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fluid mechanics

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Problem Solvers

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No. 15

Fluid Mechanics

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Fluid Mechanics

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Contents

Chapter 1

General Flow

1.1 Introduction Fluid mechanics is concerned with the behaviour of fluids (liquids or gases) in motion. One method, due to Lagrange, traces the progress of the individual fluid particles in their movement. Each particle in the continuum is labelled by its initial position vector (say) $\mathbf{a}$ relative to a fixed origin 0 at time $t = 0$. At any subsequent time $t > 0$ this position vector becomes $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ from which the particle's locus or *pathline* is determined. In general, this pathline will vary with each fluid particle. Thus every point P of the continuum will be traversed by an infinite number of particles each with its own pathline. In Figure 1.1 let A_1, A_2, A_3 be three such particles labelled by their position vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, respectively, at time $t = 0$. Travelling along their separate

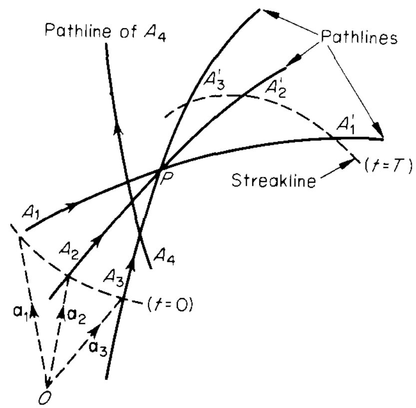


Figure 1.1

pathlines, these fluid particles will arrive at P at *different* times and continue to move to occupy the points A'_1, A'_2, A'_3, respectively, at some time $t = T$. These points, together with P, lie on a curve called the *streakline* associated with the point P. If a dye is introduced at P a thin strand of colour will appear along this streakline $PA'_1 A'_2 A'_3$ at time $t = T$. It is obvious that this streakline emanating from P will change its shape with time. A fourth fluid particle A_4 which at time $t = 0$ lies on the pathline A_3P will, in general, have a *different* pathline $A_4 A'_4$ which may *never* pass through P. The situation created by the Lagrangian approach is complicated and tells us more than we normally need to know about the fluid motion. Finally, the velocity and acceleration of the particle at any instant are given by $\partial \mathbf{r}/\partial t$ and $\partial^2 \mathbf{r}/\partial t^2$ respectively.

The method of solution mainly used is due to Euler. Attention is paid to a point P of the fluid irrespective of the particular particle passing through. In this case the solutions for velocity $\mathbf{q}$, pressure p and density ρ etc are expressed in the form $\mathbf{q} = \mathbf{q}(\mathbf{r}, t)$, $p = p(\mathbf{r}, t)$, $\rho = \rho(\mathbf{r}, t)$ respectively where $\mathbf{r} = \mathbf{OP}$ is the position vector of the point P referred to a fixed origin and t is the time. If these solutions are independent of time t, the flow is said to be *steady*, otherwise the flow is *unsteady* and varies with time at any fixed point in the continuum. In the Eulerian approach the pathline is replaced by the *streamline* defined as follows.

Definition. A line drawn in the fluid so that the tangent at every point is the direction of the fluid velocity at the point is called a *streamline.*

In unsteady flow these streamlines form a continuously changing pattern. If, on the other hand, motion is time independent, i.e. steady, the streamlines are fixed in space and in fact coincide with the pathlines.

Definition. A *stream surface* drawn in a fluid has the property that, at every point on the surface, the normal to the surface is perpendicular to the direction of flow at that point.

A stream surface, therefore, contains streamlines.

Definition. Given any closed curve C, a *streamtube* is formed by drawing the streamline through every point of C.

Definition. A *stream filament* is a streamtube whose cross-sectional area is infinitesimally small.

To obtain the equation of the streamlines or, as they are sometimes called, the lines of flow we write

$$\mathbf{q}(\mathbf{r}, t) = u(\mathbf{r}, t)\mathbf{i} + v(\mathbf{r}, t)\mathbf{j} + w(\mathbf{r}, t)\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors parallel to the fixed coordinate axes OX, OY and OZ respectively. Since, by definition, $\mathbf{q}$ is parallel to $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ we have

$$\frac{dx}{u(\mathbf{r}, t)} = \frac{dy}{v(\mathbf{r}, t)} = \frac{dz}{w(\mathbf{r}, t)} \tag{1.1}$$

Any integral of these equations must be of the form $f(\mathbf{r}, t) = $ constant, which is a stream surface. Its intersection with a second independent solution, $g(\mathbf{r}, t) = $ constant, gives the streamline at any t.

Problem 1.1 Given that the Eulerian velocity distribution at any time t in a fluid is $\mathbf{q} = \mathbf{i} \wedge \mathbf{r} + \mathbf{j}\cos at + \mathbf{k}\sin at$ where a is a constant ($\neq \pm 1$), find the streamlines and pathlines. Discuss the special case $a = 0$.

Solution. Writing $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, we find that $u = 0$, $v = -z + \cos at$,

$w = y + \sin at$. So the streamlines at any given time t are determined by the equations

$$\frac{dx}{0} = \frac{dy}{-z + \cos at} = \frac{dz}{y + \sin at}$$

One solution is $x = F$ where F is arbitrary, i.e. a family of planes. The solution of $(y + \sin at)dy = (-z + \cos at)dz$ is the family of circular cylinders forming the second system of stream surfaces whose equation is $y^2 + z^2 + 2y\sin at - 2z\cos at = G$ where G is arbitrary. The intersections are circles, the required streamlines. When $a \neq 0$ these form a continuously changing pattern, the motion being *time dependent.* In the special case $a = 0$, the flow is steady with $\mathbf{q} = (-z + 1)\mathbf{j} + y\mathbf{k}$ and the streamlines are *fixed circles* given by the equations $x = $ constant, $y^2 + z^2 - 2z = $ constant. The pathlines are the solutions of

$$u = \partial x/\partial t = 0, \qquad v = \partial y/\partial t = -z + \cos at, \qquad w = \partial z/\partial t = y + \sin at$$

from which we obtain $x = $ constant. Eliminating $\partial z/\partial t$ by differentiating, the equation for y is

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial z}{\partial t} - a\sin at = -y - (a+1)\sin at$$

Since we are given that $a \neq \pm 1$, the solution is

$$y = A\cos t + B\sin t + C\sin at$$

where A and B are arbitrary constants and $C = 1/(a-1)$. Also, from the equation for v we have

$$z = -\frac{\partial y}{\partial t} + \cos at = A\sin t - B\cos t - C\cos at$$

In the special case $a = 0$, $C = -1$, $\cos at = 1$, $\sin at = 0$, and $y^2 + (z-1)^2 = A^2 + B^2 = $ constant. Since also $x = $ constant, the pathlines are circles coincident with the streamlines in steady flow. □

Next we consider the concept of *pressure* in a fluid. Referring to Figure 1.2, let P be any point in the fluid and δA any infinitesimally small plane area containing the point with $\mathbf{PQ} = \mathbf{n}$ representing the unit normal *from*

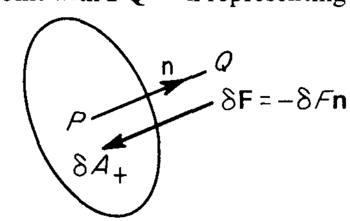


Figure 1.2

one side δA_+ of δA *into* the fluid. Let $\delta\mathbf{F}$ denote the force exerted by the fluid on δA_+.

The fluid is defined to be *inviscid* when $\delta\mathbf{F}$ has no component in the plane of δA for *any orientation* of $\mathbf{n}$. If in addition $\delta\mathbf{F}$ is anti-parallel to $\mathbf{n}$ and has a magnitude $\delta F = |\delta\mathbf{F}|$ which in the limit as $\delta A_+ \to 0$ is independent of the direction of $\mathbf{n}$, the fluid is said to be *perfect*. Moreover, the pressure at P is $p = p(\mathbf{r}, t)$ where

$$p\mathbf{n} = \lim_{\delta A_+ \to 0} \delta\mathbf{F}/\delta A_+ \tag{1.2}$$

When motion is steady $p = p(\mathbf{r})$ instead.

1.2 The mobile operator D/Dt In the Eulerian system where the velocity $\mathbf{q} = \mathbf{q}(\mathbf{r}, t)$, $\partial\mathbf{q}/\partial t$ *does not* represent the acceleration of a particle but is simply the rate of change of $\mathbf{q}$ at a *fixed point* $\mathbf{r}$ which is being traversed by *different* particles. To evaluate this acceleration we need to find the rate of change of the velocity $\mathbf{q}$ *momentarily following a labelled particle*. We write this rate of change as $D\mathbf{q}/Dt$. Similarly, if any other quantity, such as temperature T, is carried by a fluid particle its rate of change would be DT/Dt.

Suppose $\mathscr{H} = \mathscr{H}(\mathbf{r}, t)$ denotes any differentiable vector or scalar function of $\mathbf{r}$ and t then we may write, in Cartesian terms,

$$\mathscr{H} = \mathscr{H}(\mathbf{r}, t) \equiv \mathscr{H}(x, y, z; t)$$

Hence, at time δt later the increase $\delta\mathscr{H}$ in $\mathscr{H}$ is

$$\delta\mathscr{H} = \mathscr{H}(x+\delta x, y+\delta y, z+\delta z; t+\delta t) - \mathscr{H}(x, y, z; t)$$

However, when we *follow* the fluid particle we must write $\delta x = u\,\delta t$, $\delta y = v\,\delta t$, $\delta z = w\,\delta t$ (correct to the *first* order in δt) where $u = u(x, y, z; t)$ etc. are the Cartesian components of the velocity $\mathbf{q}$ so that

$$\begin{aligned}\delta\mathscr{H} &= \mathscr{H}(x+u\,\delta t, y+v\,\delta t, z+w\,\delta t; t+\delta t) - \mathscr{H}(x, y, z; t)\\ &= \frac{\partial\mathscr{H}}{\partial t}\delta t + \frac{\partial\mathscr{H}}{\partial x}u\,\delta t + \frac{\partial\mathscr{H}}{\partial y}v\,\delta t + \frac{\partial\mathscr{H}}{\partial z}w\,\delta t + O(\delta t^2)\end{aligned}$$

It follows that in taking limits,

$$\frac{D\mathscr{H}}{Dt} = \lim_{\delta t \to 0} \frac{\delta\mathscr{H}}{\delta t} = \frac{\partial\mathscr{H}}{\partial t} + u\frac{\partial\mathscr{H}}{\partial x} + v\frac{\partial\mathscr{H}}{\partial y} + w\frac{\partial\mathscr{H}}{\partial z} \tag{1.3}$$

In vector terms, since $\nabla \equiv \mathbf{i}\,\partial/\partial x + \mathbf{j}\,\partial/\partial y + \mathbf{k}\,\partial/\partial z$, we have

$$u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z \equiv \mathbf{q}\,.\nabla,$$

therefore,

$$\frac{D\mathscr{H}}{Dt} = \frac{\partial\mathscr{H}}{\partial t} + (\mathbf{q}\,.\nabla)\mathscr{H} \tag{1.4}$$

The first term on the right-hand side is the time rate of change at a fixed point P and the second term $(\mathbf{q}\,.\nabla)\mathscr{H}$ is the *convective* rate of change due to the particle's changing position. In particular, the fluid acceleration $\mathbf{f}$ is

$$\mathbf{f} = \frac{D\mathbf{q}}{Dt} = \frac{\partial\mathbf{q}}{\partial t} + (\mathbf{q}\,.\nabla)\mathbf{q} \tag{1.5}$$

Moreover, it can now be seen that in terms of this mobile operator the fluid velocity in the Eulerian system is simply

$$\mathbf{q} = \frac{D\mathbf{r}}{Dt} = (\mathbf{q}\,.\nabla)\mathbf{r} \equiv \mathbf{q}$$

since here $\partial\mathbf{r}/\partial t \equiv 0$.

Problem 1.2 A fluid flows steadily from infinity with velocity $-U\mathbf{i}$ past the fixed sphere $|\mathbf{r}| = a$. Given that the resultant velocity $\mathbf{q}$ of the fluid at any point is $\mathbf{q} = -U(1+a^3r^{-3})\mathbf{i} + 3a^3r^{-5}xU\mathbf{r}$, find the acceleration $\mathbf{f}$ at any point $\mathbf{r} = b\mathbf{i}$ $(b > a)$ and evaluate the maximum value of $|\mathbf{f}|$ for variation in b.

Solution. Since the motion is steady $\mathbf{f} = (\mathbf{q}\,.\nabla)\mathbf{q}$. At $\mathbf{r} = b\mathbf{i}$, $\mathbf{q} = -U(1+a^3b^{-3})\mathbf{i} + 3a^3b^{-3}U\mathbf{i} = (2a^3b^{-3}-1)U\mathbf{i}$. Hence, $\mathbf{q}\,.\nabla \equiv q\,\partial/\partial x$, $\mathbf{f} = U(2a^3b^{-3}-1)\,\partial\mathbf{q}/\partial x$. Differentiating $\mathbf{q}$,

$$\frac{\partial\mathbf{q}}{\partial x} = -U\left(-3a^3r^{-4}\frac{\partial r}{\partial x}\right)\mathbf{i} + 3a^3U\left(r^{-5}\mathbf{r} - 5r^{-6}\frac{\partial r}{\partial x}x\mathbf{r} + xr^{-5}\frac{\partial\mathbf{r}}{\partial x}\right)$$

But $\partial r/\partial x = x/r$ and $\partial\mathbf{r}/\partial x = \mathbf{i}$ so that at $\mathbf{r} = b\mathbf{i}$

$$\frac{\partial\mathbf{q}}{\partial x} = -6Ua^3b^{-4}\mathbf{i} \qquad \text{and} \qquad \mathbf{f} = 6U^2(b^3-2a^3)a^3b^{-7}\mathbf{i}$$

The maximum value of $f = |\mathbf{f}|$ occurs when $(d/db)(b^{-4}-2a^3b^{-7}) = 0$ for which $b = (72)^{\frac{1}{3}}a$; it is a maximum because $(d^2/db^2)f$ is negative. Finally, $f_{\max} = 9(2/7)^{\frac{7}{3}}U^2/a$. □

1.3 Flux through a surface Given that $\mathscr{H} = \mathscr{H}(\mathbf{r}, t)$ is some physical (scalar or vector) quantity per unit volume which is carried by the fluid particles in their motion, the flux (rate of flow) of the quantity outward through a fixed geometrical (nonsolid) surface S is $\int_S \mathscr{H}(\mathbf{q}\,.\,\mathbf{dS})$, where $\mathbf{dS}$ is an outward normal elemental vector area of S. Choosing $\mathscr{H} = 1$, the *volume flux* through S is $\int_S \mathbf{q}\,.\,\mathbf{dS}$. With $\mathscr{H} = \rho$, the *mass flux* is $\int_S \rho\mathbf{q}\,.\,\mathbf{dS}$ and the *momentum flux* is $\int_S \rho\mathbf{q}(\mathbf{q}\,.\,\mathbf{dS})$ when $\mathscr{H} = \rho\mathbf{q}$.

1.4 Equation of continuity This states that the total fluid mass is conserved within any volume V bounded by a fixed geometrical surface S provided V does not enclose any fluid source or sink (where fluid is injected or drawn away respectively). Adding the contributions of mass change due to density variation within V to the outward flow across S we have

$$\int_V \frac{\partial\rho}{\partial t}\,d\tau + \int_S \rho\mathbf{q}\,.\,d\mathbf{S} = \int_V \left\{\frac{\partial\rho}{\partial t} + \nabla\,.\,(\rho\mathbf{q})\right\} d\tau = 0$$

where Gauss's theorem has been applied to the surface integral with $d\tau$ representing an element of volume. In the absence of sources and sinks the result is true for all subvolumes of V in which case

$$\frac{\partial\rho}{\partial t} + \nabla\,.\,(\rho\mathbf{q}) = 0 \tag{1.6}$$

This is called the *equation of continuity* or the *mass-conservation* equation. It must be satisfied at every point of a source-free region $\mathscr{R}_s$. An alternative form is found by appeal to the identities $\nabla\,.\,(\rho\mathbf{q}) = \rho\nabla\,.\,\mathbf{q} + (\mathbf{q}\,.\,\nabla)\rho$ and $\partial\rho/\partial t + (\mathbf{q}\,.\,\nabla)\rho = D\rho/Dt$ leading to

$$\frac{1}{\rho}\cdot\frac{D\rho}{Dt} + \nabla\,.\,\mathbf{q} = 0 \tag{1.7}$$

This simplifies to

$$\nabla\,.\,\mathbf{q} = \operatorname{div}\mathbf{q} = 0 \tag{1.8}$$

in the case of an incompressible liquid for which $D\rho/Dt = 0$ because here the density change of an element followed in its motion is zero. In Cartesian coordinates, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ for all time t we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.9}$$

at every point $P \in \mathscr{R}_s$. Whenever this relation is *not* satisfied, say at a set of points Q, liquid must be inserted or extracted.

