y), \tag{5.1.2.36}$$

where V denotes a fluctuation. In the presence of a point particle with mass μ at the origin, the non relativistic limit of the linearized equation for V reads

$$\left[\Delta_3 + e^{-2k|y|}\left(\partial_y^2 + 4k\delta(y) - 4k^2\right)\right] V(x, y) = G\mu\, \delta^{(3)}(x)\, \delta(y), \tag{5.1.2.37}$$

where G is the five dimensional Newton constant. The fact that V is indeed the Newton potential can be confirmed by studying the geodesic equation of a point particle probe and comparing it with the Newton equation of motion. The equation (5.1.2.37) is the warped geometry analogon of equation (4.4.1.16). (The normalization of the higher dimensional Newton constant is not really important here.) It is useful to redefine the coordinate y according to

$$z \equiv \frac{sgn(y)}{k}\left(e^{k|y|} - 1\right). \tag{5.1.2.38}$$

With

$$\bar{V} = V(x, y)\, e^{\frac{k|y|}{2}} \tag{5.1.2.39}$$

equation (5.1.2.37) takes the form

$$\left[\Delta_3 + \partial_z^2 - \frac{15k^2}{4\left(k|z| + 1\right)^2} + 3k\delta(z)\right] \bar{V} = G\mu\, \delta^{(3)}(x)\, \delta(z) \tag{5.1.2.40}$$

Analogous to section 4.4.1 we plan to expand the solution V into a series of eigenfunctions, i.e. in the case at hand we are looking for solutions of the differential equation

$$\left[\partial_z^2 - \frac{15k^2}{4\left(k|z| + 1\right)^2} + 3k\delta(z)\right] \psi(m, z) = -m^2 \psi(m, z), \tag{5.1.2.41}$$

where we expect a continuous eigenvalue m now, since the "internal space" is not compact. Let us discuss first the zero mode, i.e. the solution to (5.1.2.41) with $m^2 = 0$. The zero mode is found to be[2]

$$\psi_0(z) \equiv \psi(0, z) = \frac{N_0}{(k|z| + 1)^{\frac{3}{2}}}, \tag{5.1.2.42}$$

where N_0 is an integration constant to be fixed later. Note that

$$\partial_z |z| = sgn(z) \quad , \quad \partial_z sgn(z) = 2\delta(z). \tag{5.1.2.43}$$

Now, we take $m > 0$. For $z > 0$ the general solution to the above equation can be written as a superposition of Bessel functions

$$\psi(m, z) = \sqrt{|z| + \frac{1}{k}} \left(c_1 J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) + c_2 Y_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \right), \tag{5.1.2.44}$$

where J_ν denotes the Bessel functions of the first kind whereas Y_ν stands for the Bessel functions of the second kind and $c_{1,2}$ are constants to be fixed below. Because the solution (5.1.2.44) is written as a function of $|z|$, the second derivative with respect to z in (5.1.2.41) will yield a term containing a $\delta(z)$ (and other terms). One can fix the ratio c_1/c_2 by matching the factor in front of this delta function with the factor in front of the delta function in (5.1.2.44). We will do this in an approximate way. The most severe corrections to Newton's law are to be expected from gravitons with small m (because they carry interactions over longer distances). In matching the coefficients of the delta functions, only a neighborhood around $z = 0$ matters. Therefore, we replace the Bessel functions by their asymptotics for small arguments, which are

$$J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \sim \frac{m^2 \left(|z| + \frac{1}{k} \right)^2}{8}, \tag{5.1.2.45}$$

$$Y_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \sim -\frac{4}{\pi m^2 \left(|z| + \frac{1}{k} \right)^2} - \frac{1}{\pi}. \tag{5.1.2.46}$$

Plugging the asymptotic approximation into (5.1.2.44) and then into (5.1.2.41) one finds that the overall coefficient in front of the delta function vanishes if

$$\frac{c_1}{c_2} = \frac{4k^2}{\pi m^2}. \tag{5.1.2.47}$$

Hence, our general solution (5.1.2.44) reads

$$\psi(m, z) = N_m \sqrt{|z| + \frac{1}{k}} \left[Y_2 \left(m|z| + \frac{1}{k} \right) + \frac{4k^2}{\pi m^2} J_2 \left(m \left(|z| + \frac{1}{k} \right) \right) \right], \tag{5.1.2.48}$$

[2]In forthcoming expressions we will always imply that $m > 0$ when writing $\psi(m, z)$. The zero mode will be denoted by $\psi_0(z)$ from now on.

where we replaced $c_2 = N_m$ because this remaining integration constant will turn out to depend on the eigenvalue m.

Recall that the extra dimension y (or z) is not compact. Thus the eigenvalue m is continuous. Therefore, we normalize

$$\int dz\, \psi\left(m,z\right)\psi\left(m',z\right) = \delta\left(m-m'\right), \tag{5.1.2.49}$$

for $m, m' > 0$. For $m \geq 0$ we impose the normalization condition

$$\int dz\, \psi_0\left(z\right)\psi\left(m,z\right) = \delta_{m,0}, \tag{5.1.2.50}$$

such that the completeness relation reads

$$\psi_0\left(z\right)\psi_0\left(z'\right) + \int_0^\infty dm\, \psi\left(m,z\right)\psi\left(m,z'\right) = \delta\left(z-z'\right). \tag{5.1.2.51}$$

The orthonormalization condition (5.1.2.49) fixes N_m. It turns out that the computation simplifies essentially in the approximation where the arguments of the Bessel functions are large, since the corresponding asymptotics yields plane waves. Explicitly, for large mz the Bessel functions are approximated by

$$\sqrt{z}J_2\left(mz\right) \sim \sqrt{\frac{2}{\pi m}}\cos\left(mz-\frac{5\pi}{4}\right), \tag{5.1.2.52}$$

$$\sqrt{z}Y_2\left(mz\right) \sim \sqrt{\frac{2}{\pi m}}\sin\left(mz-\frac{5\pi}{4}\right). \tag{5.1.2.53}$$

Because we are mainly concerned about large ($> \mu m \gg 1/M_p$) distance modifications of Newton's law we focus on the contribution of the "light" modes ($\frac{m^2}{k^2} \ll 1$). (Recall that k is of the order of the Planck mass.) Then (5.1.2.49) yields for the normalization constant (for $m > 0$)

$$N_m = \frac{\pi m^{\frac{5}{2}}}{(4k^2)}. \tag{5.1.2.54}$$

The condition (5.1.2.50) is satisfied for $m > 0$ to a good approximation. Evaluating (5.1.2.50) for $m = 0$ fixes

$$N_0 = \sqrt{k}. \tag{5.1.2.55}$$

Now, we expand $\bar{V}\left(x,z\right)$ into eigenfunctions $\psi_0\left(z\right)$ and $\psi\left(m,z\right)$ with x dependent coefficients $\varphi_m\left(x\right)$

$$\bar{V}\left(x,z\right) = \varphi_0\left(x\right)\psi_0\left(z\right) + \int_0^\infty dm\, \varphi_m\left(x\right)\psi\left(m,z\right). \tag{5.1.2.56}$$

By plugging the ansatz (5.1.2.56) into (5.1.2.40), we find that for $m \geq 0$ and $r = |x|$

$$\varphi_m (x) = -\frac{G\mu}{r} e^{-mr} a_m, \tag{5.1.2.57}$$

with the constants a_m taken such that

$$a_0 \psi_0 (z) + \int dm \, a_m \psi (m, z) = \delta (z) \, . \tag{5.1.2.58}$$

Comparison with (5.1.2.51) yields

$$a_0 = \psi_0 (0) \quad , \quad a_m = \psi (m, 0) \, . \tag{5.1.2.59}$$

In the current setup we are interested in corrections to Newton's law as an observer on the brane at the origin would measure them. Defining the four dimensional Newton constant G_4 as

$$G_4 = Gk, \tag{5.1.2.60}$$

we find from (5.1.2.56)

$$\bar{V} (x, 0) = V (x, 0) = -\frac{G_4 \mu}{r} \left(1 + \int_0^\infty dm \, \frac{m}{k^2} e^{-mr} \right), \tag{5.1.2.61}$$

where once again we took into account only modes with $m/k \ll 1$ such that we could use the asymptotics (5.1.2.45) and (5.1.2.46) in order to evaluate $\psi (m, 0)$. Finally, performing the integral in (5.1.2.61) leads to

$$V (x, 0) = -G_4 \frac{\mu}{r} \left(1 + \frac{1}{r^2 k^2} \right) . \tag{5.1.2.62}$$

For k being of the order of the Planck mass (5.1.2.62) is in very good agreement with the experimental values. This may look a bit surprising. Even though the extra dimension is not compact, we obtain a four dimensional Newton potential for observers who live on the brane at $y = 0$. This non trivial result finds its explanation in the exponentially warped geometry. It is this geometry which is responsible for the fact that the amplitude of the zero mode has its maximum at the brane and vanishes rapidly for finite z. On the other hand, the massive modes reach their maximal amplitudes asymptotically far away from the brane. Therefore, they have very little influence on the gravitational interactions on the brane, although the masses of the extra gravitons can be arbitrarily small.