Problem 1.3 Find an expression for the equation of continuity in terms of cylindrical coordinates r, θ, z defined by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Solution. Here we write the velocity $\mathbf{q} = u\mathbf{r} + v\boldsymbol{\theta} + w\mathbf{k}$ where $\mathbf{r}$, $\boldsymbol{\theta}$ are the radial and transverse *unit* vectors in the plane whose normal is parallel to OZ, the $\mathbf{k}$-axis. We recall equation 1.7 for which we evaluate

$$\nabla\,.\,\mathbf{q} = \left(\mathbf{r}\frac{\partial}{\partial r} + \frac{\boldsymbol{\theta}}{r}\frac{\partial}{\partial\theta} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(u\mathbf{r} + v\boldsymbol{\theta} + w\mathbf{k}) \tag{1.10}$$

Using suffixes to denote partial derivatives $(\partial/\partial r)(u\mathbf{r}) = u_r\mathbf{r} + u\mathbf{r}_r$ etc and since $\mathbf{r}_r = \boldsymbol{\theta}_r = \mathbf{k}_r = \mathbf{r}_z = \boldsymbol{\theta}_z = \mathbf{k}_z = \mathbf{k}_\theta = \mathbf{0}$, whilst $\mathbf{r}_\theta = \boldsymbol{\theta}$, $\boldsymbol{\theta}_\theta = -\mathbf{r}$ (proved in elementary textbooks on vectors) it follows that

$$\begin{aligned}\nabla\,.\,\mathbf{q} &= \mathbf{r}\,.\,(u_r\mathbf{r} + v_r\boldsymbol{\theta} + w_r\mathbf{k}) + (\boldsymbol{\theta}/r)\,.\,(u_\theta\mathbf{r} + v_\theta\boldsymbol{\theta} + w_\theta\mathbf{k} + u\boldsymbol{\theta} - v\mathbf{r})\\ &\qquad + \mathbf{k}\,.\,(u_z\mathbf{r} + v_z\boldsymbol{\theta} + w_z\mathbf{k})\\ &= u_r + \{(v_\theta + u)/r\} + w_z\end{aligned}$$

From equation 1.7 the equation of continuity is

$$\frac{r}{\rho}\cdot\frac{D\rho}{Dt} + ru_r + u + v_\theta + rw_z = 0$$

or, since $D\rho/Dt = \rho_t + (\mathbf{q}\,.\,\nabla)\rho = \rho_t + u\rho_r + (v/r)\rho_\theta + w\rho_z$, we have

$$r\rho_t + r(u\rho)_r + u\rho + (v\rho)_\theta + r(w\rho)_z = 0 \quad \square \tag{1.11}$$

Problem 1.4 If A is the cross-section of a stream filament show that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho q A) = 0$$

where ds is an element of arc in the direction of flow, q is the speed and ρ is the density of the fluid.

Solution. If P is the section at $s = s$ and Q the neighbouring section at $s = s + ds$, the mass of fluid which enters at P during the time δt is $A\rho q\,\delta t$ and the mass which leaves at Q is $Apq\,\delta t + (\partial/\partial s)(A\rho q\,\delta t)\delta s$. The increase in mass within PQ during the time t is therefore $-(\partial/\partial s)(A\rho q)\,\delta t\,\delta s$. Since at time t the mass of fluid within PQ is $A\rho\,\delta s$ the increase in time δt is also given by $(\partial/\partial t)(A\rho\delta s)\,\delta t = (\partial/\partial t)(A\rho)\,\delta s\,\delta t$. Hence

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho q A) = 0 \quad \square \tag{1.12}$$

Problem 1.5 Evaluate the constants a, b and c in order that the velocity $\mathbf{q} = \{(x+ar)\mathbf{i} + (y+br)\mathbf{j} + (z+cr)\mathbf{k}\}/\{r(x+r)\}$, $r = \sqrt{(x^2+y^2+z^2)}$ may satisfy the equation of continuity for a liquid.

Solution. Writing $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, the equation of continuity is $(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$. Using $\partial r/\partial x = x/r$ etc.,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{x+ar}{r(x+r)}\right\} = \frac{1+a(x/r)}{r(x+r)} + (x+ar)\left\{-\frac{x}{r^3(x+r)} - \frac{1}{r^2(x+r)}\right\}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{y+br}{r(x+r)}\right\} = \frac{1+b(y/r)}{r(x+r)} + (y+br)\left\{-\frac{y}{r^3(x+r)} - \frac{y}{r^2(x+r)^2}\right\}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left\{\frac{z+cr}{r(x+r)}\right\} = \frac{1+c(z/r)}{r(x+r)} + (z+cr)\left\{-\frac{z}{r^3(x+r)} - \frac{z}{r^2(x+r)^2}\right\}$$

Hence,

$$r^3(x+r)^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$\begin{aligned}
&= r(x+r)(r+ax+r+by+r+cz)-(x+r)\{x(x+ar)+y(y+br)+z(z+cr)\}\\
&\quad -r\{(x+r)(x+ar)+y(y+br)+z(z+cr)\}\\
&= r(x+r)(3r+ax+by+cz)-r(x+r)(r+ax+by+cz)\\
&\quad -r^2(r+ax+by+cz+x+ar)\\
&= r^2\{r(1-a)+x(1-a)-by-cz\}
\end{aligned}$$

This expression will be identically zero if and only if $a = 1$ and $b = c = 0$. □

Problem 1.6 Show that the variable ellipsoid

$$\frac{x^2}{a^2e^{-t}\cos(t+\frac{1}{4}\pi)}+\frac{y^2}{b^2e^t\sin(t+\frac{1}{4}\pi)}+\frac{z^2}{c^2\sec 2t} = 1$$

is a possible form of boundary surface of a liquid for any time t and determine the velocity components u, v and w of any particle on this boundary. Deduce that the requirements of continuity are satisfied.

Solution. Since any boundary surface with equation $F(x, y, z, t) = 0$ is made up from a time-invariant set of liquid particles we must have $DF/Dt = 0$ for all points on the boundary at any time t. Hence,

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z} = 0 \quad \text{for all } t \text{ and } (x, y, z) \in F.$$

But

$$F(x, y, z, t) \equiv \frac{x^2}{a^2}e^t\sec(t+\tfrac{1}{4}\pi)+\frac{y^2}{b^2}e^{-t}\operatorname{cosec}(t+\tfrac{1}{4}\pi)+\frac{z^2}{c^2}\cos 2t - 1 = 0$$

so that

$$\begin{aligned}
\frac{DF}{Dt} &\equiv \frac{x^2}{a^2}e^t\{\sec(t+\tfrac{1}{4}\pi)+\sec(t+\tfrac{1}{4}\pi)\tan(t+\tfrac{1}{4}\pi)\}\\
&\quad +\frac{y^2}{b^2}e^{-t}\{-\operatorname{cosec}(t+\tfrac{1}{4}\pi)-\operatorname{cosec}(t+\tfrac{1}{4}\pi)\cot(t+\tfrac{1}{4}\pi)\}\\
&\quad -\frac{2z^2}{c^2}\sin 2t+\frac{2ux}{a^2}e^t\sec(t+\tfrac{1}{4}\pi)+\frac{2vy}{b^2}e^{-t}\operatorname{cosec}(t+\tfrac{1}{4}\pi)+\frac{2wz}{c^2}\cos 2t
\end{aligned}$$

Putting $y = 0 = z$, $DF/Dt = 0$ for all x and t if

$$u = -\tfrac{1}{2}x\{1+\tan(t+\tfrac{1}{4}\pi)\} \quad (\sec(t+\tfrac{1}{4}\pi) \text{ cannot be zero})$$

Similarly, with $z = 0 = x$, $DF/Dt = 0$ for all y and t if

$$v = \tfrac{1}{2}y\{1+\cot(t+\tfrac{1}{4}\pi)\}$$

and finally, with $x = 0 = y$, we find the third velocity component

$$w = z\tan 2t.$$

For these components *on the boundary* we find that *for all t*

$$\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= -\tfrac{1}{2}\{1+\tan(t+\tfrac{1}{4}\pi)\}+\tfrac{1}{2}\{1+\cot(t+\tfrac{1}{4}\pi)\}+\tan 2t\\
&= \frac{1-\tan^2(t+\frac{1}{4}\pi)}{2\tan(t+\frac{1}{4}\pi)}+\tan 2t\\
&= \cot(2t+\tfrac{1}{2}\pi)+\tan 2t = 0
\end{aligned}$$

so that the equation of continuity is satisfied on the boundary. Moreover, the total volume within this ellipsoid is V where

$$\begin{aligned}
V^2 &= \pi^2a^2e^{-t}\cos(t+\tfrac{1}{4}\pi)b^2e^t\sin(t+\tfrac{1}{4}\pi)c^2\sec 2t\\
&= \tfrac{1}{2}\pi^2a^2b^2c^2\sin(2t+\tfrac{1}{2}\pi)\sec 2t\\
&= \tfrac{1}{2}\pi^2a^2b^2c^2 = \text{constant}
\end{aligned}$$

i.e. continuity is satisfied within. □

1.5 Rate of change of momentum The momentum $\mathbf{M}$ at time t of the particles lying within a volume V contained by a closed geometrical surface S is $\mathbf{M} = \int_V \rho\mathbf{q}\,d\tau$. Following these particles the rate of change of $\mathbf{M}$ is

$$\frac{D\mathbf{M}}{Dt} = \frac{\partial}{\partial t}\int_V \rho\mathbf{q}\,d\tau+\int_S \rho\mathbf{q}(\mathbf{q}\,.\,d\mathbf{S}) = \int_V \left\{\frac{\partial}{\partial t}(\rho\mathbf{q})+\rho\mathbf{q}(\nabla\,.\,\mathbf{q})+(\mathbf{q}\,.\,\nabla)\rho\mathbf{q}\right\}d\tau$$

by an extension of Gauss's theorem (see Table 2). Invoking the equation of continuity (1.6) the integrand on the right-hand side of this last expression is simply $\rho\, D\mathbf{q}/Dt$ so that $D\mathbf{M}/Dt = \int_V \rho(D\mathbf{q}/Dt)\,d\tau$.

To obtain an equation of motion we equate this rate of change of momentum to the total force acting upon the particles within V. If p denotes the pressure and $\mathbf{F}$ the force per unit mass we have

$$\frac{D\mathbf{M}}{Dt} = \int_V \rho\frac{D\mathbf{q}}{Dt}d\tau = -\int_S p\,d\mathbf{S}+\int_V \mathbf{F}\rho\,d\tau = \int_V (\mathbf{F}\rho-\nabla p)\,d\tau.$$

In a continuum this is true for all subvolumes of V in which case we arrive at the equation of motion

$$\rho\frac{D\mathbf{q}}{Dt} = \mathbf{F}\rho-\nabla p \tag{1.13}$$

Problem 1.7 By integrating the equation of motion find an expression for p when ρ = constant, $\mathbf{F} = \mathbf{0}$, assuming that flow is steady with $\mathbf{q} = \boldsymbol{\omega} \wedge \mathbf{r}$ where $\boldsymbol{\omega}$ is a constant vector.

Solution Since $\partial \mathbf{q}/\partial t \equiv 0$ and ρ = constant,

$$\frac{D\mathbf{q}}{Dt} = (\mathbf{q}\,.\,\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p = -\nabla\left(\frac{p}{\rho}\right)$$

Now $(\mathbf{q}\,.\,\nabla)\mathbf{q} = \nabla(\tfrac{1}{2}\mathbf{q}^2) - \mathbf{q} \wedge (\nabla \wedge \mathbf{q})$

where

$$\begin{aligned} \nabla \wedge \mathbf{q} &= \nabla \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \\ &\equiv (\mathbf{r}\,.\,\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\,.\,\nabla)\mathbf{r} - \mathbf{r}(\nabla\,.\,\boldsymbol{\omega}) + (\nabla\,.\,\mathbf{r}) \\ &= \mathbf{0} \qquad -\boldsymbol{\omega} \qquad -\mathbf{0} \qquad +3\boldsymbol{\omega} \\ &= 2\boldsymbol{\omega} \end{aligned}$$

Hence

$$\begin{aligned} -\nabla(p/\rho) &= \nabla(\tfrac{1}{2}\mathbf{q}^2) + 2\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \\ &= \nabla(\tfrac{1}{2}\mathbf{q}^2) + 2(\boldsymbol{\omega}\,.\,\mathbf{r})\boldsymbol{\omega} - 2\boldsymbol{\omega}^2\mathbf{r} \end{aligned}$$

Taking the scalar product with $\mathbf{dr}$

$$-\nabla(p/\rho)\,.\,\mathbf{dr} = -d(p/\rho) = d(\tfrac{1}{2}\mathbf{q}^2) + d(\boldsymbol{\omega}\,.\,\mathbf{r})^2 - d(\boldsymbol{\omega}^2\mathbf{r}^2)$$

Integrating

$$p/\rho = -\tfrac{1}{2}\mathbf{q}^2 + \boldsymbol{\omega}^2\mathbf{r}^2 - (\boldsymbol{\omega}\,.\,\mathbf{r})^2 + \text{constant}$$

or, since $\boldsymbol{\omega}^2\mathbf{r}^2 - (\boldsymbol{\omega}\,.\,\mathbf{r})^2 = |\boldsymbol{\omega} \wedge \mathbf{r}|^2 = |\mathbf{q}|^2 = \mathbf{q}^2$, we have, finally,

$$\frac{p}{\rho} = \tfrac{1}{2}|\boldsymbol{\omega} \wedge \mathbf{r}|^2 + \text{constant}$$ □

1.6 Motion of a fluid element Let (x, y, z) denote the Cartesian coordinates of a point P in the fluid at which point the velocity is $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where $u = u(x, y, z)$ etc. Let Q be a neighbouring point whose coordinates are $(x+\delta x, y+\delta y, z+\delta z)$. Assuming the velocity field is continuous the corresponding velocity at Q will be of the form $\mathbf{q}+\boldsymbol{\delta}\mathbf{q}$ where $\boldsymbol{\delta}\mathbf{q} = \boldsymbol{\delta}u\mathbf{i} + \boldsymbol{\delta}v\mathbf{j} + \boldsymbol{\delta}w\mathbf{k}$

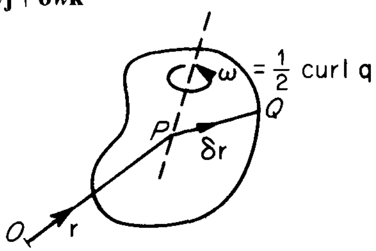


Figure 1.3

and

$$\begin{aligned} \delta u(x, y, z) &= \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\partial z \\ &= e_{xx}\delta x + (e_{xy} - \omega_z)\delta y + (e_{xz} + \omega_y)\delta z \end{aligned}$$

where

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Moreover, since $\operatorname{curl}\mathbf{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$,

$$\omega_z = \mathbf{k}\,.\,\boldsymbol{\omega}, \qquad \omega_y = \mathbf{j}\,.\,\boldsymbol{\omega} \quad \text{where } \boldsymbol{\omega} = \tfrac{1}{2}\operatorname{curl}\mathbf{q}$$

and

$$\begin{aligned} \omega_y\delta z - w_z\delta y = \mathbf{j}\,.\,\boldsymbol{\omega}\,\delta z - \mathbf{k}\,.\,\boldsymbol{\omega}\,\delta y &= (\boldsymbol{\delta}\mathbf{r} \wedge \mathbf{i})\,.\,\boldsymbol{\omega} \\ &= \mathbf{i}\,.\,(\boldsymbol{\omega} \wedge \boldsymbol{\delta}\mathbf{r}) \end{aligned}$$

Hence $\delta u = \delta u_S + \delta u_R$ where

$$\delta u_S = e_{xx}\delta x + e_{xy}\delta y + e_{xz}\delta z, \qquad \delta u_R = \mathbf{i}\,.\,(\boldsymbol{\omega} \wedge \boldsymbol{\delta}r).$$

δu_S is the contribution to δu from the *local* rate of strain (change of shape) of the element whereas δu_R is the contribution due to the *local* angular velocity $\boldsymbol{\omega} = \tfrac{1}{2}\operatorname{curl}\mathbf{q}$. In fact, if the element were *frozen* it would rotate with this angular velocity $\boldsymbol{\omega}$ which varies throughout the medium with the velocity *curl*.

The *vorticity vector* ζ is defined by $\zeta = 2\boldsymbol{\omega} = \mathit{curl}\,\mathbf{q}$. Motion is said to be *irrotational* when the vorticity ζ is zero in which case the *local* angular velocity $\boldsymbol{\omega}$ is zero.

Denoting the whole of fluid space by $\mathscr{R}$, the vortex-free space by $\mathscr{R}_v$ the remainder $\mathscr{R}_v^* = \mathscr{R} - \mathscr{R}_v$ is the space occupied by particles possessing vorticity, i.e. $\boldsymbol{\zeta} \neq \mathbf{0}$ when $P \in \mathscr{R}_v^*$ and $\boldsymbol{\zeta} = \mathbf{0}$ when $P \in \mathscr{R}_v$. The rest of physical space may either be empty or occupied by solids.

A circuit (closed curve) $\mathscr{C} \in \mathscr{R}_v$ is said to be *reducible* if it can be contracted to a point without passing out of the region $\mathscr{R}_v$. If in the contraction the circuit $\mathscr{C}$ intersects $\mathscr{R}_v^*$, or a solid, or empty space the circuit is termed *irreducible.*

A region $\mathscr{R}_v$ for which *every* circuit $\mathscr{C} \in \mathscr{R}_v$ is reducible is said to be *simply connected.* A region $\mathscr{R}_v$, in general, can be made simply connected by inserting barriers to prevent circuits having access to $\mathscr{R}_v^*$ or solids.

The necessary and sufficient condition for the irrotationality of a region $\mathscr{R}_v$ is the existence of a scalar point function φ from which the velocity can be derived by $\text{grad}\,\varphi = -\mathbf{q}$. This function φ is called the *velocity potential.*

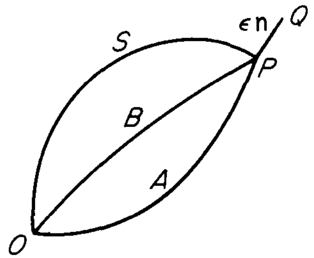


Figure 1.4

Proof. The condition is evidently sufficient for, when φ exists with $\mathbf{q} = -\text{grad}\,\varphi$, $\text{curl}\,\mathbf{q} = -\text{curl grad}\,\varphi \equiv \mathbf{0}$. To prove that the condition is also necessary, let 0 be a fixed point (Figure 1.4) and P variable in the vortex-free region $\mathscr{R}_v$ in which $\text{curl}\,\mathbf{q} = \mathbf{0}$. We assume also that $\mathscr{R}_v$ is simply connected. Join O to P by two paths OAP, OBP, both in $\mathscr{R}_v$ and construct a surface S in $\mathscr{R}_v$ having the circuit $OAPBO$ as rim. This circuit $OAPBO$ is denoted by $\mathscr{C}$ and is reducible. The *circulation Γ in $\mathscr{C}$* is defined by $\Gamma = \int_{\mathscr{C}} \mathbf{q}.\mathbf{dr}$. By Stokes's theorem applied to $\mathscr{C}$ and its spanning surface S, on which $\text{curl}\,\mathbf{q} = \mathbf{0}$, we have

$$\Gamma = \int_{\mathscr{C}} \mathbf{q}\,.\,\mathbf{dr} = \int_S \text{curl}\,\mathbf{q}\,.\,\mathbf{dS} = 0$$

Hence,

$$\left(\int_{OAP} + \int_{PBO}\right)\mathbf{q}\,.\,\mathbf{dr} = 0$$

or

$$\int_{OAP} \mathbf{q}\,.\,\mathbf{dr} = \int_{OBP} \mathbf{q}\,.\,\mathbf{dr} = -\varphi(\mathbf{O}P)$$

because, through the independence of the paths OAP, OBP, the integrals are scalar functions of the point P only. If now we choose a second point Q close to P with $\mathbf{PQ} = \varepsilon\boldsymbol{\eta}$ ($|\boldsymbol{\eta}| = 1$, $\varepsilon \ll 1$) provided $PQ \in \mathscr{R}_v$

$$\int_{OAQ} \mathbf{q}\,.\,\mathbf{dr} - \int_{OAP} \mathbf{q}\,.\,\mathbf{dr} = \int_{PQ} \mathbf{q}\,.\,\mathbf{dr} = -\varphi(\mathbf{OQ}) + \varphi(\mathbf{O}P)$$
$$= -\varepsilon\boldsymbol{\eta}.\nabla\varphi + O(\varepsilon^2)$$

Denoting the fluid velocity at P by $\mathbf{q}_P$, on PQ we can write $\mathbf{q} = \mathbf{q}_P + \mathbf{O}(\varepsilon)$ so that

$$\int_{PQ} \mathbf{q}\,.\,\mathbf{dr} = \int_{PQ} \{\mathbf{q}_P + \mathbf{O}(\varepsilon)\}\,.\,\mathbf{dr} = \varepsilon\boldsymbol{\eta}.\mathbf{q}_P + O(\varepsilon^2)$$

Equating the two evaluations of $\int_{PQ} \mathbf{q}\,.\,\mathbf{dr}$,

$$\varepsilon\boldsymbol{\eta}.\mathbf{q}_P = -\varepsilon\boldsymbol{\eta}.\nabla\varphi + O(\varepsilon^2)$$

Allowing $\varepsilon \to 0$ with $\boldsymbol{\eta}$ arbitrary, we find that

$$\mathbf{q}_P = -\nabla\varphi$$

i.e. we have shown that the condition is *necessary*. For the given reducible circuit $\mathscr{C}$ the circulation Γ is zero. In a simply connected region $\mathscr{R}_v$ not only does φ exist, it is also single valued and the ensuing fluid motion is termed *acyclic*.

To discuss vortex fields we first define a *vortex line* by the property that its tangent at every point is parallel to the vorticity vector ζ at the same point. It follows that every particle on this line is instantaneously rotating about an axis coincident with the tangent. The equation of this line will be of the same form as equation 1.1 with $u(\mathbf{r}, t)$ etc. replaced by the Cartesian components of ζ.

A *vortex tube* of finite cross-section with boundary C at some station is constructed by drawing a vortex line through each and every point of C (if they exist). If the area enclosed by C has negligible dimensions, the tube becomes a *vortex filament*.

We can show that vortices cannot originate or terminate anywhere other than on fluid boundaries or else they form closed circuits. Applying

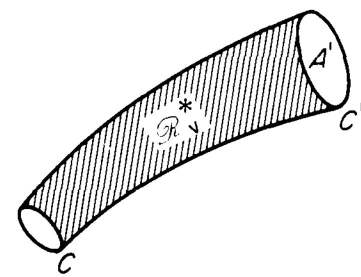


Figure 1.5

Stokes's theorem to the vortex space $\mathscr{R}_v^*$ within the vortex tube (Figure 1.5) enclosed by the sections whose boundary curves are C and C'

$$\int_C \mathbf{q}\,.\,\mathbf{dr} = \int_S \text{curl}\,\mathbf{q}\,.\,\mathbf{dS} = \int_S \zeta\,.\,\mathbf{dS}$$

where S spans C, i.e. S is the area A' enclosed by C' plus the vortex surface between C and C'. However, by construction, $\zeta\,.\,\mathbf{dS} = 0$ on the vortex (curved) surface so that the circulation Γ^* around C satisfies

$$\Gamma^* = \int_C \mathbf{q}\,.\,\mathbf{dr} = \int_{A'} \zeta\,.\,\mathbf{dS}$$

Since the section C' is arbitrary $\int_{A'} \boldsymbol{\zeta} \cdot d\mathbf{S}$ is constant along the vortex tube and is referred to as the *strength of the tube*. This result implies that ζ cannot vanish in the interior of the fluid space $\mathscr{R}$.

Any circuit $\mathscr{C}$ (Figure 1.6a) which encircles a vortex ring or similarly

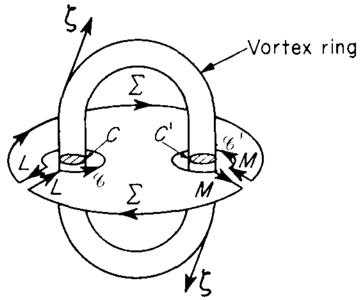
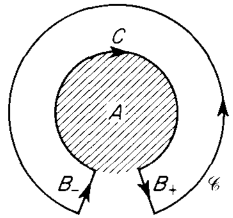


Figure 1.6a



Plan

Figure 1.6b

shaped obstacle is irreducible since the circuit cannot be contracted beyond (inside) C without moving outside $\mathscr{R}_v$. This region $\mathscr{R}_v$ can be made simply connected by the insertion of a barrier B with two sides B_+, B_- bridging C with $\mathscr{C}$ as shown in Figure 1.6b representing a plan through $\mathscr{C}$. The shaded area A enclosed by C is the intersection of the vortex ring with the plane. Consider the circuits $\mathscr{L}$ made up as follows

$$\mathscr{L} \equiv \mathscr{C}\uparrow + B_-\uparrow + C\downarrow + B_+\downarrow$$

$\mathscr{L} \in \mathscr{R}_v$ and does not cross the barrier B, therefore the circulation in $\mathscr{L}$ is zero. Consequently,

$$\int_{\mathscr{L}} \mathbf{q} \cdot d\mathbf{r} = 0 = \left\{ \int_{\mathscr{C}\uparrow} + \int_{B_-\uparrow} + \int_{C\downarrow} + \int_{B_+\downarrow} \right\} \mathbf{q} \cdot d\mathbf{r}$$

In the limit when B_+ coincides with B_- the sum of the contributions of these bridge passages to $\int \mathbf{q} \cdot d\mathbf{r}$ is zero. Hence the circulation in $\mathscr{C}$ is

$$\Gamma = \int_{\mathscr{C}\uparrow} \mathbf{q} \cdot d\mathbf{r} = \int_{C\uparrow} \mathbf{q} \cdot d\mathbf{r} = \Gamma^* \tag{1.14}$$

where Γ^* is the vortex strength. Alternatively, writing $\mathbf{q} = -\text{grad}\,\varphi$ $\mathbf{q} \cdot d\mathbf{r} = -\text{grad}\,\varphi \cdot d\mathbf{r} = -d\varphi$, i.e. $\Gamma = \int -d\varphi = -[\varphi]_{B_+}^{B_-} = \varphi_+ - \varphi_-$. In crossing the barrier, φ increases by Γ^* in which case φ is *not single valued* for an irreducible circuit. The motion is termed *cyclic*.

It should be noticed that we obey the right-hand screw rule for the sense of integration for the line integral over C in relation to the sense of direction of $\boldsymbol{\zeta}$. Hence, for the circuit C' on the other arm of the vortex ring

we have

$$\int_{C'\uparrow} \mathbf{q} \cdot d\mathbf{r} = -\int_{C'\downarrow} \mathbf{q} \cdot d\mathbf{r} = -\Gamma^*$$

consequently the circulation Γ' in $\mathscr{C}'$ is

$$\Gamma' = \int_{\mathscr{C}'\uparrow} \mathbf{q} \cdot d\mathbf{r} = \int_{C'\uparrow} \mathbf{q} \cdot d\mathbf{r} = -\Gamma^*$$

We note also that the sum of the circulations for the circuits $\mathscr{C}$ and $\mathscr{C}'$ is $\Gamma + \Gamma' = \Gamma^* - \Gamma^* = 0$.

Suppose Σ is a circuit (Figure 1.6a) which lies *outside* both arms of the vortex ring without threading either arm. Using bridges L and M it is seen that

$$\int_\Sigma \mathbf{q} \cdot d\mathbf{r} = \left\{ \int_{\mathscr{C}} + \int_{\mathscr{C}'} \right\} \mathbf{q} \cdot d\mathbf{r} = 0$$

This is precisely what should have been expected since Σ is a *reducible* circuit whose circulation is therefore zero.

We have stated that the necessary and sufficient condition for irrotational motion is the existence of a scalar point function φ such that grad $\varphi = -\mathbf{q}$. When the fluid is incompressible the equation of continuity for a source-free region is $\text{div}\,\mathbf{q} = 0$ so that φ, when it exists, satisfies Laplace's equation,

$$\text{div grad}\,\varphi \equiv \nabla^2\varphi = 0 \tag{1.15}$$

If on the other hand $\boldsymbol{\zeta} =$ constant then $\text{curl}\,\mathbf{q} =$ constant $= 2\boldsymbol{\omega}$ (say). Writing $\mathbf{q} = \boldsymbol{\omega} \wedge \mathbf{r} + \mathbf{q}_0$, since $\text{curl}\,\boldsymbol{\omega} \wedge \mathbf{r} = 2\boldsymbol{\omega}$, we find that $\text{curl}\,\mathbf{q}_0 = 0$, therefore $\mathbf{q} = \boldsymbol{\omega} \wedge \mathbf{r} - \text{grad}\,\varphi_0$ where φ_0 is *any* scalar point function.

Problem 1.8 Show that $\varphi = xf(r)$ is a possible form for the velocity potential of an incompressible liquid motion. Given that the liquid speed $q \to 0$ as $r \to \infty$, deduce that the surfaces of constant speed are $(r^2 + 3x^2)r^{-8} =$ constant.

Solution. When $\varphi = xf(r)$, $\mathbf{q} = -\text{grad}\,\varphi = -f(r)\,\text{grad}\,x - x\,\text{grad}\,f(r)$. Hence, with primes denoting differentiation with respect to r,

$$\mathbf{q} = -f\mathbf{i} - xf'\mathbf{r}/r \qquad f \equiv f(r) \tag{1.16}$$

and

$$\begin{aligned}
\text{div}\,\mathbf{q} &= -\text{div}\,f\mathbf{i} - (xf'/r)\text{div}\,\mathbf{r} - \mathbf{r} \cdot \text{grad}(xf'/r) \\
&= -\frac{\partial f}{\partial x} - 3x\frac{f'}{r} - \mathbf{r} \cdot \left(\frac{f'}{r}\mathbf{i} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\frac{\mathbf{r}}{r} \right) \\
&= -f'\frac{x}{r} - 3\frac{f'}{r}x - x\frac{f'}{r} - xr\left(\frac{f''}{r} - \frac{f'}{r^2}\right) \\
&= -x\left(f'' + 4\frac{f'}{r}\right)
\end{aligned}$$

For an incompressible liquid div $\mathbf{q} = 0$, consequently,

$$f'' + \frac{4f'}{r} = 0, \quad \text{or} \quad \frac{f''}{f'} + \frac{4}{r} = 0$$

Integrating, $f' = Ar^{-4}$ where A = constant. Integrating again $f = -\frac{1}{3}Ar^{-3} + B$ where B = constant. Hence by equation 1.16,

$$\mathbf{q} = \left(\frac{A}{3r^3} - B\right)\mathbf{i} - \frac{Ax\mathbf{r}}{r^5}$$

When $r \to \infty$, $\mathbf{q} \to -B\mathbf{i}$ so that $B = 0$ leaving

$$\mathbf{q} = \frac{A}{3r^3}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)$$

and

$$q^2 = \mathbf{q}\cdot\mathbf{q} = \frac{A^2}{9r^6}\left(1 - \frac{6rx\mathbf{r}\cdot\mathbf{i}}{r^2} + \frac{9x^2r^2}{r^4}\right).$$

Hence q^2 = constant when $r^{-8}(r^2 + 3x^2)$ = constant. □

Problem 1.9 Examine the liquid motions for which φ the velocity potential, equals:

$$\text{(i)}\ \frac{m}{r}, \qquad \text{(ii)}\ \frac{m_1}{|\mathbf{r}-\mathbf{r}_1|} + \frac{m_2}{|\mathbf{r}-\mathbf{r}_2|}, \qquad \text{(iii)}\ \mu\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

where m, m_1, m_2, μ are constant scalars and $\mathbf{r}_1$ and $\mathbf{r}_2$ are constant vectors.