In the following subsection we are going to rederive (5.1.2.62) in a different way.

5.1.2.2 ... and the holographic principle

In section 4.3 we have described a duality between a field theory living on the boundary of an AdS_5 space and a theory living in the bulk of an AdS_5 space. This correspondence is sometimes called the holographic principle since it allows to reproduce bulk data from boundary data (and vice versa). Now we are going to apply this principle to the RS2 setup. Before doing so, we will establish that the RS2 setup has something to do with an AdS_5 space (namely it is a slice of an AdS_5 space). To this end, we first write down the RS2 metric (5.1.2.35) in terms of the coordinate z defined in (5.1.2.38). This results in

$$ds^2_{RS2} = \frac{1}{(k|z|+1)^2}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2. \tag{5.1.2.63}$$

For symmetry reasons the coordinate z can be restricted to the half interval between zero and infinity. The singularity at $z = 0$ is caused by the brane.

Now, let us recall from section 4.3 that the AdS_5 metric is (see (4.3.1.12))

$$ds^2_{AdS} = \frac{U^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + R^2\frac{dU^2}{U^2}. \tag{5.1.2.64}$$

Changing the coordinates according to ($-R < z < \infty$)

$$U = \frac{R^2}{z+R} \tag{5.1.2.65}$$

yields an AdS metric of the form

$$ds^2_{Ads} = \frac{1}{(1+\frac{z}{R})^2}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2. \tag{5.1.2.66}$$

Comparing (5.1.2.66) with (5.1.2.63), we observe that the RS2 geometry describes a slice of an AdS_5 space. The radius of the AdS_5 space is $1/k$, and the space is cut off at $z = 0$. Since the boundary of the AdS_5 space is situated at $U \to \infty$, the cutoff at $z = 0$ means that we lost the region between $U = R$ and the boundary. Hence, the position of the brane in the RS2 setup can be viewed as an infrared cutoff for gravity on an AdS_5 space. This suggests that we may apply the AdS/CFT conjecture on the RS2 scenario. (Note however, that we do not have any supersymmetry now. Without supersymmetry the AdS/CFT conjecture has passed less consistency checks. Nevertheless, let us assume that the conjecture is correct also without supersymmetry.) The field theory dual of the RS2 setup is thus a conformal field theory with a UV cutoff[3] given by k. (The cutoff actually breaks the conformal invariance. The conformal anomaly

[3] Recall from section 4.3.2.1 that an IR cutoff in the bulk theory corresponds to a UV cutoff in the dual field theory.

induces a coupling of the field theory to gravity.) In particular, we plan to employ the AdS/CFT duality conjecture for the computation of corrections to Newton's law. As a preparation let us sketch how Newton's potential is related to the gravity propagator in four dimensions. If we did not have an extra dimension, the gravity propagator in momentum space is (up to a polarization tensor) $1/\left(M_p^2 p^2\right)$. The Newton potential can be obtained from this propagator by formally setting the p_0 component to zero and Fourier transforming[4] with respect to the spatial momentum components. The result in position space is then $1/(M_p^2 r)$. Therefore, we will use the AdS/CFT duality conjecture to compute the corrected graviton propagator and deduce the corrected Newton potential via the above description.

The dual picture for the RS2 setup is that we have four dimensional gravity plus the CFT dual of gravity on AdS_5 with a UV cutoff k. Corrections to four dimensional gravity are caused by the interaction of gravity with the CFT. The effective graviton propagator is obtained by integrating over the CFT degrees of freedom. The one loop corrected graviton propagator will be schematically of the form

$$\frac{1}{M_p^2 p^2}\left(1 + \langle T_{CFT}\left(p\right) T_{CFT}\left(-p\right)\rangle \frac{1}{M_p^2 p^2}\right), \tag{5.1.2.67}$$

where T_{CFT} stands for the energy momentum tensor of the CFT dual. (The coupling of gravity to the CFT fields is given by the energy momentum tensor.) For any four dimensional CFT, the two point function of the energy momentum tensor is fixed to be of the form

$$\langle T_{CFT}\left(p\right) T_{CFT}\left(-p\right)\rangle = cp^4, \tag{5.1.2.68}$$

where we imposed that the UV cutoff is k. We will not derive this result here, but just give two comments. First, notice that (5.1.2.68) is the four dimensional analogon of (2.1.3.51). The number c quantifies the conformal anomaly. The second remark is, that the reader may get some impression on how the expression (5.1.2.68) arises by computing it explicitly for pure gauge theory. We are interested in the order of magnitude of c. This has been computed in[241] to be

$$c \approx \frac{M_5^3}{k^3} = \frac{M_p^2}{k^2}, \tag{5.1.2.69}$$

where M_5 denotes the five dimensional Planck mass and the radius of the AdS space dual to the CFT is $1/k$. In the second equality of (5.1.2.69) we used the relation

[4] We use the following prescription for performing the Fourier transformation. Transforming the equation $\Delta_3 f\left(x\right) = \delta^{(3)}\left(x\right)$ one finds that $1/p^2$ transforms into $1/r$. Later we will also have to compute Fourier transforms with additional powers of p in the numerator or denominator. An additional power of p in the numerator transforms into ∂_r whereas powers of p in the denominator can be generated by ∂_p which in turn transforms into r.

between the five and four dimensional Planck mass (M_5 and M_p) which, in the RS2 setup, is obtained by taking $r_c \to \infty$ in (5.1.1.17).

The corrected Newton potential is obtained by setting formally p_0 to zero in (5.1.2.67) and performing a three dimensional Fourier transformation to the position space. Thus, the effect of integrating over the CFT fields results in the following replacement of the Coulomb (Newton) potential

$$-\frac{1}{r} \to -\frac{1}{r}\left(1+\frac{1}{k^2 r^2}\right). \tag{5.1.2.70}$$

This result agrees with the expression (5.1.2.62) computed in the previous section. Thus, we have learned that integrating over the CFT fields yields the same corrections to four dimensional gravity as taking into account the massive "Kaluza-Klein" gravitons. Employing the AdS/CFT correspondence, the computational effort decreases substantially. We will make use of this fact when we combine the RS1 with the RS2 scenario in the next subsection.

5.1.2.3 The RS2 model with two branes

In the previous two subsections we have seen that the RS2 setup has the exciting feature of giving rise to effectively four dimensional gravitational interactions even though the extra dimension is not compact. On the other hand, we observed before that the RS1 model is capable to explain the hierarchy between the Planck scale and the weak scale without introducing large numbers. How can we combine these two models? We should introduce a brane with the observers at $y = \pi r_c$ into the RS2 setup. However, this brane should not cause a change of the RS2 metric (5.1.2.35). The observers on the additional brane (at $y = \pi r_c$) can achieve this by performing a fine tuning such that the vacuum energy on their brane vanishes. In the following we will call the brane at $y = \pi r_c$ the SM (Standard Model) brane. The SM brane can be viewed as a probe in the RS2 background. The hierarchy can now be explained in the same way as it is explained in the RS1 setup. What we should worry about are the gravitational interactions as viewed by an observer on the SM brane. In principle, these can be computed along the lines of section 5.1.2.1. The situation is, however, slightly more complicated since the approximation has to be refined. In particular, replacing the Bessel functions by their plane wave asymptotics in the computation of N_m is too rough an estimate. Now, this would imply that the observer on the SM brane sees the Bessel functions as plane waves. As argued in[332] this is not the case, in particular for the light continuum modes. The authors of[332] refined the approximation and obtained the result

$$V\left(r, y=\pi r_c\right)=-\frac{G_4 \mu}{r}\left(1+\frac{1}{k^2 r^2}\right)-\frac{\mu}{M_w^8 r^7} \tag{5.1.2.71}$$

for the Newton potential observed on the SM brane. Here, M_w is of the order of a TeV if we take r_c such that the hierarchy problem is solved. Instead of going through the tedious refinement of the approximations performed in section 5.1.2.1, we employ the AdS/CFT correspondence to motivate (5.1.2.71). The introduction of the SM brane modifies the RS2 dual such that it consists out of four dimensional gravity, the CFT dual of the RS2 AdS_5 slice and the Standard Model of the probe brane. Note that $y_c = r_c\pi$ is translated to $U_0 - U_c = TeV$ in the course of the coordinate transformations (5.1.2.38) and (5.1.2.65), where U_0 denotes the position of the brane at the origin and U_c the position of the SM brane. This means that SM fields and CFT fields interact via fields with masses of the order of a TeV.[5] Integrating out those fields yields effective coupling terms between SM fields and CFT fields. (This is analogous to generating the Fermi interaction via integrating out the W and Z bosons.) The structure of the possible interaction terms is restricted by symmetries to[37]