Solution. (i) $\varphi = m/r$, $\mathbf{q} = -\text{grad}(m/r) = m\mathbf{r}/r^3$, div $\mathbf{q} = m(3r^2 - 3\mathbf{r}\cdot\mathbf{r})/r^5 = 0$. Motion is irrotational everywhere *except* at $\mathbf{r} = \mathbf{0}$ and the equation of continuity is satisfied everywhere except at $\mathbf{r} = \mathbf{0}$. Velocity is radial from $\mathbf{r} = \mathbf{0}$ with magnitude $q = |\mathbf{q}| \to 0$ as $r \to \infty$ and $q \to \infty$ as $r \to 0$. Moreover, the flux of volume flow across the sphere $|\mathbf{r}| = a$ equals

$$Q = \int_{r=a} \mathbf{q}\cdot d\mathbf{S} = ma^{-3}\int \mathbf{r}\cdot d\mathbf{S} \qquad d\mathbf{S} = (\mathbf{r}/a)dS$$

$$= ma^{-4}\int \mathbf{r}\cdot\mathbf{r}\,dS = ma^{-2}\int dS = 4\pi m.$$

The flux is independent of the radius a; this fact follows directly from the equation of continuity for

$$\int_S \mathbf{q}\cdot d\mathbf{S} = \int_V \text{div}\,\mathbf{q}\,d\tau \begin{cases} = 0 & \text{if } V \not\subset \mathbf{r} = \mathbf{0} \quad \text{for then div}\,\mathbf{q} = 0 \text{ for all } V \\ = 4\pi m & \text{if } V \subset \mathbf{r} = \mathbf{0} \quad \text{(just proved)} \end{cases}$$

Thus $\varphi = m/r$ corresponds to the liquid motion induced by a *source*

of output $4\pi m$ at $\mathbf{r} = 0$; this source is said to be of *strength m*. If m is negative the source becomes a sink of strength $-m$.

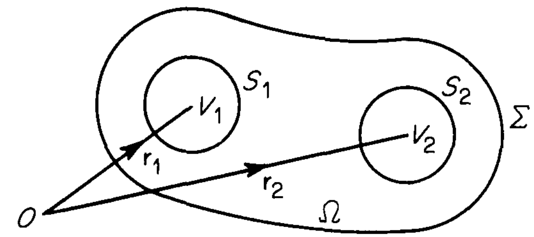


Figure 1.7

(ii) In Figure 1.7 let S_1 and S_2 be two nonintersecting spheres centred at $\mathbf{r} = \mathbf{r}_1$ and $\mathbf{r} = \mathbf{r}_2$ respectively and enclosing volumes V_1 and V_2 respectively. Σ is a surface enclosing a volume Ω containing both V_1 and V_2. We have

$$\mathbf{q} = -\text{grad}\varphi = \mathbf{q}_1 + \mathbf{q}_2$$

where

$$\mathbf{q}_1 = m_1(\mathbf{r}_1 - \mathbf{r}_1)/|\mathbf{r} - \mathbf{r}_1|^3, \quad \mathbf{q}_2 = m_2(\mathbf{r} - \mathbf{r}_2)|\mathbf{r} - \mathbf{r}_2|^3$$

Using the results proved in (i), div $\mathbf{q}_1 = 0$ except at $\mathbf{r} = \mathbf{r}_1$, and div $\mathbf{q}_2 = 0$ except at $\mathbf{r} = \mathbf{r}_2$; hence,

$$\int_{V_1} \text{div}\,\mathbf{q}\,d\tau = 4\pi m_1 \quad \text{because } V_1 \subset \mathbf{r} = \mathbf{r}_1, \text{ but } V_1 \not\subset \mathbf{r} = \mathbf{r}_2$$

$$\int_{V_2} \text{div}\,\mathbf{q}\,d\tau = 4\pi m_2 \quad \text{because } V_2 \subset \mathbf{r} = \mathbf{r}_2, \text{ but } V_2 \not\subset \mathbf{r} = \mathbf{r}_2$$

$$\int_{\Omega - V_1 - V_2} \text{div}\,\mathbf{q}\,d\tau = 0 \quad \text{since } \Omega - V_1 - V_2 \not\subset \mathbf{r} = \mathbf{r}_1 \text{ or } \mathbf{r} = \mathbf{r}_2$$

It follows that this flow corresponds to a source of output $4\pi m_1$ at $\mathbf{r}_1$ and a source of output $4\pi m_2$ at $\mathbf{r}_2$.

(iii) When $\epsilon = |\boldsymbol{\epsilon}|$ is very small, neglecting ϵ^2, we have for constant m

$$\frac{m}{|\mathbf{r}+\boldsymbol{\epsilon}|} - \frac{m}{r} \approx \frac{m}{(\mathbf{r}^2 + 2\mathbf{r}\cdot\boldsymbol{\epsilon})^{\frac{1}{2}}} - \frac{m}{r} \approx -\frac{m\boldsymbol{\epsilon}\cdot\mathbf{r}}{r^3} = m\boldsymbol{\epsilon}\cdot\nabla\left(\frac{1}{\mathbf{r}}\right)$$

Now put $\boldsymbol{\epsilon} = \epsilon\mathbf{i}$ and $m = \mu/\epsilon$, then

$$\mu\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \lim_{\epsilon \to 0}\left\{\frac{(\mu/\epsilon)}{|\mathbf{r}+\epsilon\mathbf{i}|} - \frac{(\mu/\epsilon)}{r}\right\}$$

The right-hand side is the limit as $\epsilon \to 0$ of a source of strength μ/ϵ at $\mathbf{r} = -\epsilon\mathbf{i}$ together with a sink of equal strength μ/ϵ at $\mathbf{r} = \mathbf{0}$. This limit is termed a *doublet* of strength μ with its axis parallel to $\mathbf{i}$.

1.7 Pressure equation

From equation 1.13 the equation of motion is

$$\frac{D\mathbf{q}}{Dt} \equiv \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

or using

$$(\mathbf{q}\cdot\nabla)\mathbf{q} \equiv \nabla(\tfrac{1}{2}\mathbf{q}^2) - \mathbf{q} \wedge (\nabla \wedge \mathbf{q})$$

we have

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \wedge \boldsymbol{\zeta} = \mathbf{F} - \nabla(\tfrac{1}{2}\mathbf{q}^2) - \frac{1}{\rho}\nabla p, \qquad \boldsymbol{\zeta} = \nabla \wedge \mathbf{q}$$

At constant entropy, p is a function of ρ only so that $\nabla p/\rho = \nabla \int dp/\rho$. If also $\mathbf{F}$ is derivable from a potential Ω, $\mathbf{F} = -\operatorname{grad}\Omega$, and

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \wedge \boldsymbol{\zeta} = -\nabla\chi \quad \text{where } \chi = \int dp/\rho + \tfrac{1}{2}q^2 + \Omega \tag{1.17}$$

In steady motion $\mathbf{q} \wedge \boldsymbol{\zeta} = \nabla\chi$ in which case the surfaces χ = constant contain both streamlines and vortex lines. When flow is irrotational $\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \mathbf{0}$ so that $\mathbf{q} = -\operatorname{grad}\varphi$, $\partial/\partial t(-\nabla\varphi) = -\nabla\chi$ or $\nabla(-\partial\varphi/\partial t + \chi) = 0$. Integrating

$$-\frac{\partial\varphi}{\partial t} + \chi \equiv -\frac{\partial\varphi}{\partial t} + \int\frac{dp}{\rho} + \tfrac{1}{2}q^2 + \Omega = A(t), \quad A(t) \text{ is arbitrary} \tag{1.18}$$

For steady motion $\partial\varphi/\partial t = 0$, $A(t)$ = constant leading to *Bernoulli's* equation

$$\int dp/\rho + \tfrac{1}{2}q^2 + \Omega = \text{constant} \tag{1.19}$$

Problem 1.10 The velocity $\mathbf{q}$ at any point is expressed by $\mathbf{q} = -\nabla\varphi + \lambda\nabla\mu$ where φ, λ and μ are independent scalar point functions of position. Show that vortex lines are at the intersections of surfaces λ = constant with μ = constant. From the equation of motion deduce that when

$$H = \partial\varphi/\partial t - \lambda\partial\mu/\partial t - \chi, \text{ and } \chi = \int dp/\rho + \tfrac{1}{2}q^2,$$

then

$$\nabla H = \nabla\mu(D\lambda/Dt) - \nabla\lambda(D\mu/Dt)$$

and prove that H is constant along a vortex line. Show also that $\boldsymbol{\zeta}\cdot\nabla(D\lambda/Dt) = 0 = \boldsymbol{\zeta}\cdot\nabla(D\mu/Dt)$ and deduce that $D\lambda/Dt$, $D\mu/Dt$ and ∇H are all identically zero. Interpret this result.

Solution. From $\mathbf{q} = -\nabla\varphi + \lambda\nabla\mu$, $\boldsymbol{\zeta} = \nabla \wedge \mathbf{q} = \nabla \wedge (\lambda\nabla\mu) = \lambda\nabla\wedge\nabla\mu + \nabla\lambda\wedge\nabla\mu$ or, since $\nabla\wedge\nabla\mu$ is identically zero, $\boldsymbol{\zeta} = \nabla\lambda\wedge\nabla\mu$. This means that vortex lines lie at the intersections of the surface λ = constant with μ = constant. Using the equation of motion from Section 1.7 in the form $\partial\mathbf{q}/\partial t - \mathbf{q}\wedge\boldsymbol{\zeta} = -\nabla\chi$, since

$$\frac{\partial\mathbf{q}}{\partial t} = -\nabla\left(\frac{\partial\varphi}{\partial t}\right) + \left(\frac{\partial\lambda}{\partial t}\right)\nabla\mu + \lambda\nabla\left(\frac{\partial\mu}{\partial t}\right)$$

and

$$\mathbf{q}\wedge\boldsymbol{\zeta} = \mathbf{q}\wedge(\nabla\lambda\wedge\nabla\mu) = (\mathbf{q}\cdot\nabla\mu)\nabla\lambda - (\mathbf{q}\cdot\nabla\lambda)\nabla\mu$$

we have

$$\nabla\chi = \nabla\left(\frac{\partial\varphi}{\partial t} - \lambda\frac{\partial\mu}{\partial t}\right) - \left(\frac{\partial\lambda}{\partial t} + \mathbf{q}\cdot\nabla\lambda\right)\nabla\mu + \left(\frac{\partial\mu}{\partial t} + \mathbf{q}\cdot\nabla\mu\right)\nabla\lambda$$

or

$$\nabla\left(\frac{\partial\varphi}{\partial t} - \lambda\frac{\partial\mu}{\partial t} - \chi\right) \equiv \nabla H = \frac{D\lambda}{Dt}\nabla\mu - \frac{D\mu}{Dt}\nabla\lambda \tag{1.20}$$

Now, since $\boldsymbol{\zeta} = \nabla\lambda\wedge\nabla\mu$, we have $\boldsymbol{\zeta}\cdot\nabla H = 0$ so that for each instant of time H is constant along a vortex line. Also, from the identity $\nabla\wedge\nabla H \equiv 0$,

$$\nabla\wedge\left(\frac{D\lambda}{Dt}\nabla\mu\right) = \nabla\wedge\left(\frac{D\mu}{Dt}\nabla\lambda\right)$$

or

$$\nabla\left(\frac{D\lambda}{Dt}\right)\wedge\nabla\mu = \nabla\left(\frac{D\mu}{Dt}\right)\wedge\nabla\lambda$$

Multiplying scalarly in turn by $\nabla\mu$ and $\nabla\lambda$ we have

$$\boldsymbol{\zeta}\cdot\nabla\left(\frac{D\mu}{Dt}\right) = 0 = \boldsymbol{\zeta}\cdot\nabla\left(\frac{D\lambda}{Dt}\right) \tag{1.21}$$

We have already shown that a vortex line lies on a surface $\mu = A$ = constant but, by equation 1.21, this line is contained at the same time in the surface $\mu + D\mu/Dt = A$ which is possible only if $D\mu/Dt = 0$; similarly we have $D\lambda/Dt = 0$ and equation 1.20 reduces to $\nabla H = 0$. Thus H is a function of time t only and any surface μ = constant or λ = constant contains the same fluid particles, which leads to the fact that any vortex line also contains the same fluid particles as it moves throughout the fluid. □

Problem 1.11 An open-topped tank of height c with base of length a and width b is quarter filled with water. The tank is made to rotate with uniform angular velocity ω about the vertical edge of length c. To ensure that there is no spillage show that ω must not exceed $\frac{3}{2}\{cg/(a^2+b^2)\}^{\frac{1}{2}}$.

Solution. Take axes OX and OY along the base edges of lengths a and b respectively with OZ along the vertical axis of rotation. Assuming that in the steady motion any liquid particle at (x, y, z) at some instant describes a horizontal circle with centre on OZ and radius $R = \sqrt{(x^2+y^2)}$, the liquid acceleration $D\mathbf{q}/Dt = -\omega^2\mathbf{R}$ where $\mathbf{R} = x\mathbf{i}+y\mathbf{j}$. The body force $\mathbf{F} = -g\mathbf{k}$ so that the equation of motion becomes

$$-\omega^2\mathbf{R} = g\mathbf{k} - \nabla p/\rho$$

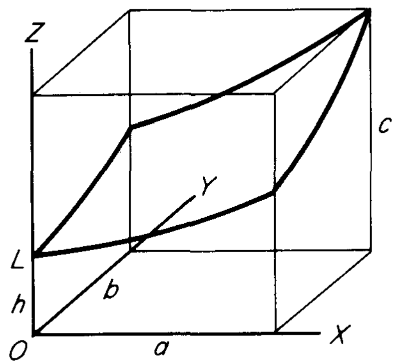


Figure 1.8

Multiplying throughout scalarly by $d\mathbf{r} = dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$ and using $\nabla p . d\mathbf{r} = dp$ we have

$$-\omega^2(x\,dx+y\,dy) = -g\,dz - dp/\rho$$

Integrating,

$$p/\rho = \tfrac{1}{2}\omega^2(x^2+y^2) - gz + A$$

where A = constant because motion is steady. On the liquid surface p = the constant atmospheric pressure, in which case its equation is

$$\tfrac{1}{2}\omega^2(x^2+y^2) - gz = \text{constant} = B \quad \text{(say)}$$

The minimum value of $z(=h)$ on this surface will occur on the axis at L where $x = y = 0$ and the maximum value of z $(=H)$ will be reached when x^2+y^2 is maximum, i.e. on the vertical edge $x = a$, $y = b$. Hence, the constant $B = -\text{gh} = \tfrac{1}{2}\omega^2(a^2+b^2) - gH$. We can evaluate H and h in terms of ω using the condition that volume is conserved in the absence of spillage.