$$\frac{1}{M_w^4} T_{SM}^{\mu\nu} T_{\mu\nu CFT}. \tag{5.1.2.72}$$

Note the similarity to the coupling of the SM fields to gravitons. Apart from charged interactions, the SM fields interact via gravitons and via CFT fields. This suggests that for an observer on the SM brane the effective graviton propagator is

$$\frac{1}{M_p^2 p^2}\left(1+\left\langle T_{CFT}(p) T_{CFT}(-p)\right\rangle \frac{1}{M_p^2 p^2}\right)+\frac{1}{M_w^8}\left\langle T_{CFT}(p) T_{CFT}(-p)\right\rangle, \tag{5.1.2.73}$$

where the first two terms are the same as in (5.1.2.67), and the last term means that the observer will interpret the interaction (5.1.2.72) as gravitational interaction. In computing the contribution due to the last term we use (5.1.2.68). Applying the recipe of the previous section, we obtain out of the propagator the modified Newton potential (5.1.2.71). This potential is still in agreement with the observational bounds on deviations from Newton's law. Hence, adding a probe brane at $y = \pi r_c$ in the RS2 setup one obtains a model which explains the hierarchy and possesses effectively four dimensional gravitational interactions, even though there is a non compact extra dimension. However, we should remark that we discussed the setup only classically and showing its stability against quantum corrections may be a problematic issue. This

[5] One may view the field theory dual as a stack of D-branes on which the CFT lives and the SM probe brane separated by a distance $1/TeV$ from the CFT branes. The interaction between the CFT and the SM can be thought of as arising due to open strings stretching between the corresponding branes.

corresponds to the technical hierarchy problem which can be solved by supersymmetry in conventional four dimensional models. Supersymmetric versions of the RS model appear in the literature listed in chapter 6.

5.2 Inclusion of a bulk scalar

In this section, we are going to modify the Randall Sundrum models of the previous section by introducing a bulk scalar Φ which couples also to the branes. Actually, we have considered this modification already in section 5.1.1.1, where we neglected the back reaction of the scalar on the geometry. In the current section we are going to take this back reaction into account. We will not return to the stabilization mechanism of section 5.1.1.1, though. (The inclusion of back reaction into the Golberger Wise mechanism is discussed in[128], with the result that the mechanism works also when the back reaction is included.) Instead of addressing the question of how a scalar helps to stabilize the inter brane distance, we want to consider another question. As we will see the cosmological constant problem is reformulated in a brane world setup. We will investigate whether a scalar can help to find a solution to the cosmological constant problem. Before doing so, we briefly present a solution generating technique and consistency conditions on the solutions.

5.2.1 A solution generating technique

Introducing a bulk scalar Φ modifies the action (5.1.1.1) to[6]

$$S = \int d^4x \int dy \sqrt{-G}\left(R - \frac{4}{3}\left(\partial\Phi\right)^2 - V\left(\Phi\right)\right) - \sum_i \int_{b_i} d^4x \sqrt{-g^{bi}} f_i\left(\Phi\right), \qquad (5.2.1.1)$$

where y is the coordinate labeling the extra dimension, and the sum over i stands for a sum over the branes. The index b_i at the integral means that y is fixed to the position (y_i) of the brane b_i. The function $V(\Phi)$ is a bulk potential for the scalar and $f_i\left(\Phi\right)$ is the coupling function of the scalar to the brane b_i.

For later use let us also generalize the ansatz (5.1.1.8) to

$$ds^2 = e^{2A(y)}\bar{g}_{\mu\nu}dx^\mu dx^\nu + dy^2, \qquad (5.2.1.2)$$

where $\bar{g}_{\mu\nu}$ denotes the metric of a four dimensional maximally symmetric space, i.e.

$$\bar{g}_{\mu\nu} = \begin{cases} diag\left(-1,1,1,1\right) & \text{for} \quad M_4 \\ diag\left(-1, e^{2\sqrt{\Lambda}t}, e^{2\sqrt{\Lambda}t}, e^{2\sqrt{\Lambda}t}\right) & \text{for} \quad dS_4 \\ diag\left(-e^{2\sqrt{-\Lambda}x^3}, e^{2\sqrt{-\Lambda}x^3}, e^{2\sqrt{-\Lambda}x^3}, 1\right) & \text{for} \quad AdS_4 \end{cases} . \qquad (5.2.1.3)$$

[6]For simplicity we set the five dimensional Planck mass to one. It can be introduced if needed by a simple analysis of the mass dimensions.

The constant $\bar{\Lambda}$ is related to the constant curvature of the de Sitter (dS_4) and the anti de Sitter (AdS_4) slices.

Let us first discuss the simplest case with $\bar{\Lambda} = 0$. As usual we consider fields which depend only on y and denote a derivative with respect to y by a prime. The equations of motion for $\bar{\Lambda} = 0$ are

$$\begin{aligned} \frac{8}{3}\Phi'' + \frac{32}{3}A'\Phi' - \frac{\partial V}{\partial \Phi} - \sum_i \frac{\partial f_i}{\partial \Phi}\,\delta(y-y_i) &= 0, & (5.2.1.4)\\ 6\left(A'\right)^2 - \frac{2}{3}\left(\Phi'\right)^2 + \frac{V}{2} &= 0, & (5.2.1.5)\\ 3A'' + \frac{4}{3}\left(\Phi'\right)^2 + \frac{1}{2}\sum_i f_i\,\delta(y-y_i) &= 0. & (5.2.1.6) \end{aligned}$$

First, we analyze this system of equations in absence of the branes. We start with the ansatz

$$A' = W(\Phi)\,. \qquad (5.2.1.7)$$

Equation (5.2.1.6) fixes then

$$\Phi' = -\frac{9}{4}\frac{\partial W}{\partial \Phi}. \qquad (5.2.1.8)$$

the second equation (5.2.1.5) yields

$$V = \frac{27}{4}\left(\frac{\partial W}{\partial \Phi}\right)^2 - 12W^2. \qquad (5.2.1.9)$$

Finally, the first equation (5.2.1.4) is satisfied automatically.

With view on (5.2.1.9), we could formally call W a superpotential because such a relation is known from five dimensional gauged supergravity[192]. A solution in the absence of branes can now be constructed as follows. Equation (5.2.1.9) determines W up to an integration constant. With a given W, one can solve (5.2.1.8) for Φ up to another integration constant. Equation (5.2.1.7) fixes A up to an integration constant. So altogether, there are three integration constants in the general solution.

Now, we take into account the source terms caused by the presence of the branes. We are looking for solutions in which the fields are continuous. Therefore, the first derivatives of the fields A and Φ are finite arbitrarily close to the position of the branes. However, the first derivatives must jump when y passes a y_i. Integrating (5.2.1.6) and (5.2.1.4) over $y = y_i - \epsilon \ldots y_i + \epsilon$ and taking the limit $\epsilon \to 0$, one finds the jump conditions

$$\begin{aligned} 3\left(A'\left(y_i+0\right) - A'\left(y_i-0\right)\right) &= -\frac{1}{2}f_i, & (5.2.1.10)\\ \frac{8}{3}\left(\Phi'\left(y_i+0\right) - \Phi'\left(y_i-0\right)\right) &= \frac{\partial f_i}{\partial \Phi}. & (5.2.1.11) \end{aligned}$$

For the "superpotential" W, this implies

$$3\left(W_{|y=y_i+0} - W_{|y=y_i-0}\right) = -\frac{1}{2}f_i, \tag{5.2.1.12}$$

$$\frac{3}{2}\left(\frac{\partial W}{\partial \Phi}_{|y=y_i+0} - \frac{\partial W}{\partial \Phi}_{|y=y_i-0}\right) = -\frac{\partial f_i}{\partial \Phi}. \tag{5.2.1.13}$$

This means that there are two additional conditions per brane. If we safely want to obtain four dimensional gravity in the effective theory, we should compactify the extra dimension. For an interval compactification we need at least two branes. The length of the interval (the inter brane distance) enters the ansatz as a further integration constant (e.g. r_c in (5.1.1.8) now appears in (5.1.1.5)). Therefore, four integration constants are to be fixed by four conditions. However, we should take into account that one of the integration constants corresponds to constant shifts in A which can be absorbed into a rescaling of x. A enters the equation of motions and the jump conditions only with its derivatives. Therefore, one of the integration constants is not fixed by the jump conditions. This means that in a two (or more) brane setup at least one fine tuning of the model parameters (appearing in $V(\Phi)$ and $f_i(\Phi)$) is necessary for the existence of a solution with $\bar{\Lambda} = 0$.

For example in the RS1 model, we obtained two fine tuning conditions (5.1.1.13). The fact that there is one more fine tuning condition than expected by naive counting is related to the fact that the inter brane distance r_c is a modulus of the solution. This feature is closely connected with the observation that we can remove the second brane and still obtain four dimensional effective gravity. Even after removing one brane the Randall Sundrum model requires one fine tuning. We will come back to this point in section 5.2.3.