But

$$\text{volume } V = \tfrac{1}{4}abc = \iint_A z\,dx\,dy$$

where A is the base of the tank, $z = h+\lambda(x^2+y^2)$, $\lambda = \omega^2/2g$, i.e.

$$V = \int_0^a dx \int_0^b \{h+\lambda(x^2+y^2)\}dy = \int_0^a \{(h+\lambda x^2)b + \tfrac{1}{3}\lambda b^3\}dx$$
$$= ab\{h + \tfrac{1}{3}\lambda(a^2+b^2)\}$$

Since

$$V = \tfrac{1}{4}abc \text{ and } h = H - \lambda(a^2+b^2),$$
$$H = \tfrac{1}{4}c + \tfrac{2}{3}\lambda(a^2+b^2)$$

To prevent spillage $H \leqslant c$, hence

$$\lambda = \omega^2/2g \leqslant 9c/\{8(a^2+b^2)\}$$

or

$$\omega \leqslant \tfrac{3}{2}\{cg/(a^2+b^2)\}^{\frac{1}{2}}$$

□

Problem 1.12 A shell formed by rotating the curve $ay = x^2$ about a vertical axis OY is filled with a large quantity of water. A *small* horizontal circular hole of radius a/n is opened at the vertex and the water allowed to escape. Assuming that (i) the flow is steady, (ii) the ensuing jet becomes cylindrical at a *small* depth c below the hole, (iii) this cylinder has a vertical axis and cross-sectional area $\alpha(<1)$ times the area of the hole, show that the time taken for the depth of water to fall from h to $\tfrac{1}{2}h$ when h is very large is approximately

$$\frac{n^2}{3\alpha a}\left(\frac{2h^3}{g}\right)^{\frac{1}{2}}\{(1+\beta)^{\frac{1}{2}} - (\tfrac{1}{2}+\beta)^{\frac{1}{2}}\}\{\tfrac{3}{2} - \beta + (1+\beta)^{\frac{1}{2}}(\tfrac{1}{2}+\beta)^{\frac{1}{2}}\}, \qquad \beta = c/h.$$

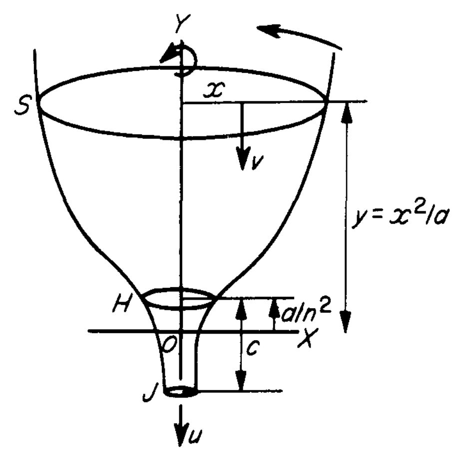


Figure 1.9

Solution. Since the radius of the hole H is a/n it is cut at a height $y = (a/n)^2/a = a/n^2$ above the vertex. The ensuing jet has area $\alpha\pi a^2/n^2$ at level

J, a depth c below H. When the upper surface S of water is at a height $y - a/n^2$ above the hole, assuming y is great enough, the surface will remain plane and fall steadily with vertical speed $v = -dy/dt$. At the same instant at J, where the exist jet becomes cylindrical, the vertical speed is u. Applying the equation of continuity at levels S and J we have

$$\pi x^2 v = \alpha\pi a^2 u/n^2$$

The pressures at levels S and J are equal to the atmospheric pressures at those levels and we shall assume they are the same, i.e. the air density will be neglected in comparison with the liquid density ρ. Hence Bernoulli's equation gives

$$\Pi/\rho + \tfrac{1}{2}v^2 + gy = \Pi/\rho + \tfrac{1}{2}u^2 - g(c - a/n^2)$$

where Π is the common atmospheric pressure. Eliminating u we have

$$v^2\left(\frac{n^4x^4}{\alpha^2a^4} - 1\right) = 2g\left(y + c - \frac{a}{n^2}\right)$$

or

$$-\frac{dy}{dt}\left(\frac{n^4y^2}{\alpha^2a^2} - 1\right)^{\frac{1}{2}} = \left(2g(y + c - \frac{a}{n^2})\right)^{\frac{1}{2}}$$

Hence the time from $y = h + a/n^2$ to $y = \frac{1}{2}h + a/n^2$ is T where

$$T = -\int_{h+a/n^2}^{\frac{1}{2}h+a/n^2}\left\{\frac{n^4y^2 - \alpha^2a^2}{2g\alpha^2a^2(y + c - a/n^2)}\right\}^{\frac{1}{2}} dy$$

$$= \int_{\frac{1}{2}h}^{h}\left\{\frac{(n^2z + a)^2 - \alpha^2a^2}{2g\alpha^2a^2(z + c)}\right\}^{\frac{1}{2}} dz \quad \text{where } z = y - a/n^2$$

$$= \frac{n^2}{\sqrt{(2g\alpha^2a^2)}}\int_{\frac{1}{2}h}^{h}\frac{z}{\sqrt{(z + c)}}\,dz + O(n)$$

Using $z/\sqrt{(z+c)} = \sqrt{(z+c)} - c/\sqrt{(z+c)}$ and neglecting the term $O(n)$ we have, approximately,

$$T = \frac{n^2}{\alpha a\sqrt{(2g)}}\left[\tfrac{2}{3}(z + c)^{\frac{3}{2}} - 2c(z + c)^{\frac{1}{2}}\right]_{\frac{1}{2}h}^{h}$$

$$= \frac{2n^2}{3\alpha a\sqrt{(2g)}}\{(h + c)^{\frac{3}{2}} - (\tfrac{1}{2}h + c)^{\frac{3}{2}} - 3c(h + c)^{\frac{1}{2}} + 6c(\tfrac{1}{2}h + c)^{\frac{1}{2}}\}$$

$$= \frac{2n^2}{3\alpha a\sqrt{(2g)}}\{(h + c)^{\frac{1}{2}} - (\tfrac{1}{2}h + c)^{\frac{1}{2}}\}\{h + c + (h + c)^{\frac{1}{2}}(\tfrac{1}{2}h + c)^{\frac{1}{2}} + \tfrac{1}{2}h + c - 3c\}$$

$$= \frac{n^2}{3\alpha a}\left(\frac{2h^3}{g}\right)^{\frac{1}{2}}\{(1 + \beta)^{\frac{1}{2}} - (\tfrac{1}{2} + \beta)^{\frac{1}{2}}\}\{\tfrac{3}{2} - \beta + (1 + \beta)^{\frac{1}{2}}(\tfrac{1}{2} + \beta)^{\frac{1}{2}}\}$$ □

Problem 1.13 A liquid of constant density ρ flows steadily with speed U under constant pressure P through a cylindrical tube with uniform circular section of area A. A semi-infinite axisymmetric body is placed in the cylinder with its axis along the axis of the tube. Given that the area of the section of the body tends asymptotically to a show that the force on the body is $a\{P - \frac{1}{2}\rho U^2 a/(A - a)\}$.

Solution. From the equation of continuity the liquid speed downstream will tend to a value V where $AU = (A - a)V$. Moreover, by Bernoulli's equation the pressure downstream tends to $P' = P + \frac{1}{2}\rho(U^2 - V^2)$. The force on the body is, by symmetry, parallel to the common axis. If F denotes this force in the downstream direction the reaction force on the liquid is $-F$ so that the total force on the liquid is $PA - P'(A - a) - F$ where the first two terms are the contributions from the upstream and downstream pressures respectively. Equating this force to the momentum flux we have

$$PA - P'(A - a) - F = \rho V^2(A - a) - \rho U^2 A$$

Using $V = AU/(A - a)$ and $P' = P + \frac{1}{2}(U^2 - V^2) = P + \frac{1}{2}U^2[1 - \{A/(A - a)\}^2]$ we have,

$$F = PA - P(A - a) - \tfrac{1}{2}\rho U^2\left\{A - a - \frac{A^2}{A - a} + \frac{2A^2}{A - a} - 2A\right\}$$

$$= Pa - \tfrac{1}{2}\rho U^2 a^2/(A - a)$$ □

1.8 One-dimensional gas dynamics We assume that a gas moves steadily in an axisymmetric tube with OX as axis and $A = A(x)$ is the normal circular cross-sectional area of the tube at any station x. Using primes to denote differentiation with respect to x we also assume that $A'(0) = 0$, $A'(x)/x > 0$ for all $|x| > 0$ with $A'(x)$ everywhere small. In this case we may neglect any component of velocity perpendicular to OX compared with the parallel component $u = u(x)$ so that $\mathbf{q} = u(x)\mathbf{i}$. Henceforth we shall refer to such a tube as a *Laval* tube. To solve any problem we need the following equations:

(i) *Equation of continuity.* From equation 1.12 when flow is steady we have

$$\frac{\partial}{\partial x}(\rho u A) = 0 \quad \text{or mass flux } \rho u A = \text{constant} = Q \qquad (1.22)$$

(ii) *Bernoulli's equation.* For steady flow with zero body force, equation 1.19 becomes

$$\int dp/\rho + \tfrac{1}{2}u^2 = \text{constant} \quad \text{or } dp + \rho u\,du = 0 \qquad (1.23)$$

(iii) *Thermodynamic equations.* For unit mass of gas

$$p = R\rho T, \quad R = \text{gas constant} = c_p - c_V \tag{1.24}$$

where c_p, c_V are the specific heat capacities at constant pressure and constant volume respectively. The acoustic speed is a where

$$a = \sqrt{(dp/d\rho)} \tag{1.25}$$

If entropy is constant along a line of flow then p and ρ are related by the adiabatic law

$$p = k\rho^\gamma \quad (k = \text{constant}, \gamma = c_p/c_V) \tag{1.26}$$

Such a flow is said to be *isentropic.*

Problem 1.14 Find an expression for the local acoustic speed in terms of the fluid speed.

Solution. When $p = k\rho^\gamma$,

$$a^2 = dp/d\rho = k\gamma\rho^{\gamma-1} = \gamma p/\rho$$

Also, by equation 1.23,

$$\int dp/\rho + \tfrac{1}{2}u^2 = \text{constant} = \int k\gamma\rho^{\gamma-2}\,d\rho + \tfrac{1}{2}u^2$$
$$= k\gamma\rho^{\gamma-1}/(\gamma-1) + \tfrac{1}{2}u^2$$

i.e. $\quad a^2/(\gamma-1) + \tfrac{1}{2}u^2 = \text{constant} = A$

If $a = a_0$ when $u = 0$, $A = a_0^2/(\gamma-1)$ and

$$a^2 = a_0^2 - \tfrac{1}{2}(\gamma-1)u^2 \quad \square \tag{1.27}$$

Problem 1.15 Prove that if a gas moves unsteadily in a Laval tube (described in Section 1.8) then $\partial^2\rho/\partial t^2 = (\partial^2/\partial x^2)(p + \rho u^2)$.

Solution. In this tube we have $u = u(x, t)$, $p = p(x, t)$ and $\rho = \rho(x, t)$. With suffixes denoting partial differentiation, the equations of motion and continuity are

$$u_t + uu_x = -p_x/\rho \tag{1.28}$$

$$\rho_t + (\rho u)_x = 0 \tag{1.29}$$

Adding ρ times equation 1.28 to u times equation 1.29 we have

$$(\rho u)_t + (\rho u^2 + p)_x = 0 \tag{1.30}$$

Differentiating equation 1.29 partially with respect to t and equating with the result of differentiating equation 1.30 partially with respect to x we have

$$(\rho u)_{tx} = (\rho u)_{xt} = -\rho_{tt} = -(p + \rho u^2)_{xx} \quad \square$$

Problem 1.16 Deduce that for a steady isentropic flow of a gas in a Laval tube the mass flux density $j = \rho u$ is maximum when the fluid speed is sonic. Prove that this maximum in terms of stagnation values is $\rho_0 a_0\{2/(\gamma+1)\}^{\frac{1}{2}(\gamma+1)/(\gamma-1)}$.

Solution. In steady flow we can regard ρu as a function of u, hence, differentiating and using Bernoulli's theorem in the form $dp + \rho u\,du = 0$, we have

$$\frac{d}{du}(\rho u) = \rho + u\frac{d\rho}{du} = \rho\left(1 - u^2\frac{d\rho}{dp}\right) = \rho\left(1 - \frac{u^2}{a^2}\right).$$

For an extremum either $\rho = 0$ (ignored) or $u = a$, i.e. the speed is sonic. Since $d(\rho u)/du$ is positive or negative according as u is $<$ or $>$ a, $j = \rho u$ is a *maximum* j_{max} when $u = a$. Since $a^2 = dp/d\rho = \gamma p/\rho = \gamma k\rho^{\gamma-1}$ ($p = k\rho^\gamma$, k = constant), $a^2/a_0^2 = (\rho/\rho_0)^{\gamma-1}$ so that $j_{max} = \rho a = \rho_0 a \times (a/a_0)^{2/(\gamma-1)}$. Again, using equation 1.27 with $u = a$, $a^2 = a_0^2 - \frac{1}{2}(\gamma-1)a^2$ or $a^2 = 2a_0^2/(\gamma+1)$. Finally, in terms of the stagnation values,

$$j_{max} = \rho_0 a_0\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}+1/(\gamma-1)}$$
$$= \rho_0 a_0\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}(\gamma+1)/(\gamma-1)} \quad \square$$

Problem 1.17 Investigate the variation of fluid speed u for steady flow along a Laval tube.

Solution. From equation 1.22, $(d/dx)\ln(\rho u A) = 0$, i.e. $\rho'/\rho + u'/u + A'/A = 0$ where primes denote differentiation with respect to x. From Bernoulli's equation and the definition of a, we have $dp = -\rho u\,du = a^2 d\rho$ so that $\rho'/\rho = -u\,u'/a^2$. Substituting we have

$$\frac{u'}{u}\left(\frac{u^2}{a^2} - 1\right) = \frac{u'}{u}(M^2 - 1) = \frac{A'}{A}$$

where $M = u/a$ is the *local Mach number*. At the *throat* of the tube $x = 0$ where $A'(x) = 0$, *either* $u' = 0$ (an extreme value of u) *or* $u = \pm a$, i.e. the fluid speed is sonic. It is convenient to assume that when $x \ll 0$, $u \to 0$ ($\Rightarrow A \to \infty$) so that $p \to p_0$ and $\rho \to \rho_0$, the *stagnation values*. As $Q = \rho u A$, the constant mass flux (as far as variation in x is concerned), is slowly increased from zero, initially, we would have $u < a$ for all x (flow is entirely subsonic). The condition is expressed by

$$u^2 < a^2 = a_0^2 - \tfrac{1}{2}(\gamma-1)u^2$$

or $u^2 < 2a_0^2/(\gamma+1)$.

In this case, at the throat, $x = 0$ where $A' = 0$ the only possible root is

$u' = 0$. Moreover, since u' and A' have opposite signs (because $u < a$) this extreme value of u is a maximum u_m. This subsonic regime is ensured by $u^2 < u_m^2 < 2a_0^2(\gamma+1)$.

As Q is further increased u_m will increase until $u_m = a$ (for $u = a$ can occur only when $A' = 0$). The channel is now *choked* because Q has reached its maximum $\rho_t u_t A_t$ (see Problem 1.16, suffix t denoting values at the throat). For $x > 0$ the flow will be supersonic ($u > a$) or subsonic ($u < a$) according to the exit pressure. For this region $A' > 0$ so that $u'(M^2-1) > 0$. If $M > 1$ (supersonic) $u' > 0$, i.e. u increases with x while (from Bernoulli's equation) ρ and p decrease. If, on the other hand, $M < 1$ (subsonic) in $x > 0$, $u' < 0$, $\rho' > 0$ and $p' > 0$. Finally if the external pressure cannot be adjusted to the correct value in terms of the shape of the tube the continuous flow will break down and *shocks* will occur.

Problem 1.18 A perfect gas flows steadily with subsonic speed in an axisymmetric tube formed by rotating the curve $y = 1+\epsilon(x)$, $|\epsilon(x)| \ll 1$ for all x, $\epsilon(0) = 0$ about the axis OX. Neglecting second-order terms prove (i) $u = u_1\{1-2\epsilon/(1-M_1^2)\}$, (ii) $M = M_1\{1-\epsilon(2+(\gamma-1)M_1^2)/(1-M_1^2)\}$ where u and M are the fluid speed and Mach number respectively at any point, the suffix 1 denoting their values at $x = 0$. Find also an expression for the temperature.

Solution. We write $u = u_1(1+\Delta)$ and the acoustic speed $a = a_1(1+\delta)$, where Δ and δ will each be of order ϵ so that, to a first-order approximation, we may neglect Δ^2, δ^2 compared with unity. From equation 1.27,

$$a^2+\tfrac{1}{2}(\gamma-1)u^2 = \text{constant} = a_1^2+\tfrac{1}{2}(\gamma-1)u_1^2$$

Neglecting δ^2 and Δ^2, $a^2 = a_1^2(1+2\delta)$, $u^2 = u_1^2(1+2\Delta)$ so that

$$a_1^2(1+2\delta)+\tfrac{1}{2}(\gamma-1)(1+2\Delta)u_1^2 = a_1^2+\tfrac{1}{2}(\gamma-1)u_1^2$$

i.e.

$$2\delta a_1^2+\Delta(\gamma-1)u_1^2 = 0$$

or

$$2\delta+\Delta(\gamma-1)M_1^2 = 0 \quad \text{where } M_1 = u_1/a_1 \tag{1.31}$$

By equation 1.22, the equation of continuity is $\rho u A = $ constant where $A = \pi y^2 = \pi(1+2\epsilon)$ neglecting ϵ^2. Also $a^2 = \gamma p/\rho = k\gamma\rho^{\gamma-1}$, $k =$ constant. Therefore,

$$a^{2/(\gamma-1)}(1+2\epsilon)u = \text{constant} = a_1^{2/(\gamma-1)}u_1$$

or

$$(1+\delta)^{2/(\gamma-1)}(1+\Delta)(1+2\epsilon) = 1$$

Neglecting second-order terms,

$$1+\Delta+2\epsilon+2\delta/(\gamma-1) = 1$$

Solving for δ and Δ using equation 1.31, subject to $M_1 < 1$ (the motion is defined as subsonic),

$$\Delta = -2\epsilon/(1-M_1^2), \qquad \delta = \epsilon M_1^2(\gamma-1)/(1-M_1^2)$$

Hence

$$\frac{u}{u_1} = 1-\frac{2\epsilon}{1-M_1^2}$$

$$M = \frac{u}{a} = \frac{u_1(1+\Delta)}{a_1(1+\delta)} = M_1(1+\Delta-\delta) = M_1\left\{1-\frac{2+(\gamma-1)M_1^2}{1-M_1^2}\epsilon\right\}$$

The corresponding expression for the temperature is found by combining equation 1.24 with $a^2 = \gamma p/\rho$. Hence $a^2 = \gamma RT$ or

$$\frac{T}{T_1} = \frac{a^2}{a_1^2} = 1+2\delta = 1+\frac{2\epsilon(\gamma-1)M_1^2}{1-M_1^2}$$ □

1.9 Channel flow In problems of shallow channel flow with gravity the nondimensional *Froude number*, $F = U(gL)^{-\frac{1}{2}}$ plays a dominant role. The two following problems serve as illustrations. □

Problem 1.19 An open-channel flow is confined between two vertical planes $z = \pm c$ and a horizontal bed $y = 0$. Upstream the flow has uniform velocity $u_1\mathbf{i}$ with constant depth y_1. A *hydraulic jump* causes this stream to attain a greater height y_2 and uniform velocity $u_2\mathbf{i}$. Deduce that (i) $y_2 = \frac{1}{2}y_1\{(1+8F_1^2)^{\frac{1}{2}}-1\}$ where $F_1 = u_1(gy_1)^{-\frac{1}{2}}$, the upstream Froude number, exceeds unity, (ii) the downstream Froude number $F_2 = u_2(gy_2)^{-\frac{1}{2}}$ as a consequence is less than unity, (iii) the speed of a tidal bore of amplitude y_2-y_1 into still water of depth y_1 is $\{\frac{1}{2}(gy_1)(\Delta+\Delta^2)\}^{\frac{1}{2}}$ where $\Delta = y_2/y_1$. Using (iii) prove that the speed of infinitesimal waves on shallow water of depth y_1 is $(gy_1)^{\frac{1}{2}}$.

Solution. With liquid density ρ everywhere constant, the equation of continuity states that the volume flux Q parallel to $\mathbf{i}$, the direction of flow, is

$$Q = 2by_1u_1 = 2by_2u_2 \tag{1.32}$$

The mean hydrostatic pressure at a cross-section of area $2by_1$ normal to the flow upstream is $p_1 = \frac{1}{2}\rho gy_1$ whereas downstream the corresponding area and pressure values are $2by_2$ and $p_2 = \frac{1}{2}\rho gy_2$ respectively. The momentum flux equation is therefore

$$p_1(2by_1)-p_2(2by_2) = \rho Qu_2-\rho Qu_1$$

or, using the preceding expressions,

$$gb(y_1^2-y_2^2) = Q(u_2-u_1) = 2bu_1^2y_1(y_1-y_2)/y_2.$$

This is a cubic equation in y_2 of which $y_2 = y_1$ is one solution representing the case of uniform flow without discontinuity. For the *jump* solution the residual quadratic equation in y_2 is $g(y_1+y_2)y_2 = 2u_1^2y_1$. Ignoring the unacceptable negative root we have

$$y_2 = \tfrac{1}{2}y_1\{(1+8F_1^2)^{\frac{1}{2}}-1\}, \qquad F_1^2 = u_1^2/(gy_1) \tag{1.33}$$

so that $y_2 > y_1$, $u_1 > u_2$ when $F_1 > 1$. To evaluate the downstream Froude number F_2, we interchange y_1 and y_2 in equation 1.33 resulting in $y_1 = \frac{1}{2}y_2\{(1+8F_2^2)^{\frac{1}{2}}-1\}$. Since $y_2 > y_1$, $(1+8F_2^2)^{\frac{1}{2}}-1 < 2$ giving $F_2 < 1$.

To prove (iii) we first find an expression for u_1 in terms of y_1 and y_2. Using equation 1.33 we have

$$8F_1^2 = \frac{8u_1^2}{gy_1} = \left(2\frac{y_2}{y_1}+1\right)^2-1 = 4(\Delta^2+\Delta), \quad \text{where } \Delta = y_2/y_1$$

Consequently, $u_1 = \{\frac{1}{2}(gy_1)(\Delta^2+\Delta)\}^{\frac{1}{2}}$ and represents the upstream speed relative to a *stationary hydraulic jump*. If this discontinuity in height moves it is called a *tidal bore*. Its speed *relative* to the upstream value remains the same as if it were stationary and so u_1 is the speed of progress of a bore into still water. Furthermore, if the height y_2-y_1 tends to zero, $\Delta \to 1$ and $u_1 \to (gy_1)^{\frac{1}{2}}$ which is then the speed of an infinitesimal wave on water of constant depth y_1 (provided that this depth is small compared with the wavelength).

Problem 1.20 Choosing axes OX, OZ horizontal and OY vertically upwards, an open waterway is cut with vertical sides defined by the equations $z = \pm b(x)$ and possesses an almost level bed $y = h(x) \approx 0$ for all x. It is assumed that the curvatures of both $b(x)$ and $h(x)$ are negligible and that water flows steadily in this canal with a velocity which, to a first approximation, is everywhere parallel to OX and has speed $u = u(x)$. Find the differential equation for the surface profiles and discuss these profiles when $h(x) = 0$, $b(x) = a(1-\epsilon\cos\frac{1}{2}\pi x)$, $\epsilon \ll 1$ when $|x| \leqslant 1$, $b(x) = a$ when $|x| \geqslant 1$ given that the flux of volume flow is $\sqrt{(108ga^5)}$.

Solution. For steady flow $p+gy+\frac{1}{2}u^2 =$ constant while on the free liquid surface $y = y(x)$, the pressure p has a constant atmospheric value so that $y+u^2/2g =$ constant. The equation of continuity for steady motion

is $Su =$ constant $= Q$, where Q is the volume flux and $S(x) = 2b(y-h)$ is the sectional area normal to the flow at a station x. Eliminating u we have,

$$y+\frac{Q^2}{8b^2(y-h)^2g} = \text{constant}$$

Differentiating

$$y'-\frac{Q^2}{8g}\left(\frac{2b'}{b^3(y-h)^2}+\frac{2(y'-h')}{b^2(y-h)^3}\right) = 0$$

where $y' \equiv dy/dx$ etc, i.e.

$$y'\left(1-\frac{Q^2}{4gb^2(y-h)^3}\right) = \frac{Q^2}{4gb^2(y-h)^2}\left(\frac{b'}{b}-\frac{h'}{y-h}\right) \tag{1.34}$$

which is the required differential equation of the profiles. The different shapes are generated by varying the upstream or downstream values of y and u.

If $h(x) = 0$ for all x

$$y'\left(1-\frac{Q^2}{4gb^2y^3}\right) = \frac{Q^2b'}{4gb^3y^2} \tag{1.35}$$

from which it appears that y' is undefined when $Q^2 = 4gb^2y^3$. We denote this *critical flow profile* $\mathscr{C}$ by $y = y_c(x)$ where

$$y_c^3 = Q^2/(4gb^2) \tag{1.36}$$

For this profile $u = u_c$ where $u_c^2 = Q^2/S^2 = Q^2/(4b^2y_c^2) = gy_c$ so that u_c is the wave speed referred to in the last problem. Thus for all points on $\mathscr{C}$, the Froude number $F = 1$. From the equation of continuity at any fixed station x, $Q^2 = 4b^2u^2y^2 = 4b^2u_c^2y_c^2$ or $F^2 = u^2/gy = (y_c/y)^2$, i.e. $F < 1$ if $y > y_c$ and $F > 1$ if $y < y_c$.

In the given canal for which $Q^2 = 108ga^5$, $y_c^3 = 27a^3(1-\epsilon\cos\frac{1}{2}\pi x)^{-2}$ for $|x| \leqslant 1$ or, correct to the first order in ϵ, $y_c = a(3+2\epsilon\cos\frac{1}{2}\pi x)$ for $|x| \leqslant 1$ with $y_c = 3a$ otherwise. We can rewrite equation 1.35 in the form

$$y'(1-y_c^3/y^3) = y_c^3b'/(y^2b) \tag{1.37}$$

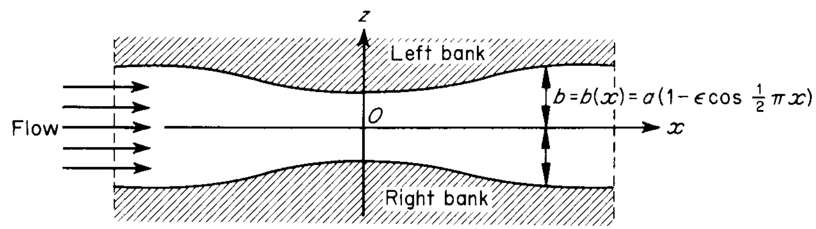


Figure 1.10—Plan

Figures 1.10 and 1.11 (not drawn to scale) illustrate the plan and elevation respectively of the canal for $|x| \leqslant 1$, the broken line $\mathscr{C}\mathscr{C}'$ in the latter is

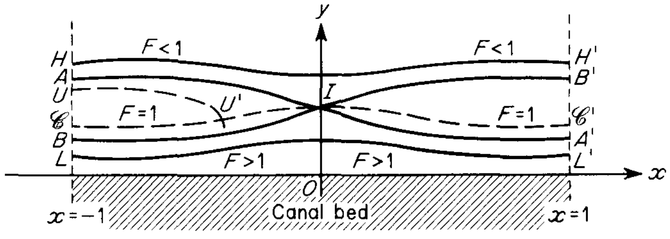


Figure 1.11—Elevation

the critical flow profile for which the Froude number $F = 1$. This profile intersects OY in the point $I(0, a(3+2\epsilon))$. Using equation 1.37 and Figure 1.11 the various cases are:

1 Profile HH' (when depth is sufficient): $y > y_c$ $(F < 1)$ for all x. By equation 1.37, $y' = 0$ at $x = 0$ where $b'(x) = 0$ and y' and b' have the *same* sign.
2 Profile LL' (low depth): $y < y_c$ $(F > 1)$ for all x. Here y' and b' have opposite sign with $y' = 0$ at $x = 0$.
3 Profile AIA': $y > y_c$ $(F < 1)$ for $x < 0$, $y < y_c$ $(F > 1)$ for $x > 0$. At the interchange $y = y_c$, unless $y' = \infty$, $b' = 0$ so that the profile passes through I.
4 Profile BIB': $y < y_c$ $(F > 1)$ for $x < 0$, $y > y_c$ $(F < 1)$ for $x > 0$ With $y' \neq \infty$ the profile passes through I.
5 Profile UU': intersects $\mathscr{C}$ orthogonally *without* passing through I. This profile is not physically possible since there would be *two* values of y for one of x.

There are obviously an infinite number of profiles according to the conditions upstream or downstream. For profile HH' any change downstream will propagate upstream since $F < 1$. If y is steadily decreased downstream the profile will eventually attain the form AIA' when conditions downstream will not penetrate beyond I. For profile LL' the shape is entirely dependent upon the upstream values. □

1.10 Impulsive motion If $\varpi = \varpi(x)$ is the impulsive pressure generated at any point P $(\overrightarrow{OP} = \mathbf{r})$ of a liquid of constant density ρ the impulsive equation of motion applied to the liquid of volume V enclosed by a geometric surface S is

$$\int_V \rho\mathbf{q}\, d\tau = -\int_S \varpi\, \mathbf{dS} = -\int_V \nabla\varpi\, d\tau \quad \text{(by Gauss's theorem)}$$

Since V is arbitrary,

$$\mathbf{q} = -\nabla(\varpi/\rho) \equiv -\nabla\varphi, \quad \text{where } \varphi = \varpi/\rho + \text{constant} \tag{1.38}$$

so that the resulting motion is irrotational.

1.11 Kinetic energy Suppose that liquid of constant density moving irrotationally with a single-valued velocity potential φ contains a solid body of surface B moving with velocity $\mathbf{U}$. The kinetic energy $\mathscr{T}$ of volume V of the liquid, which is external to B and internal to some geometrical surface Σ is

$$\mathscr{T} = \tfrac{1}{2}\rho \int_V \mathbf{q}^2\, d\tau = \tfrac{1}{2}\rho \int_V (\nabla\varphi)^2\, d\tau, \quad \text{where } \mathbf{q} = -\nabla\varphi$$

By Green's theorem since $\nabla^2\varphi = 0$

$$\mathscr{T} = \tfrac{1}{2}\rho \int_B \varphi\nabla\varphi\,.\mathbf{dS} + \tfrac{1}{2}\rho \int_\Sigma \varphi\nabla\varphi\,.\,\mathbf{dS} \tag{1.39}$$

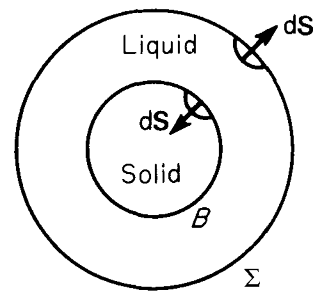


Figure 1.12

If as $|\mathbf{r}| = R \to \infty$, $\varphi \sim (\boldsymbol{\mu}\,.\,\mathbf{r})r^{-3}$ where $\boldsymbol{\mu}$ is a constant, choosing Σ as $|\mathbf{r}| = R$, we have $\int_\Sigma \varphi\nabla\varphi\,.\,\mathbf{dS} = O(R^{-3})$ for large R. In this case, for an infinite liquid $(R \to \infty)$, the kinetic energy is

$$\mathscr{T} = \tfrac{1}{2}\rho \int_B \varphi\nabla\varphi\,.\,\mathbf{dS} \tag{1.40}$$

where $\mathbf{dS}$ is *into* the *solid*.

1.12 The boundary condition If $\mathbf{n}$ is the unit normal to any point of B, the body, the boundary condition is simply

$$\mathbf{n}\,.\,\mathbf{U} = \mathbf{n}\,.\,\mathbf{q} = -\mathbf{n}\,.\,\nabla\varphi \tag{1.41}$$

In this case the kinetic energy $\mathscr{T}$ of the infinite liquid surrounding B, using equation 1.40, is

$$\mathscr{T} = -\tfrac{1}{2}\rho \int_B \varphi\mathbf{U}\,.\,\mathbf{dS} \tag{1.42}$$

Problem 1.21 Find the kinetic energy of liquid lying in the region $a \leqslant |\mathbf{r}| \leqslant b$ when motion is induced entirely by a source of output $4\pi m$ located at the origin $\mathbf{r} = \mathbf{0}$.

Solution. Using equation 1.39 the kinetic energy $\mathscr{T}$ is

$$\mathscr{T} = \tfrac{1}{2}\rho \int_E \varphi\nabla\varphi\,.\,\mathbf{dS}+\tfrac{1}{2}\rho \int_\Sigma \varphi\nabla\varphi\,.\,\mathbf{dS}$$

where E is the inner (nonsolid) boundary $r = |\mathbf{r}| = a$ and Σ the outer boundary $r = b$. Now $\nabla\varphi\,.\,\mathbf{dS} = (\partial\varphi/\partial n)dS$ where $n \equiv r$ on Σ and $n \equiv -r$ on E, n being the outward normal from the liquid. Now for the source $\varphi = m/r$ so that on Σ

$$\partial\varphi/\partial n \equiv \partial\varphi/\partial r = -m/r^2 = -m/b^2$$

whilst on E

$$\partial\varphi/\partial n \equiv -\partial\varphi/\partial r = m/r^2 = m/a^2$$

Hence,

$$\mathscr{T}= \tfrac{1}{2}\rho \int_{r=b}\left(-\frac{m^2}{b^3}\right)dS+\tfrac{1}{2}\rho\int_{r=a}\left(\frac{m^2}{a^3}\right)dS$$

$$= \tfrac{1}{2}\rho\left(-\frac{m^2}{b^3}\right)4\pi b^2+\tfrac{1}{2}\rho\left(\frac{m^2}{a^3}\right)4\pi a^2 = 2\pi\rho m^2\left(\frac{1}{a}-\frac{1}{b}\right) \qquad \square$$

Problem 1.22 A sphere of radius a moves with velocity $\mathbf{U}$ in an infinite liquid at rest at inifinity. Show that $\varphi = \tfrac{1}{2}a^3(\mathbf{U}\,.\,\mathbf{r})/r^3$ is a possible velocity potential of irrotational motion and find the kinetic energy of the liquid in this case.