The fact that our solution requires fine tuning of parameters has its origin in the $\bar{\Lambda} = 0$ condition of the ansatz we have considered so far. We can view $\bar{\Lambda}$ as an additional integration constant in the ansatz (5.2.1.2). In general, constant shifts in A can be absorbed in a rescaling of x^μ in combination with a redefinition of $\bar{\Lambda}$. This suggests that a mismatch in the fine tuning conditions results in a nonzero $\bar{\Lambda}$. In order to see this more explicitly we write down the equations of motion for $\bar{\Lambda} \neq 0$,

$$\frac{8}{3}\Phi'' + \frac{32}{3}A'\Phi' - \frac{\partial V}{\partial \Phi} - \sum_i \frac{\partial f_i}{\partial \Phi}\,\delta(y-y_i) = 0, \tag{5.2.1.14}$$

$$6\left(A'\right)^2 - \frac{2}{3}\left(\Phi'\right)^2 + \frac{V}{2} - 6\bar{\Lambda}e^{-2A} = 0, \tag{5.2.1.15}$$

$$3A'' + \frac{4}{3}\left(\Phi'\right)^2 + 3\bar{\Lambda}e^{-2A} + \frac{1}{2}\sum_i f_i\,\delta(y-y_i) = 0. \tag{5.2.1.16}$$

The jump conditions (5.2.1.10) and (5.2.1.11) are still of the same form. We observe that a constant shift in A enters the equations of motion. Hence, there is no fine tuning to be expected if we do not fix the value of $\bar{\Lambda}$ in the ansatz. For completeness, we note that the equations of motion can be reduced to a set of first order equations like in the $\bar{\Lambda} = 0$ case. The corresponding first order equations are

$$\begin{aligned} V &= \frac{27}{4}\frac{1}{\gamma(r)^2}\left(\frac{\partial W(\Phi)}{\partial\Phi}\right)^2 - 12W(\Phi)^2\,, && (5.2.1.17)\\ A' &= \gamma(r)\,W(\Phi)\,, && (5.2.1.18)\\ \Phi' &= -\frac{9}{4}\frac{1}{\gamma(r)}\frac{\partial W(\Phi)}{\partial\Phi}, && (5.2.1.19)\\ \gamma(r) &= \sqrt{1+\frac{\bar{\Lambda}}{W(\Phi)^2}e^{-2A}}. && (5.2.1.20) \end{aligned}$$

To find a solution to this system of first order equations looks more complicated than in the $\bar{\Lambda} = 0$ case. The equation (5.2.1.17) now couples to the rest of the equations due to the γ dependent factor.

5.2.2 Consistency conditions

In this subsection we are going to discuss consistency conditions which any solution to the setup of the previous subsection has to satisfy. In principle, these consistency conditions constitute nothing but a check whether there has been a computational error. They are, however, useful in cases where the envisaged solution possesses singularities. Further, consistency conditions give sometimes informations about the system without the need of constructing an explicit solution. The condition we are going do derive next is most simply expressed in words. It states that the four dimensional effective cosmological constant is compatible with the constant curvature of the four dimensional slices. (This curvature is fixed by $\bar{\Lambda}$ in (5.2.1.3).) Now let us translate this verbal statement into fromulæ.

In order to obtain the four dimensional effective cosmological constant, we need to construct an effective action for four dimensional gravity. We start with the five dimensional metric

$$ds^2 = e^{2A(y)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dy^2, \qquad (5.2.2.1)$$

where

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad (5.2.2.2)$$

is the metric on the four dimensional slices. It is taken to be independent of y, and the background metric $\bar{g}_{\mu\nu}$ is defined in (5.2.1.3). If we do not consider other fluctuations than $h_{\mu\nu}$, the action for four dimensional gravity will be of the general form

$$S_4 = M_p^2\int d^4x\sqrt{-\tilde{g}}\left(\tilde{R}^{(4)} - \lambda\right), \qquad (5.2.2.3)$$

where $\tilde{R}^{(4)}$ is the four dimensional scalar curvature computed from $\tilde{g}_{\mu\nu}$. The cosmological constant λ is fixed by the condition that $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu}$ should be a stationary point of (5.2.2.3). This yields

$$\lambda = 6\bar{\Lambda}. \tag{5.2.2.4}$$

We should also recall that the effective four dimensional Planck mass is given by

$$M_p^2 = \int dy \, e^{2A(y)}, \tag{5.2.2.5}$$

where A takes its classical value. The vacuum value of the Lagrange density in (5.2.2.3) can be easily computed to be

$$\bar{\mathcal{L}}_4 = M_p^2 \left(\bar{R}^{(4)} - \lambda \right) = 6\bar{\Lambda} M_p^2, \tag{5.2.2.6}$$

where $\bar{R}^{(4)}$ is the scalar curvature computed from $\bar{g}_{\mu\nu}$.

For consistency, $\bar{\mathcal{L}}_4$ should coincide with a result obtained in the following way. We plug the solution of the equations of motion into the five dimensional action and integrate over y. (This is exactly the prescription of obtaining the classical value of the four dimensional Lagrangian.) In order to do so, it is useful to write down part of the equations of motion in a less explicit form than before. The equations obtained from five dimensional metric variations are the five dimensional Einstein equations

$$R_{MN} - \frac{1}{2} G_{MN} R = \frac{1}{2} T_{MN}. \tag{5.2.2.7}$$

For the model defined in (5.2.1.1), the energy momentum tensor T_{MN} is

$$T_{MN} = \frac{8}{3} \partial_M \Phi \partial_N \Phi - \frac{4}{3} (\partial\Phi)^2 G_{MN} - V(\Phi) \, G_{MN} - \sum_i f_i \, \delta(y - y_i) \, g_{\mu\nu} \delta^\mu_M \delta^\nu_N, \tag{5.2.2.8}$$

where $g_{\mu\nu}$ is the metric induced on the brane (see (5.1.1.3)). The classical value of R can be easily computed by taking the trace of (5.2.2.7) with the result

$$R = \frac{4}{3} (\partial\Phi)^2 + \frac{5}{3} V(\Phi) + \frac{4}{3} \sum_i f_i \, \delta(y - y_i) \,. \tag{5.2.2.9}$$

Plugging this into (5.2.1.1), we obtain the classical value for the four dimensional Lagrangian

$$\bar{\mathcal{L}}_4 = \int dy \, e^{4A} \left(\frac{2}{3} V(\Phi) + \frac{1}{3} \sum_i f_i \, \delta(y - y_i) \right), \tag{5.2.2.10}$$

where it is understood that A and Φ satisfy the equations of motion. Comparing with (5.2.2.6) and using (5.2.2.8), we obtain finally the consistency condition

$$-\frac{1}{3}\int dy\, e^{4A}\left(T_0^0+T_5^5\right)=6\bar{\Lambda}M_p^2. \qquad (5.2.2.11)$$

We should emphasize again that (5.2.2.11) is just a consequence of the equations of motion. For $\bar{\Lambda}=0$, (5.2.2.11) implies that the vacuum energy density of the solution has to vanish.

Before closing this subsection we want to describe an alternative way to obtain the same (or equivalent) consistency conditions. First, we note that

$$\left(A'e^{nA}\right)'=e^{nA}\left(\frac{n-4}{9}\left(\Phi'\right)^2-\frac{nV}{12}+(n-1)\,\bar{\Lambda}e^{-2A}-\frac{1}{6}\sum_i f_i\,\delta(y-y_i)\right). \qquad (5.2.2.12)$$

This can be easily checked with the equations (5.2.1.15) and (5.2.1.16). With the expression (5.2.2.8) we rewrite (5.2.2.12) in the following way

$$\left(A'e^{nA}\right)'=e^{nA}\left(\frac{1}{6}T_0^0+\left(\frac{n}{12}-\frac{1}{6}\right)T_5^5+(n-1)\,\bar{\Lambda}e^{-2A}\right). \qquad (5.2.2.13)$$

Assuming that for a consistent solution the integral over the total derivative on the lhs of (5.2.2.13) vanishes we find

$$-\frac{1}{3}\int dy\, e^{nA}\left(T_0^0+\left(\frac{n}{2}-1\right)T_5^5\right)=2\,(n-1)\,\bar{\Lambda}\int dy\, e^{(n-2)A}. \qquad (5.2.2.14)$$

We observe that for $n=4$ this condition is identical to the previously derived condition (5.2.2.11). In the next subsection we will discuss solutions with singularities. For those solutions, one could argue that the preposition of condition (5.2.2.14) is not necessarily satisfied. If there are singularities, an integral over a total derivative may differ from zero, and one may not worry about (5.2.2.14) in such a case. For $n=4$, we have shown that (5.2.2.14) encodes the statement that the effective four dimensional cosmological constant is compatible with the curvature of the four dimensional slices. This should be the case also in the presence of singularities. We leave it as an exercise to verify that the Randall Sundrum models satisfy all the consistency conditions.

5.2.3 The cosmological constant problem

In this section, we are going to discuss whether it is possible to solve the cosmological constant problem within a brane world scenario containing a bulk scalar. Let us first state the problem as it arises in conventional quantum field theory. The observational bound on the value of the cosmological constant (as measured from the curvature of the universe) is

$$\lambda M_p^2\leq 10^{-120}\left(M_p\right)^4. \qquad (5.2.3.1)$$

Taking into account the leading order contribution of quantum field theory, one obtains

$$\lambda M_p^2 = \lambda_0 M_p^2 + (\text{UV-cutoff})^4\, Str\,(\mathbf{1})\,, \tag{5.2.3.2}$$

where λ_0 corresponds to a tree level contribution which can be viewed as an input parameter of the model. The size of the UV-cutoff is set by the scale up to which the effective field theory at hand is valid. The supertrace is taken over degrees of freedom which are light compared to the UV-cutoff. If for example we assume that the standard model of particle physics is a valid effective description of physics up to the Planck scale, we need to fine tune 120 digits of the input parameter $\lambda_0 M_p^2$ in order to obtain agreement with (5.2.3.1). The situation slightly improves if we assume that the standard model is a good effective description only up to a supersymmetry breaking scale at (at least) about a TeV. In this case we should take the UV-cutoff to be roughly a TeV. We still have to fine tune 60 digits in $\lambda_0 M_p^2$ in order to match the observation (5.2.3.1). To summarize, the cosmological constant problem is that a huge amount of fine tuning of input parameters is implied by the observational bound on the cosmological constant.

How could the situation improve in a brane world setup? Here, it may happen that the field theory produces a huge amount of vacuum energy which however results only in a curvature along the invisible extra dimension. In section 5.2.1 we have seen that in a two (or more) brane setup we need to fine tune input parameters such that $\bar{\Lambda} = 0$[7] is a solution of the model. (See equation (5.2.2.4) for the relation between $\bar{\Lambda}$ and λ.) Actually, the amount of fine tuning needed in a two brane setup is of the order of magnitude by which the vacuum energy on a brane deviates from the observed value (5.2.3.1) because this quantity enters the jump conditions. One may hope to find a single brane model for which a solution without fine tuning exists. This possibility is not excluded by our investigations in section 5.2.1. However, we will prove later that a single brane model with effectively four dimensional gravity requires a fine tuning (as the RS2 model of section 5.1.2 does). Before presenting the general (negative) result, we would like to demonstrate the problems at an illustrative example.

5.2.3.1 An example

The model we are going to discuss is a special case of (5.2.1.1) with a single brane at $y = 0$ as well as $V(\Phi) \equiv 0$ and $f_0(\Phi) = Te^{b\Phi}$. Hence, the action reads

$$S = \int d^5x\sqrt{-G}\left(R - \frac{4}{3}\left(\partial\Phi\right)^2\right) - \int d^4x\sqrt{-g}\,Te^{b\Phi}{}_{|y=0}\,, \tag{5.2.3.3}$$

[7]Recent observations seem to hint at a small but non zero constant. For the discussion of the fine tuning problem this value is too small to be relevant.

where b and T are constants. In what follows we will focus on the case $b \neq \pm\frac{4}{3}$. The case $b = \pm\frac{4}{3}$ is similar and discussed in[268, 267], [36], [186, 187]. We take the ansatz (5.2.1.2) with $\bar{\Lambda} = 0$. From equation (5.2.1.5) one finds that

$$A' = \pm\frac{1}{3}\Phi'. \tag{5.2.3.4}$$

We choose

$$A' = \begin{cases} \frac{1}{3}\Phi', & y < 0 \\ -\frac{1}{3}\Phi', & y > 0 \end{cases} . \tag{5.2.3.5}$$

The reader may verify that taking the same sign on both sides of the brane does not lead to a consistent solution. The only other choice is to interchange the signs in (5.2.3.5). This can be undone by redefining $y \to -y$ and hence the ansatz (5.2.3.5) is general (for $b \neq \pm\frac{4}{3}$). The rest of the equations of motion is easily solved with the result

$$\Phi(y) = \begin{cases} \frac{3}{4}\log\left|\frac{4}{3}y + c_1\right| + d_1, & y < 0 \\ -\frac{3}{4}\log\left|\frac{4}{3}y + c_2\right| + d_2, & y > 0 \end{cases} , \tag{5.2.3.6}$$

where c_i and d_i are integration constants. The condition that Φ should be continuous at $y = 0$ fixes d_2 in terms of the other integration constants. The jump conditions (5.2.1.10) and (5.2.1.11) determine c_1 and c_2 in terms of d_1 according to

$$\frac{2}{c_2} = \left(-\frac{3b}{8} - \frac{1}{2}\right) T e^{bd_1} \left|c_1\right|^{\frac{3b}{4}} , \tag{5.2.3.7}$$

$$\frac{2}{c_1} = \left(-\frac{3b}{8} + \frac{1}{2}\right) T e^{bd_1} \left|c_1\right|^{\frac{3b}{4}} . \tag{5.2.3.8}$$

Together with possible constant shifts in A, two integration constants are not fixed by the equations of motion.

The next step is to ensure that an observer will experience four dimensional gravitational interactions (plus possible small corrections). This is the case only if the four dimensional Planck mass is finite. The expression for the four dimensional Planck mass is given in (5.2.2.5). If the parameters (T and b) of the model are such that there is no singularity at some $y > 0$ ($y < 0$) the integration region in (5.2.2.5) extends to (minus) infinity. In one or both of these cases the four dimensional Planck mass diverges, and an effective four dimensional theory decouples from gravity. This is not what we are interested in since with decoupled gravity the problem of the cosmological constant does not occur. Therefore, we have to choose our parameters such that there are singularities at which we can cut off the integration over y. Explicitly this imposes the conditions

$$
\begin{aligned}
T\left(\frac{1}{2}-\frac{3b}{8}\right) &> 0, \\
T\left(-\frac{1}{2}-\frac{3b}{8}\right) &< 0.
\end{aligned}
\tag{5.2.3.9}
$$

These conditions are easy to satisfy without fine tuning of the parameters. So far, it looks as if we have achieved to find a solution with vanishing four dimensional curvature without the necessity of a severe fine tuning of input parameters.

It remains to check whether the consistency condition (5.2.2.11) is satisfied. Since we have taken the ansatz with $\bar{\Lambda} = 0$, the condition states that the vacuum energy density of our solution should vanish. The vacuum energy density is most easily computed from (5.2.2.10). To be specific, we fix the integration constant in A via $A = \frac{1}{3}\Phi$ for $y < 0$. Taking further into account that our background is static, we find for the vacuum energy density

$$
\mathcal{E} = -\frac{1}{3}Te^{4A+b\Phi}{}_{|y=0} = -\frac{2}{3}\frac{8}{4-3b}e^{\frac{4}{3}d_1} \neq 0. \tag{5.2.3.10}
$$

We see that the consistency condition is not satisfied. Since the condition of vanishing vacuum energy density $\mathcal{E} = 0$ is derived from the equation of motion, (5.2.3.10) implies that the equations of motion are not solved. Indeed, with the parameter choice (5.2.3.9), the second derivatives of Φ and A contain delta functions which are not canceled by source terms in the equations of motion. We have to cure this inconsistency by adding additional source terms to the setup, i.e. to extend the single brane scenario to a three brane scenario. From our considerations in section 5.2.1, we know already that this will lead to fine tuning conditions on the input parameters. For illustrative purposes, let us demonstrate the appearance of the fine tuning explicitly. We modify our action (5.2.3.3) by two additional source terms, i.e.

$$
S \rightarrow S + S_{+} + S_{-}, \tag{5.2.3.11}
$$

with

$$
S_{\pm} = -\int d^4x T_{\pm} e^{b_{\pm}\Phi}{}_{|y=y_{\pm}}\,. \tag{5.2.3.12}
$$

The quantities $b_{\pm}$ and $T_{\pm}$ are now input parameters of the model. The value of $y_{\pm}$ gives the locations of the singularities,

$$
y_{-} = -\frac{3}{4}c_1 \quad , \quad y_{+} = -\frac{3}{4}c_2. \tag{5.2.3.13}
$$

The additional source terms give rise to four more jump conditions to be satisfied by the solution. These jump conditions are

$$\frac{8}{3}\left(\Phi'(y_\pm+0)-\Phi'(y_\pm-0)\right) = b_\pm T_\pm e^{b_\pm\Phi(y_\pm)}, \tag{5.2.3.14}$$

$$\mp\left(\Phi'(y_\pm+0)-\Phi'(y_\pm-0)\right) = -\frac{1}{2}T_\pm e^{b_\pm\Phi(y)}. \tag{5.2.3.15}$$

Before solving these additional jump conditions we need to give a prescription how to continue our solution beyond the singularities. There are several possible descriptions. For example, one may continue in such a way that the setup becomes periodic in y. The simplest choice is to effectively cut off the space at the singularities (at $y=y_\pm$) by freezing the fields to the singularity values for $y\notin[y_-,y_+]$ such that the first derivatives vanish beyond the singularities. (The final conclusion is not affected by the particular way of continuing the solution beyond the singularities.) With our prescription the conditions (5.2.3.14) and (5.2.3.15) are solved by

$$b_\pm = \pm\frac{4}{3} \tag{5.2.3.16}$$

and

$$T_- e^{-\frac{4}{3}d_1} = T_+ e^{\frac{4}{3}d_2} = -2. \tag{5.2.3.17}$$

One should recall that d_2 is already fixed by the jump conditions at $y=0$. We observe that the input parameters need to be fine tuned.