Solution. With $\varphi = \tfrac{1}{2}a^3(\mathbf{U}\,.\,\mathbf{r})r^{-3}, \mathbf{q} = -\nabla\varphi = \tfrac{1}{2}a^3\{3(\mathbf{U}\,.\,\mathbf{r})r^{-5}\mathbf{r}-\mathbf{U}r^{-3}\}$ and $-\nabla^2\varphi = \text{div}\,\mathbf{q} = \tfrac{1}{2}a^3\,\{3(\mathbf{U}\,.\,\mathbf{r})r^{-5}+9(\mathbf{U}\,.\,\mathbf{r})r^{-5} - 15(\mathbf{U}\,.\,\mathbf{r})(\mathbf{r}\,.\,\mathbf{r})r^{-7}+ 3(\mathbf{U}\,.\,\mathbf{r})r^{-5}\} = 0$ fulfilling the equation of continuity. Since $\mathbf{q} \to \mathbf{0}$ as $r \to \infty$ the condition of rest at infinity is also satisfied. On the sphere the boundary condition is $\mathbf{q}\,.\,\mathbf{n} = \mathbf{U}\,.\,\mathbf{n}$ or, since $\mathbf{n} = \mathbf{r}/a$, $\mathbf{q}\,.\,\mathbf{r} = \mathbf{U}\,.\,\mathbf{r}$. From the above $\mathbf{q}\,.\,\mathbf{r} = a^3(\mathbf{U}\,.\,\mathbf{r})r^{-3}$ so that when $r = a$ we have the correct relation. Hence the given φ *is* a possible solution.

To find an expression for the kinetic energy we use equation 1.42. On the sphere $r = a$, $\varphi = \tfrac{1}{2}(\mathbf{U}\,.\,\mathbf{r})$, $\mathbf{dS} = -(\mathbf{r}/a)dS$, therefore

$$\mathscr{T} = \tfrac{1}{4}\left(\frac{\rho}{a}\right)\int_B (\mathbf{U}\,.\,\mathbf{r})^2\,dS$$

Choosing the axis OX parallel to $\mathbf{U}$, $\mathbf{U}\,.\,\mathbf{r} = Ux$ so that $\mathscr{T}= (\tfrac{1}{4}\rho U^2/a)\int_B x^2\,dS$.

By symmetry of the sphere B, $\int x^2\,dS = \int y^2\,dS = \int z^2\,dS = \tfrac{1}{3}\int (x^2+y^2+z^2)\,dS = \tfrac{4}{3}\pi a^4$, since $x^2+y^2+z^2 = a^2$. Hence

$$\mathscr{T} = \tfrac{1}{3}\pi\rho U^2a^3 = \tfrac{1}{4}M'U^2$$

where M' = mass of liquid displaced by B. $\square$

1.13 Expanding bubbles Gas occupies the region $|\,\mathbf{r}\,| \leqslant R$, where R is a function of time t, and liquid of constant density ρ lies outside in $|\mathbf{r}| \geqslant R$. We assume that there is contact between gas and liquid at all time, and that all motion is symmetric about the origin $\mathbf{r} = \mathbf{0}$. Hence, the liquid velocity $\mathbf{q}$ at any point P, where $\vec{OP} = \mathbf{r}$, is of the form $\mathbf{q} = q(r)\hat{\mathbf{r}}$, $(r \geqslant R)$. The equation of continuity, implying that the flux of volume flow across $|\mathbf{r}| = r$ is independent of r but not necessarily independent of time t, is given by

$$4\pi r^2 q = \text{constant} = 4\pi m$$

$$q = m/r^2$$

$$\mathbf{q} = m\mathbf{r}/r^3 \tag{1.43}$$

Here curl $\mathbf{q} = \mathbf{0}$ (the vorticity is zero everywhere by symmetry), i.e. φ exists with $\mathbf{q} = -\nabla\varphi$ where $\varphi = m/r$ and $m = m(t)$. This source strength m can be expressed in terms of R and $dR/dt \equiv \dot{R}$, for at the gas-liquid interface continuity of velocity means that $q = \dot{R}$ when $r = R$, i.e. $m/R^2 = \dot{R}$.

The liquid pressure is found from equation 1.18 with $\Omega = 0$ and $\partial\varphi/\partial t \equiv (d/dt)(R^2\dot{R})/r$ giving

$$\frac{p}{\rho}+\tfrac{1}{2}\left(\frac{R^2\dot{R}}{r^2}\right)^2 - \frac{1}{r}\,.\,\frac{d}{dt}(R^2\dot{R}) = A(t) \tag{1.44}$$

Problem 1.23 Given that a liquid extends to infinity and is at rest there with constant pressure Π, prove that the gas-interface pressure is $\Pi + \tfrac{1}{2}\rho R^{-2}(d/dR)(R^3\dot{R}^2)$. If the gas obeys the law $pV^{1+\alpha}$ = constant (α is a constant) and expands from rest at $R = a$ to a position of rest at $R = 2a$, deduce that its initial pressure is $7\alpha\Pi/(1-2^{-3\alpha})$.

Solution. From equation 1.44 with $r \to \infty$, $A(t) = \Pi/\rho$ = constant. When $r = R$

$$\frac{p}{\rho}+\tfrac{1}{2}\dot{R}^2 - \frac{1}{R}\frac{d}{dt}(R^2\dot{R}) = \frac{\Pi}{\rho}$$

i.e.

$$p = \Pi+\tfrac{1}{2}\rho\,(2R\ddot{R}+3\dot{R}^2) = \Pi+\tfrac{1}{2}\left(\frac{\rho}{R^2}\right)\frac{d}{dR}(R^3\dot{R}^2)$$

To find the gas pressure we use $pV^{1+\alpha}$ = constant where volume $V = \frac{4}{3}\pi R^3$. If $p = p_0$ when $R = a$, then $pR^{3+3\alpha}$ = constant $= p_0 a^{3+3\alpha}$. Applying continuity of pressure between the gas and liquid at the interface

$$p_0\left(\frac{a}{R}\right)^{3+3\alpha} = \Pi + \tfrac{1}{2}\left(\frac{\rho}{R^2}\right)\frac{d}{dR}(R^3\dot{R}^2)$$

Multiplying throughout by $2R^2$ and integrating,

$$-\frac{2p_0 a^{3+3\alpha}}{3\alpha R^{3\alpha}} = \tfrac{2}{3}\Pi R^3 + \rho R^3\dot{R}^2 + C, \quad C = \text{constant}$$

$\dot{R} = 0$ when $R = a$ gives

$$-\frac{2p_0 a^3}{3\alpha} = \tfrac{2}{3}\Pi a^3 + C$$

$\dot{R} = 0$ when $R = 2a$ gives

$$-\frac{2p_0 a^3}{3\alpha 2^{3\alpha}} = \tfrac{16}{3}\Pi a^3 + C$$

Subtracting, to eliminate C, we obtain the result

$$\frac{2p_0 a^3}{3\alpha}\left(1 - \frac{1}{2^{3\alpha}}\right) = \tfrac{14}{3}\Pi a^3$$

or

$$p_0 = \frac{7\alpha\Pi}{1 - 2^{-3\alpha}}. \qquad \square$$

Problem 1.24 A solid sphere centre O and radius a is surrounded by liquid of density ρ to a depth $(a^3+b^3)^{\frac{1}{3}} - a$. Π is the external pressure and and the whole lies in a field of attraction μr^2 per unit mass towards O. Show that if the solid sphere is suddently annihilated the velocity $\dot{R}$ of the inner surface when its radius is R is given by

$$9\dot{R}^2R^3\{(R^3+b^3)^{\frac{1}{3}} - R\}\rho = 2(3\Pi + \rho\mu b^3)(a^3 - R^3)(R^3+b^3)^{\frac{1}{3}}$$

Solution. The volume of liquid is $\frac{4}{3}\pi(a^3+b^3) - \frac{4}{3}\pi a^3 = \frac{4}{3}\pi b^3$. Hence at time $t > 0$ when the internal radius is $R < a$ the extreme radius is E where $E^3 - R^3 = b^3$. We shall apply the principle of energy starting at time $t = 0$ *after* annihilation of the sphere.

The kinetic energy of the liquid at time t using the result of Problem 1.21 is

$$\mathscr{T} = 2\pi\rho m^2\left(\frac{1}{R} - \frac{1}{E}\right) = 2\pi\rho R^4\dot{R}^2\left(\frac{1}{R} - \frac{1}{E}\right)$$

since $m = R^2\dot{R}$. The work done by the external pressure when E, the external radius, reduces from $E_0 = (a^3+b^3)^{\frac{1}{3}}$ to E is

$$W_1 = -\int_{E_0}^{E} 4\pi\Pi r^2\,dr = \tfrac{4}{3}\pi\Pi(E_0^3 - E^3)$$

The work done by the attractive force μr^2 per unit mass in a displacement from $r = 0$ to $r = r$ is $-\frac{1}{3}\mu r^3$. The total work done by this force to produce the initial configuration of the liquid is $\frac{1}{3}\mu\int_{E_0}^{a} 4\pi r^5\rho\,dr$. Similarly the work done to produce the configuration at time t is $\frac{1}{3}\mu\int_{E}^{R} 4\pi r^5\rho\,dr$ so that the difference is

$$W_2 = \tfrac{4}{3}\pi\mu\rho\left(\int_E^R r^5\,dr - \int_{E_0}^{a} r^5\,dr\right) = \tfrac{2}{9}\pi\mu\rho(R^6 - E^6 - a^6 + E_0^6)$$

$$= \tfrac{2}{9}\pi\mu\rho b^3(-R^3 + E^3 + a^3 + E_0^3)$$

since $E^3 - R^3 = b^3 = E_0^3 - a^3$, the energy balance is finally $\mathscr{T} = W_1 + W_2$, i.e.

$$2\pi\rho\dot{R}^2R^4\left(\frac{1}{R} - \frac{1}{E}\right) = \tfrac{4}{3}\pi\Pi(E_0^3 - E^3) + \tfrac{2}{9}\pi\mu\rho b^3(a^3 + E_0^3 - R^3 - E^3)$$

$$= \tfrac{4}{3}\pi\Pi(a^3 - R^3) + \tfrac{2}{9}\pi\mu\rho b^3(2a^3 - 2R^3)$$

so that

$$9\dot{R}^2R^3\{(R^3+b^3)^{\frac{1}{3}} - R\}\rho = 2(3\Pi + \mu\rho b^3)(a^3 - R^3)(R^3+b^3)^{\frac{1}{3}} \qquad \square$$

EXERCISES

1. In a given fluid motion every particle moves on a spherical surface on which its position is defined by the latitude α and longitude β. If ω and Ω denote the corresponding angular velocities deduce that the equation of continuity may be written in the form

$$\frac{\partial\rho}{\partial t}\cos\alpha + \frac{\partial}{\partial\alpha}(\rho\omega\cos\alpha) + \frac{\partial}{\partial\beta}(\rho\Omega\cos\beta) = 0$$

2. A gas for which $p = k\rho$ moves in a conical pipe. Assuming the particle paths are straight lines radiating from the vertex reaching an exit speed v where the diameter is D, show that the particle speed is $\alpha^2 v$ when the diameter is $(D/\alpha)\exp\{(\alpha^4 - 1)v^2/4k\}$.

3. Show that when the velocity potential ϕ exists the fluid acceleration may be expressed in the form $\nabla\{-(\partial\varphi/\partial t) + \frac{1}{2}q^2\}$. Using this result show that for a source of variable strength m moving with variable speed along the OX axis the fluid acceleration at a point distant x ahead of the source is $x^{-4}(d/dt)(mx^2) - 2m^2x^{-5}$.

4. The gas within an expanding spherical bubble surrounded by liquid at rest at infinity obeys the law $pV^{\frac{4}{3}} = \text{constant}$. If initially its radius R is a with $\dot{R} = 0$ and $p = \lambda p_\infty$ where p_∞ is the liquid pressure at infinity, show that R will oscillate between a and μa where μ is the positive root of the cubic $\mu(\mu^2+\mu+1) = 3\lambda$.

Chapter 2

Two-Dimensional Steady Flow

2.1 Fundamentals In this chapter we shall assume that everywhere in the fluid (i) flow is steady ($\partial/\partial t \equiv 0$), (ii) the fluid density ρ is constant, (iii) flow is two-dimensional and independent of the z-coordinate ($\partial/\partial z \equiv 0$), (iv) volume quantities such as volume flux, forces on two-dimensional bodies, kinetic energy etc. which *do* involve the z-dimension are measured in terms of unit thickness parallel to OZ. Using suffixes to denote partial differentiation, the main features of flow are:

the *velocity vector*

$$\mathbf{q} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j} + 0\mathbf{k} \tag{2.1}$$

the *lines of flow* $\mathscr{C}$ or *streamlines*, from equation 1.1, are integrals of

$$\frac{dx}{u(x, y)} = \frac{dy}{v(x, y)} \tag{2.2}$$

or $v\,dx - u\,dy = 0$.

In any source-free region $\mathscr{R}_s$ the *mass-conservation* equation or *equation of continuity* from equation 1.9 is

$$u_x + v_y = 0 \tag{2.3}$$

This is the necessary and sufficient condition that $v\,dx - u\,dy$ is the exact total differential of some function $\psi = \psi(x, y)$, for then

$$v\,dx - u\,dy = d\psi = \psi_x\,dx + \psi_y\,dy \quad \text{for all } x, y$$

implies

$$u = -\psi_y, \qquad v = \psi_x \tag{2.4}$$

The lines of flow $\mathscr{C}$ are then given by $v\,dx - u\,dy = 0 = d\psi$, i.e.

$$\psi(x, y) = \text{constant}$$

ψ is called the *stream function* (or specifically, the *Earnshaw* stream function). When it exists, the equation of continuity is automatically satisfied and conversely ψ exists at all points P of a source-free region $\mathscr{R}_s$. Since, by assumption, the motion is steady, the streamlines $\psi = \text{constant}$ are *fixed* curves in two-dimensional space and coincide *exactly* with the pathlines.

The *flux of volume flow Q* across any plane curve joining $A\ (a, b)$ to $P\ (x, y)$ in Figure 2.1 is

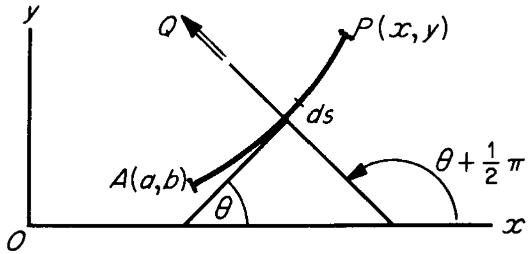


Figure 2.1

$$Q = \int_A^P (-u\sin\theta + v\cos\theta)\,ds = \int_A^P \left(\psi_y \frac{dy}{ds} + \psi_x \frac{dx}{ds}\right) ds$$

$$= \int_A^P d\psi = \psi(x, y) - \psi(a, b) \tag{2.5}$$

Q is positive when measured in the sense right to left with respect to an observer at A looking towards P. In particular, the locus of points P satisfying the condition $Q = 0$ is $\psi(x, y) = \psi(a, b) =$ constant, which is simply the streamline passing through A.

For two-dimensional flow the *vorticity vector* $\boldsymbol{\zeta}$ is given by

$$\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ u & v & 0 \end{vmatrix} = (v_x - u_y)\mathbf{k}$$

i.e. $\boldsymbol{\zeta} = \zeta\mathbf{k}$ where $\zeta = v_x - u_y$. If ψ exists ζ may be expressed in terms of it, for

$$\zeta = v_x - u_y = (\psi_x)_x - (-\psi_y)_y = \psi_{xx} + \psi_{yy} = \nabla^2\psi \tag{2.6}$$

∇ is the two-dimensional nabla operator because all quantities are independent of z.