The contribution of the branes at $y=y_\pm$ to the vacuum energy density is

$$\begin{aligned}\mathcal{E}_+ + \mathcal{E}_- &= -\frac{1}{3}\left(T_+ e^{4A+b_+\Phi}{}_{|y=y_+} + T_- e^{4A+b_-\Phi}{}_{|y=y_-}\right)\\ &= \frac{2}{3}e^{\frac{4}{3}d_1}\frac{8}{4-3b},\end{aligned} \tag{5.2.3.18}$$

where we have employed the jump conditions and fixed an integration constant in A by the choice $A=\frac{1}{3}\Phi$ for $y<0$. Hence, the contribution (5.2.3.10) is exactly canceled by the additional branes and the model is consistent now. However, we failed to construct a brane setup yielding a vanishing effective four dimensional cosmological constant without fine tuning of the parameters. If the fine tuning is not satisfied, there exist $\bar{\Lambda}\neq 0$ solutions[267]. (The situation is slightly different in the $b=\pm\frac{4}{3}$ case where the possible value of $\bar{\Lambda}$ is fixed by the bulk potential V, which needs to be fine tuned to zero for $\bar{\Lambda}=0$ to be a solution[187]. In addition there is a fine tuning due to the necessity of additional branes for $b=\pm\frac{4}{3}$, too.)

In the next subsection we will show that our failure to find a $\bar{\Lambda}=0$ solution without fine tuning is not caused by an unfortunate choice of the model we started with but rather a generic feature of brane models with a bulk scalar.

5.2.3.2 A no go theorem

The prepositions for the no go theorem for a "brany" solution to the cosmological constant problem are:

- The model contains a single brane and $\bar{\Lambda} = 0$.
- The four dimensional Planck mass is finite.
- The model does not contain singularities apart from the one corresponding to the single brane source.
- The bulk potential V can be expressed in terms of the "superpotential" W according to (5.2.1.9).

In a first step, we are going to show that these prepositions imply that the five dimensional space must be asymptotically (for large $|y|$) an AdS space. Suppose the warp factor asymptotically shows a power like behavior,

$$e^A \sim |y|^{-\alpha} . \tag{5.2.3.19}$$

The four dimensional Planck mass is computed in (5.2.2.5). With a single brane and no further singularities the integration is taken over $y \in (-\infty, \infty)$. A necessary condition for

$$\int_{-\infty}^{\infty} dy\, e^{2A} < \infty \tag{5.2.3.20}$$

is

$$\alpha > \frac{1}{2}. \tag{5.2.3.21}$$

On the other hand, equation (5.2.1.6) tells us that in the bulk (in particular asymptotically)

$$A'' < 0 \Longrightarrow \alpha < 0. \tag{5.2.3.22}$$

We conclude that e^A cannot fall off with a power of $|y|$ as $|y| \to \infty$.

Therefore, we assume an exponential fall off, i.e. for large $|y|$

$$A = -k\,|y|\,, \tag{5.2.3.23}$$

with k being a positive constant. In the following we will show that in this case there is a fine tuning similar to the fine tuning of the RS2 model. Before going into the details, let us sketch the outline of the proof. The asymptotic behavior (5.2.3.23) suffices to reproduce the superpotential for all y. Plugging this into the matching conditions (5.2.1.12) and (5.2.1.13) will show that the input parameters of the model need to be fine tuned. Let us now present the details of the slightly tedious construction of W from its asymptotics.

From equation (5.2.1.6) we learn that Φ must be asymptotically constant. We denote the asymptotic values of Φ by $\Phi_c^{\pm}$ corresponding to the limits $y \to \pm\infty$. Equation (5.2.1.8) implies that

$$\frac{\partial W}{\partial \Phi}_{|\Phi=\Phi_c^{\pm}} = 0. \tag{5.2.3.24}$$

Plugging (5.2.3.23) into (5.2.1.7) yields

$$W\left(\Phi_c^{+}\right) < 0 \quad \text{and} \quad W\left(\Phi_c^{-}\right) > 0. \tag{5.2.3.25}$$

Let us look again at equation (5.2.1.8)

$$\Phi' = -\frac{9}{4}\frac{\partial W}{\partial \Phi}, \tag{5.2.3.26}$$

and view Φ' as a function of Φ. Φ should reach its asymptotic values in a dynamical way which means that Φ' should be monotonically decreasing (increasing) as Φ approaches Φ_c^{+} (Φ_c^{-}). We obtain the conditions

$$\frac{\partial^2 W}{\partial \Phi^2}_{|\Phi=\Phi_c^{+}} > 0 \quad , \quad \frac{\partial^2 W}{\partial \Phi^2}_{|\Phi=\Phi_c^{-}} < 0. \tag{5.2.3.27}$$

(Equation (5.2.3.26) can be viewed as a renormalization group equation, where the renormalization group scale is related to Φ. W is proportional to the running coupling, and Φ' (viewed as a function of Φ) is the beta function. The conditions (5.2.3.27) mean that $\Phi = \Phi_c^{+}$ ($\Phi = \Phi_c^{-}$) correspond to stable UV (IR) fixed points.) Equations (5.2.1.9) and (5.2.3.24) fix the asymptotic values of the superpotential according to

$$V\left(\Phi_c^{-}\right) = -12 W\left(\Phi_c^{-}\right)^2 \quad , \quad V\left(\Phi_c^{+}\right) = -12 W\left(\Phi_c^{+}\right)^2. \tag{5.2.3.28}$$

This implies that the asymptotic values of V must be negative. Further note that the asymptotic values of W are fixed in a unique way with the additional conditions (5.2.3.25). So far, we know the asymptotic value of W in terms of the input parameters and the asymptotics of the first derivative of W (5.2.3.24).

In order to compute the higher derivatives of W, it is useful to express the nth derivative of V in terms of W via (5.2.1.9). The corresponding expression is

$$\frac{\partial^n V}{\partial \Phi^n} = \sum_{k=1}^{n} 2\binom{n-1}{k-1}\frac{\partial^k W}{\partial \Phi^k}\left(\frac{27}{4}\frac{\partial^{n-k+2} W}{\partial \Phi^{n-k+2}} - 12\frac{\partial^{n-k} W}{\partial \Phi^{n-k}}\right). \tag{5.2.3.29}$$

This formula is most easily proven in the following way. First, apply the Leibniz rule (F and G are arbitrary functions of Φ)

$$\frac{\partial^n (FG)}{\partial \Phi^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k F}{\partial \Phi^k} \frac{\partial^{n-k} G}{\partial \Phi^{n-k}} \tag{5.2.3.30}$$

on $\frac{\partial W^2}{\partial \Phi^2} = 2W \frac{\partial W}{\partial \Phi}$ in order to show that

$$\frac{\partial^n W^2}{\partial \Phi^n} = \sum_{k=1}^{n} 2 \binom{n-1}{k-1} \frac{\partial^{n-k} W}{\partial \Phi^{n-k}} \frac{\partial^k W}{\partial \Phi^k}. \tag{5.2.3.31}$$

In a second step use (5.2.3.31) with W replaced by its first derivative and redefine the summation index $k \to n+1-k$.

In the following we will employ (5.2.3.29) to compute the asymptotics of all derivatives of W. Since there are no singularities between the brane and the asymptotic region, this will enable us to expand W in a Taylor series yielding its value arbitrarily close to the brane.

The second derivative of W needs some separate discussion. With the result (5.2.3.24) we obtain the relation

$$\left[\frac{27}{2}\left(\frac{\partial^2 W}{\partial \Phi^2}\right)^2 - 24W \frac{\partial^2 W}{\partial \Phi^2} - \frac{\partial^2 V}{\partial \Phi^2}\right]_{|\Phi=\Phi_c^\pm} = 0. \tag{5.2.3.32}$$

This equation has real solutions for the asymptotics of the second derivative of W provided that

$$\frac{\partial^2 V}{\partial \Phi^2}_{|\Phi=\Phi_c^\pm} > \frac{8}{9} V\left(\Phi_c^\pm\right). \tag{5.2.3.33}$$

Taking into account that the asymptotic value of W is fixed uniquely by (5.2.3.28) and (5.2.3.25), and that the sign of the asymptotic value of the second derivative of W is determined by (5.2.3.27), one finds that (5.2.3.32) can be solved in a unique way. Note, that (5.2.3.25) and (5.2.3.27) imply

$$\frac{\partial^2 V}{\partial \Phi^2}_{|\Phi=\Phi_c^\pm} > 0. \tag{5.2.3.34}$$

The computation of the higher derivatives of W in the large $|y|$ region is somewhat simpler. First, we notice that asymptotically on the rhs of (5.2.3.29) the nth derivative of W is the highest occurring derivative (see (5.2.3.24)). Terms containing the nth derivative correspond to $k = 2, n$. The expression (5.2.3.29) evaluated at $\Phi_c^\pm$ takes the form ($n > 2$)

$$\frac{\partial^n V}{\partial \Phi^n}_{|\Phi=\Phi_c^\pm} = \left[\frac{\partial^n W}{\partial \Phi^n}\left(\frac{27}{2} n \frac{\partial^2 W}{\partial \Phi^2} - 24W\right) + \ldots\right]_{|\Phi=\Phi_c^\pm}, \tag{5.2.3.35}$$

where the dots stand for terms containing lower derivatives of W. The relation (5.2.3.35) allows to determine all derivatives of W provided that the coefficient at the nth derivative of W differs from zero. This is ensured by equations (5.2.3.32) and (5.2.3.34). Indeed, requiring the coefficient in front of the nth derivative of W to vanish yields

$$\left[\frac{\partial^2 V}{V\partial\Phi^2}\right]_{|\Phi=\Phi_c^{\pm}} = -\frac{32}{9n}\left(1-\frac{1}{n}\right), \tag{5.2.3.36}$$

which is not compatible with (5.2.3.34) and (5.2.3.28). We conclude that in the Taylor expansion

$$W(\Phi) = \begin{cases} \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n W}{\partial\Phi^n}_{|\Phi=\Phi_c^-}\left(\Phi-\Phi_c^-\right)^n, & y<0 \\ \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n W}{\partial\Phi^n}_{|\Phi=\Phi_c^+}\left(\Phi-\Phi_c^+\right)^n, & y>0 \end{cases} \tag{5.2.3.37}$$

all the coefficients are fixed uniquely by the model parameters. Then the jump condition (5.2.1.12) will fix the value of Φ at $y=0$, whereas (5.2.1.13) imposes generically a fine tuning of the model parameters.

It may look somewhat disappointing to close a review with the proof of a no go theorem. However, often no go theorems help to find a way leading to the desired aim. This way should then start with a model not satisfying the prepositions of the no go theorem. Indeed, there have been proposals for not fine tuned solutions with $\bar{\Lambda}=0$. These proposals are based on the idea of introducing more integration constants without increasing the number of jump conditions. We provide the corresponding references in section 6. Here, we should remark that (so far) there is no commonly accepted solution to the cosmological constant problem within brane world setups. The explanation of the observed value of the cosmological constant remains a great challenge. Whether branes will be helpful in a solution of this problem has to be seen in the future.

Chapter 6

Bibliography and further reading

Throughout the text I have already given some references. However, this I did only when I felt that a direct hint on results obtained in the literature would be useful for the reader at that particular point. Of course, these notes are based on many more publications than already given in the text. In the present chapter, I will provide all the references I used and give suggestions for further reading. However, there are many more contributions to this field. I apologize to all those authors whose work could have been listed but is not.

6.1 Chapter 2

6.1.1 Books

In [222, 223], [87], [331], [371], [269, 270] I list the textbooks on string theory of which I am aware. In section 2.1 I used mainly[222] but also[331]. For the discussion of orbifold planes in section 2.2 I borrowed some results presented in[371]. D-branes and orientifolds are also covered in[371]. Since string theory is a conformal field theory the book[191] may be also a useful reference. The subject of Calabi-Yau compactifications entered the text rather as a side remark. Apart from the discussions presented in the above mentioned textbooks on strings, the book[251] is perhaps a useful reference for people who are interested in Calabi-Yau spaces. Let me also mention three books on supersymmetry. Often the conventions of the standard reference[41] are used in the literature. Ref.[456] contains (at least in its second edition) a discussion of supersymmetry in two dimensions. Finally, [399] is not really a textbook but a collection of papers dealing with supergravity in various dimensions. For each dimension there is a summary of the possible supermultiplets.

6.1.2 Review articles

Four recent review articles on perturbative string theory are [20], [368], [359] and [295]. In the context of perturbative string theory, the CFT lectures[205] may be also useful. The computation of beta functions in nonlinear sigma models is reviewed in[445]. Various aspects of T-duality are presented in[211, 14]. Orbifold compactifications are covered in most of the textbooks mentioned in 6.1.1. Two review articles on orbifolds are listed in[354], [139]. K3 and other Calabi-Yau compactifications are e.g discussed in[249], [224], [38], [350]. There are various reviews on D-branes: [373, 370], [39, 40], [437], [439], [263, 264], [130, 131]. Readers who are interested in D-branes on Calabi-Yau spaces should consult[146] (and references therein). In[397], [118] two lecture notes on orientifolds are listed. Phenomenological aspects of string theory are reviewed in [324, 325], [380, 381], [135]. There are quite a few reviews on supersymmetry, e.g. [355], [429], [333]. Ref. [127] presents supergravities in various dimensions.

6.1.3 Research papers

For early papers on string theory I refer to the excellent commented bibliography given in [222, 223]. Although there is still some overlap with the references in[222], I want to start with section 2.1.3. Here, we presented details which at some points differ from the discussion in[222]. A list of references about beta functions in string sigma models (some of them about the open string (section 2.3.3.1)) is [18], [79], [326, 327], [189, 190], [98], [419], [99], [1], [44], [447, 446], [315], [111], [142], [49] and many others. The normal coordinate expansion technique in section 2.1.3 is taken from[18]. In a slightly different version it can be found in[79]. The Fischler Susskind mechanism is developed in [170, 171] and also discussed in e.g.[327], [48]. Like the present text, most of the articles do not include the discussion of non trivial backgrounds for massive string modes. The corresponding sigma model is not renormalizable. Some papers on beta functions for massive string modes are [311], [89], [162], [314], [176], [90], [91].

Concerning section 2.1.4 I give some references related to the construction of the supergravity theories. The existence of ten dimensional type II supergravities (and also 11 dimensional supergravity) was suggested in[349]. The explicit construction has been carried out in[219] (see also[411, 405, 250]). Anomalies are discussed in[21]. That $N = 1$ ten dimensional supergravity coupled to $E_8 \times E_8$ or $SO(32)$ gauge theory is anomaly free was demonstrated in[220].

T-duality for the circle compactified bosonic string is discussed in[290, 398]. For compactifications on higher dimensional tori see[352, 425, 212]. The presentation in section 2.1.5.3 follows closely[392]. T-duality in non trivial backgrounds with abelian isometries was originally studied in[94, 95]. Some related papers are: [339], [213], [209],

[296], [12, 11], [448], [207], [298]. T-duality has been also discussed for backgrounds with non Abelian isometries e.g. in [126], [11, 13], [214], [297], [305], [160], [113, 114, 112], [17], [450], [9], [343], [242]. The T-duality relation between type IIA and type IIB strings can be found in [136], [122], [57], [289]. The connection between compactified $E_8 \times E_8$ and $SO(32)$ strings is presented in [206].

The techniques for orbifold compactifications of string theory have been developed in [137, 138]. More papers on orbifolds are (including explicit constructions of phenomenological interest): [451], [237], [42], [286], [256], [257], [253], [174], [173], [255], [318], [175]. T-duality for orbifold compactifications is for example discussed in [312, 313].

The importance of D-branes was realized in[369], where the connection to BPS solutions of supergravity was discovered. In the text I have given conditions imposed by the requirement unbroken supersymmetry on the number of ND directions. More generally, D-branes can intersect at certain angles[59]. The computation of D-brane interactions is presented in [372], [92, 93]. References concerning the beta function approach are given together with the other references for the beta function approach to effective field theories, above. D-brane actions are also e.g. discussed in[400]. The interchange of Dirichlet with Neumann boundary conditions via T-duality has been pointed out in [122], [243], [217]. For general backgrounds, T-duality for open strings with respect to abelian isometries is presented in[15], [143]. T-duality with respect to non-Abelian isometries has been performed in [184], [77]. (The boundary Lagrange multiplier has been introduced in [184, 185].) A different method of performing T-duality transformations in general backgrounds has been proposed in [306]. The Wess-Zumino term in the D-brane action has been derived (in steps) in [145], [320], [61, 62], [218]. Our discussion of open strings and non commutative geometry follows closely (the introductory section of) [418]. Constant B-fields and non commutative geometry have been connected earlier in e.g. [147], [106], [404]. The connection between non commutativity and the renormalization scheme is further investigated e.g. in [24, 23]. A more abstract conformal field theory approach to D-branes can for example be found in [194], [388]. There are many more aspects of D-branes for which I would like to ask the reader to consult one of the given reviews and the references therein.

Orientifolds were introduced in[396]. For early papers on orientifold constructions see also [378], [216], [244], [63, 64]. The cancellation of divergences in string diagrams of type I $SO(32)$ strings is observed in[221]. The model of section 2.4.3.2 has been first constructed in [63, 64]. The presentation in the text follows [204]. Indeed, it has been the paper [204] which triggered an enormous amount of research devoted to orientifolds. This research resulted in a lot of papers out if which I list only “a few”: [120, 121], [203], [60], [68], [272], [469], [179], [178], [357], [5], [71, 70], [115], [377],

[183], [254], [31], [25], [8], [26], [28], [27], [383], [302], [300], [301], [72], [73], [182], [382], [74], [7, 6], [117, 116].

6.2 Chapter 3

As far as I know there are no books devoted to solutions of ten dimensional supergravity.

6.2.1 Review articles

There are quite a few review articles to be mentioned in the context of brane solutions to supergravity. In the text I used mainly results presented in[152]. BPS solutions to ten dimensional supergravity are also derived in [97], [431]. The theories on the worldvolumes of the branes are discussed e.g. in [444]. Intersecting brane solutions are e.g. reviewed in[196]. In the text I did not discuss the relevance of the brane solutions to black hole physics. A nice introductory review to black holes is[442]. Branes in the context of black hole physics are reviewed e.g. in[337, 338], [46], [464], [362], [428], [151], [344, 345], [348].

6.2.2 Research Papers

The elementary string solution was found in[119]. The five brane solution has been considered e.g. in [433], [154], [97]. The general p-brane solutions are presented in [248]. For more references on the topic of brane solutions to supergravity I would like to ask the reader to consult the review articles mentioned in section 6.2.1.

6.3 Chapter 4

The presented applications of branes are not a subject of a book. A discussion of string dualities can be found in[371].

6.3.1 Review articles

There are many reviews devoted to the subject of string dualities: [441, 443], [38], [407, 408], [452], [188], [127], [265], [423], [356], [235].

A comprehensive review on the relation between brane setups and field theory dualities is listed in[210]. (Another (shorter) review is [282].) In the text I mentioned only duality relations in $N = 1$ supersymmetric field theories. Such dualities are summarized in[414, 261], [208], [365], [426]. $N = 2$ supersymmetric field theories are considered in[65], [129], [317], [19], [33]. The duality of $N = 4$ super Yang-Mills theory is presented in [358], [239].

The standard review article on the AdS/CFT correspondence is[2]. Two more introductory notes are listed in [366], [299]. Lecture notes dealing with Wilson loops in the context of the AdS/CFT correspondence are e.g.[430].

Settings where the string scale is the TeV scale are reviewed in[32], [29].

6.3.2 Research papers

Early proposals of strong/weak coupling duality appear within the context of the compactified heterotic string[172], [389]. This conjecture was supported by observations reported in[420, 422, 421], [410, 409]. The existence of 11 dimensional supergravity was suggested in[349]. The explicit construction was carried out in[108]. The M-theory picture was developed in the papers[252], [440], [458]. The duality between $SO\,(32)$ type I and heterotic strings was proposed in[374]. The $SL\,(2, \mathbb{Z})$ duality of type IIB strings is discussed in[406]. The relation between the $E_8 \times E_8$ heterotic string and eleven dimensional supergravity is worked out in[246, 245].

Dualities in field theories were conjectured in[346], and shown to be exact in $N = 4$ supersymmetric Yang Mills theory in[463], [360], [457]. Strong coupling results in $N = 2$ gauge theories are presented in [416], [417], for $SU\,(2)$. Extensions to other gauge groups are discussed in e.g.[304], [303], [34], [167], [166]. The $N = 1$ field theory dualities have been conjectured in[412, 413]. Some out of many subsequent papers are[316], [258], [260], [259], [156, 157], [309], [310], [88]. Studying field theories via manipulations in brane setups was initiated in[238]. The discussion in the text follows[158]. There are many related works. Some examples are: [125], [159], [165], [82], [3], [436], [283]. The connection between $N = 2$ supersymmetric gauge theories and M-theory branes is considered in[459]. There is also a larger list of literature dealing with brane setups for $N = 2$ theories, for which, however, I would like to ask the reader to consult one of the reviews since this would lead to far away from the subjects discussed in the text.

The AdS/CFT correspondence is conjectured in[335], and further elaborated in[233], [460]. The computation of Wilson loops within the conjecture is described in[336], [391]. Differently shaped Wilson loops are discussed in[54], [148]. Breaking supersymmetry by a finite temperature one can observe the confinement of quarks[461]. Related papers are[81], [390], [80], [230], [234], [144], [361] and many others. The string action on $AdS_5 \times S^5$ is constructed in[340]. This action is discussed further in[276], [275], [273], [364], [277], [384]. The construction of[340] leads also to the result that the $AdS_5 \times S^5$ background is exact. Different arguments for this statement are given in[43]. The discussion of the stringy corrections to the Wilson loop follows[180, 181]. A similar approach (in the conformal gauge) and more examples are discussed in[149]. This paper also addresses the problem of the divergence and gives a numerical estimate of the correction. String fluctuations as a source for corrections to the Wilson loop are also discussed in[226], [351], [466], [294], [262], [322]. Corrections to the field theory calculation are derived in[163], [467], [164],[367]. An attempt to apply the techniques for computing corrections to the Wilson loop on the M5 brane case is reported in[177].

That branes allow constructions with the string scale at a TeV has been pointed out in[30]. (Relating the hierarchy problem to the size of extra dimensions has been proposed before in a field theory context[35].) The argument that in compactifications of the perturbative heterotic string the size of the compact space is of the order of the Planck size is given in[281]. Our discussion of corrections to Newton's law follows[287].

6.4 Chapter 5

Since there are no books on the subject of brane world setups I start directly with a list of review articles.

6.4.1 Review articles

The review articles on brane world setups with warped transverse dimensions I am aware of are[266], [393], [133], [334]. An overview on the cosmological constant problem is presented in[455], [462], [66].

6.4.2 Research papers

Brane world models have been proposed already sometime back in[394], [4]. The model discussed in section 5.1.1 is presented in[386]. The stabilization mechanism is proposed in[215]. The model of section 5.1.2 is taken from[385]. An early paper on connecting the Randall Sundrum model with the holographic principle is[454]. The computation of the Newton potential via the holographic principle has been pointed out by Witten in the discussion session in a Santa Barbara Conference in 1999. (I have not been there.) The presentation in the text is taken from[232] (see also[153]). The inclusion of the second brane into the RS2 scenario is performed in[332]. The computation of the Newton potential via the AdS/CFT correspondence is taken from[37] (see also [201]). More discussions of the RS models from a holographic perspective can be found e.g. in[22], [78], [387], [363] , [284, 285], [427], [197], [107]. Supersymmetry within the context of the Randall Sundrum model is discussed in[274], [47], [10], [198], [168, 169], [58], [468].

Section 5.2.1 is closely related to[128]. The consistency condition that the effective cosmological constant should be compatible with the metric on the brane is also mentioned in[128]. The derivation and form of the consistency condition in section 5.2.2 is presented in[187]. The alternative method of integrating a total derivative is developed in[161]. The connection between the two conditions has been pointed out in[187]. The complete set of consistency conditions (as it appears in the text) is given in[199]. (Different consistency conditions are discussed in[279].)

That the cosmological constant problem is rephrased within a brane world setup is discussed in[395]. The example of section 5.2.3.1 (and a closely related example) appear in[268], [36]. That the effective cosmological constant does not vanish in this models is observed (simultaneously) in [465], [186]. To reach consistency by adding branes and consequently fine tuning input parameters is proposed in [186]. (Problems with singularities in warped compactifications are considered e.g. also in[231], [291].) The proof of the no go theorem is taken from[110].

There are too many papers on warped brane world scenarios to be listed. Therefore, the following list is restricted to papers dealing with the cosmological constant problem (and most likely this list is also incomplete): [101], [123], [280], [124], [247], [227], [45], [308, 307], [105], [67], [321], [271], [278], [104], [55], [85], [100], [103], [134], [86], [240]. Papers containing proposals on avoiding the fine tuning problem of the cosmological constant by going beyond the prepositions of the no go theorem (section 5.2.3.2) are [288], [292, 293], [109, 229]. A different proposal for addressing the cosmological constant problem in brane world scenarios is put forward in[453], [401, 402, 403].

Warped compactifications in the context of string theory are e.g. discussed in[330, 329], [53], [150], [69], [84].

Observational bounds on extra dimension scenarios are e.g. presented in[323], [342], [341].

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