In a vortex-free region $\mathscr{R}_v$, $\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \mathbf{0}$ and motion is irrotational. For all points P of this region a velocity potential $\varphi = \varphi(x, y)$ exists and the velocity components are derived from it by $\mathbf{q} = -\operatorname{grad}\varphi$, which, for two dimensions, gives

$$u = -\varphi_x, \qquad v = -\varphi_y \tag{2.7}$$

To summarise, we have:

1 For all $P \in \mathscr{R}_s$, ψ exists, $u = -\psi_y$, $v = \psi_x$, $\nabla^2\psi = \zeta$.

2 For all $P \in \mathscr{R}_v$, φ exists, $u = -\varphi_x$, $v = \varphi_y$, $\zeta = 0$, $\nabla^2\varphi = -(u_x + v_y)$.

Defining $\mathscr{R}_{sv}$ as the region which is both source-free and vortex-free, i.e. the intersection of regions $\mathscr{R}_s$ and $\mathscr{R}_v$, we have,

3 For all $P \in \mathscr{R}_{sv}$ both φ and ψ exist, $\nabla^2\psi = \zeta = 0$, $\nabla^2\varphi = -(u_x + v_y) = 0$, by equation 2.3, i.e. φ and ψ are harmonic functions and

$$-u = \varphi_x = \psi_y, \qquad -v = \varphi_y = -\psi_x \tag{2.8}$$

These constitute the Cauchy–Riemann equations which form the necessary and sufficient conditions that the harmonic functions φ and ψ are the real and imaginary parts of (some) complex function w of the complex variable $z = x + iy$, i.e.

$$w(z) = \varphi(x, y) + i\psi(x, y), \qquad z = x + iy \text{ for all } P \in \mathscr{R}_{sv} \tag{2.9}$$

$w(z)$ is called the *complex potential* of liquid motion; it ceases to exist *at* points occupied by sources, sinks or vortices for which $P \notin \mathscr{R}_{sv}$. Differentiating with respect to z,

$$\frac{dw}{dz} = \varphi_x + i\psi_x = \psi_y - i\varphi_y$$

$$= -u + iv$$

$$= -q\,e^{-i\lambda} \quad \text{where } u = q\cos\lambda,\ v = q\sin\lambda \tag{2.10}$$

$q = |dw/dz|$ is the magnitude of the liquid velocity and λ is its inclination to the positive axis. At points of *liquid stagnation* $u = v = 0$, given by $dw/dz = 0 = d\bar{w}/d\bar{z}$. The fact that *any* complex function $w(z)$ represents *some* liquid motion produces a convenient method of generating liquid motions.

It should be noticed that the level curves $\varphi =$ constant and $\psi =$ constant are orthogonal. Their gradients are respectively

$$\left.\frac{dy}{dx}\right|_{\varphi=\text{const}} = -\frac{\varphi_x}{\varphi_y}, \qquad \left.\frac{dy}{dx}\right|_{\psi=\text{const}} = -\frac{\psi_x}{\psi_y}$$

At the points of intersection, by the Cauchy–Riemann equations (2.8) we have

$$\left(-\frac{\varphi_x}{\varphi_y}\right)\left(-\frac{\psi_x}{\psi_y}\right) = -1$$

In terms of plane polar coordinates r and θ, $z = re^{i\theta}$, $\varphi = \varphi(r, \theta)$, $\psi = \psi(r, \theta)$. We denote the radial and transverse components of the velocity by q_r and q_θ respectively. From the Argand diagram of velocity representation we have $u + iv = (q_r + iq_\theta)e^{i\theta}$, hence taking the complex conjugate and using equation 2.10,

$$-\frac{dw}{dz} = u - iv = (q_r - iq_\theta)e^{-i\theta} \tag{2.11}$$

Again, from $\mathbf{q} = -\operatorname{grad}\varphi$, we have $q_r = -\partial\varphi/\partial r$ and $q_\theta = -\partial\varphi/r\partial\theta$. Next, we express these velocity components in terms of ψ as follows. In Figure 2.2, let $T(r+\delta r, \theta+\delta\theta)$ be a point neighbouring $P(r,\theta)$ such

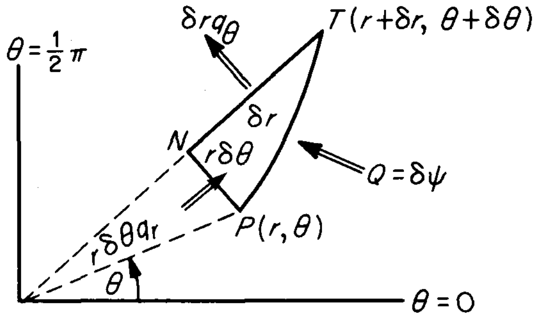


Figure 2.2

that if ψ is the value of the stream function at P, $\psi+\delta\psi$ is the corresponding value at T. By equation 2.5, the flux of volume flow across *any* curve PT is $\delta\psi$. Complete the elemental polar triangle PNT where $NT = \delta r$ is drawn radially and $PN = r\,\delta\theta$ is drawn transversely. The flux, to the first order, out of this triangle across NT is $\delta r\, q_\theta$ and across NP the flux is $-r\,\delta\theta\, q_r$. Therefore, in the absence of sources or sinks within PNT, for all r, θ, we have

$$\delta\psi = \frac{\partial\psi}{\partial r}\delta r + \frac{\partial\psi}{\partial\theta}\delta\theta = q_\theta\,\delta r - q_r r\,\delta\theta$$

leading to

$$q_\theta = \frac{\partial\psi}{\partial r}, \qquad q_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \tag{2.12}$$

2.2 Elementary complex potentials

2.2.1 ***Uniform stream*** Here $u = U\cos\alpha$, $v = U\sin\alpha$ where U and α are constants representing the magnitude and inclination respectively of the stream. Since

$$\frac{dw}{dz} = -u + \mathrm{i}v = -U(\cos\alpha - \mathrm{i}\sin\alpha) = -U\,\mathrm{e}^{-\mathrm{i}\alpha}$$

integrating and acknowledging the physical insignificance of the constant of integration,

$$w = -(U\,\mathrm{e}^{-\mathrm{i}\alpha})z = \varphi + \mathrm{i}\psi$$

The real and imaginary parts are

$$\varphi = -U(x\cos\alpha + y\sin\alpha), \qquad \psi = U(x\sin\alpha - y\cos\alpha)$$

The lines of equipotential and streamlines form mutually orthogonal networks of parallel straight lines.

2.2.2 ***Two-dimensional source*** Given that $z = 0$ is a source of volume output $2m\pi$, by symmetry on the circle $z = r$, the velocity components due to this source alone are $q_\theta = 0$, $q_r = 2m\pi/2r\pi = m/r$. $\mathscr{R}_s$ is the *whole* of the z-plane excluding $z = 0$. Both φ and ψ exist everywhere except at $z = 0$. Using equations 2.11 and 2.12

$$q_r = \frac{m}{r} = -\frac{\partial\varphi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$q_\theta = 0 = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{\partial\psi}{\partial r}$$

Integrating

$$\varphi = -m\ln r, \qquad \psi = -m\theta$$

or

$$w = \varphi + \mathrm{i}\psi = -m\ln z \tag{2.13}$$

The singularity at $z = 0$ is due to the source there. The test for the presence of a source at any point is given by evaluating $\oint_C d\psi = [\psi]$ where C is a *tight* circuit enclosing the point. For $z = 0$, in this particular instance, we have $[\psi] = -2m\pi$ corresponding to a source of output $2m\pi$ or *strength* m.

Similarly, a source of equal strength m at $z = z_0$ has a complex potential $w = -m\ln(z - z_0)$ from which

$$\varphi = -m\ln|z - z_0| \text{ and } \psi = -m\arg(z - z_0).$$

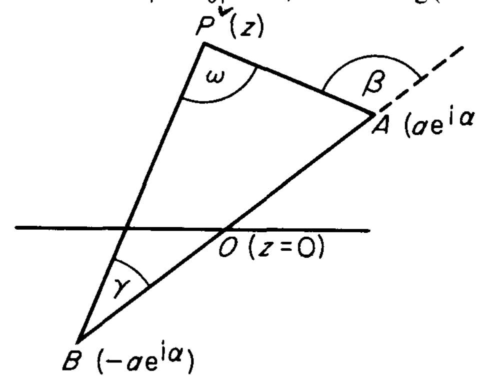


Figure 2.3

2.2.3 *Source and sink of equal strengths* The complex potential of a source of strength m at $A\,(z = ae^{i\alpha})$ with an equal sink at $B\,(z = -ae^{i\alpha})$ is $w = -m\ln(z-ae^{i\alpha})+m\ln(z+ae^{i\alpha})$. Writing $\arg(z-ae^{i\alpha}) = \beta$ and $\arg(z+ae^{i\alpha}) = \gamma$ it follows that $\psi = m(\gamma-\beta) = -m\omega$ where $\omega = \angle BPA$ (Figure 2.3). The streamlines ψ = constant are circles through A and B. If (u, v) are the Cartesian velocity components at P,

$$-u+iv = \frac{dw}{dz} = \frac{m}{z+ae^{i\alpha}} - \frac{m}{z-ae^{i\alpha}} = -\frac{2mae^{i\alpha}}{(z+ae^{i\alpha})(z-ae^{i\alpha})}$$

from which the magnitude of the velocity is

$$q = \left|\frac{dw}{dz}\right| = \frac{2ma}{|AP|.|BP|}$$

since $|z-ae^{i\alpha}| = AP$ etc. We can show that the uniform stream is the limiting case of this source-sink pair when both a and m tend to infinite values with $2m/a$ remaining constant. When a is large compared with $|z|$ the above complex potential w is written in the form,

$$\begin{aligned} w &= m\ln\{(z/a)e^{i\alpha}+1\} - m\ln\{1-(z/a)e^{i\alpha}\} + \text{constant} \\ &\sim m\{(z/a)e^{-i\alpha}-\ldots\} - m\{-(z/a)e^{-i\alpha}-\ldots\} + \text{constant} \\ &\sim 2m(z/a)e^{-i\alpha} + \text{constant} \end{aligned}$$

If $a \to \infty$ and $m \to \infty$ such that $2m/a$ = constant, we obtain the uniform stream, $w = Uze^{-i\alpha}$ + constant.

2.2.4 *Two-dimensional doublet* A two-dimensional doublet at $z = z_0$ of strength μ and direction α is defined as the limit, in which $a \to 0$, $m \to \infty$ with $ma = \mu$, of a sink of strength m at z_0 with a source of equal strength at $z_0 + ae^{i\alpha}$. The complex potential is, therefore,

$$\begin{aligned} w &= \lim_{a\to 0}\{-(\mu/a)\ln(z-z_0-ae^{i\alpha}) + (\mu/a)\ln(z-z_0)\} \\ &= \lim_{a\to 0}\left\{-\frac{\mu}{a}\ln\left(1-\frac{ae^{i\alpha}}{z-z_0}\right)\right\} \\ &= \lim_{a\to 0}\frac{\mu}{a}\left\{\frac{ae^{i\alpha}}{z-z_0} + O(a^2)\right\} \\ &= \frac{\mu e^{i\alpha}}{z-z_0} \end{aligned} \tag{2.14}$$

When $z_0 = 0$ we have

$$w = \varphi + i\psi = \mu(\cos\alpha + i\sin\alpha)/(x+iy)$$

from which

$$\varphi = \frac{\mu(x\cos\alpha + y\sin\alpha)}{x^2+y^2} \qquad \psi = \frac{\mu(x\sin\alpha - y\cos\alpha)}{x^2+y^2}$$

The equipotentials and streamlines form mutually orthogonal systems of circles.

2.2.5 *Two-dimensional vortex* Consider $w = ik\ln z$, k real. Putting $z = re^{i\theta}$ and taking real and imaginary parts,

$$\varphi = -k\theta, \qquad \psi = k\ln r \tag{2.15}$$

The streamlines are the circles $r = |z|$ = constant and the equipotentials are their radii θ = constant. The only singular point in the finite z-plane is the origin $z = 0$. Consequently, expecting possibly this point, the flow is everywhere source-free and vortex-free. The volume flux across r = constant = a is zero since ψ = constant on $r = a$ for all a. Hence $\mathscr{R}_s$ is the whole of the z-plane. On $r = a$,

$$q_r = -\frac{\partial\varphi}{\partial r} = 0, \qquad q_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{k}{a}$$

so that for the circuit $r = a$, $\int \mathbf{q}\,.\,d\mathbf{r} = \int q_\theta\, a\, d\theta = 2\pi k$. This result, which is independent of a however small, implies a circulation $\Gamma = 2\pi k$ about *any* circuit containing the origin $z = 0$. This denotes the presence of a vortex of strength k at $z = 0$. Because φ exists and is single valued everywhere else, all other circuits *not enclosing the origin* will have zero circulation. Consequently $\mathscr{R}_v$ is the whole of the z-plane *excluding* the point $z = 0$ occupied by the vortex.

Problem 2.1 Prove that for the complex potential $\tan^{-1} z$ the streamlines and equipotentials are circles. Determine the velocity at any point and examine the singularities at $z = \pm i$.

Solution. From $w = \varphi + i\psi = \tan^{-1} z$, $w = \varphi - i\psi = \tan^{-1}\bar{z}$, we have

$$2i\psi = \tan^{-1} z - \tan^{-1}\bar{z} = \tan^{-1}\{(z-\bar{z})/(1+z\bar{z})\}$$

or

$$z-\bar{z} = 2iy = (1+z\bar{z})\tan 2i\psi = (1+x^2+y^2)i\tanh 2\psi$$

The streamlines ψ = constant are the circles $x^2+y^2+1 = 2y\coth 2\psi$ or, in complex terms, $|z - i\coth 2\psi| = \operatorname{cosech}|2\psi|$. Similarly,

$$2\varphi = \tan^{-1} z + \tan^{-1}\bar{z} = \tan^{-1}\{(z+\bar{z})/(1-z\bar{z})\}$$

or

$$1-x^2-y^2 = 2x\cot 2\varphi \qquad \text{i.e.} \quad |z+\cot 2\varphi| = \operatorname{cosec}|2\varphi|$$

Consequently the equipotentials φ = constant are also circles which are

D

orthogonal to ψ = constant and form a coaxial system with limit points at $z = \pm i$. The velocity components (u, v) are given by

$$-u+iv = \frac{dw}{dz} = \frac{1}{z^2+1}$$

Since the denominator is zero at $z = \pm i$, there are singularities at these points. Near $z = i$ put $z = i+z'$ where $|z'|$ is very small. Neglecting $|z'|^2$

$$-u+iv = \frac{dw}{dz} = \frac{dw}{dz'} = \frac{1}{1+(-1+2iz')} = \frac{1}{2iz'}$$

Integrating, $w = -\frac{1}{2}i\ln z'$. From equation 2.15 the singularity at $z = i$ is a vortex of strength $k = -\frac{1}{2}$ with circulation $-\pi k$. Similarly, near $z = -i$ putting $z = -i+z''$, $w \approx \frac{1}{2}i\ln z''$ so that the singularity at $z = -i$ is a vortex of strength $k = \frac{1}{2}$. □

Problem 2.2 Show that when $w = Vf(z)-a\lambda\ln f(z)$, where $f(z) = z-2\sqrt{(az)}$, $(V, \lambda, a$ real), part of the streamline $\psi = -a\lambda\pi$ is a parabola. Interpret the motion and prove that provided $0 < V < 2\lambda$, the pressure at infinity must exceed $\rho(V-2\lambda)^2(4V+\lambda)/54\lambda$ to prevent cavitation on the parabola.

Solution. We have $\psi = \operatorname{Im} w = V\operatorname{Im} f(z)-a\lambda\arg f(z)$. When $\operatorname{Im} f(z) = 0$ $\arg f(z) = 0$ or π according as $\operatorname{Re} f(z)$ is positive or negative. Putting $z = re^{i\theta}$, $f(re^{i\theta}) = r(\cos\theta+i\sin\theta)-2\sqrt{(ar)}(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta)$, so that $\operatorname{Im} f(z) = 0$ implies $r\sin\theta = 2\sqrt{(ar)}\sin\frac{1}{2}\theta$, i.e. $\sin\frac{1}{2}\theta = 0$ $(\theta = 0)$ or $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$, a parabola with focus at $r = 0$. When $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$, $\operatorname{Re} f(z) = r\cos 2\sqrt{(ar)}\cos\frac{1}{2}\theta = a\cos\theta\sec^2\frac{1}{2}\theta-2a = -a\sec^2\frac{1}{2} < 0$, in which case $\arg f(z) = \pi$ leaving $\psi = -a\lambda\pi$ on the parabola.

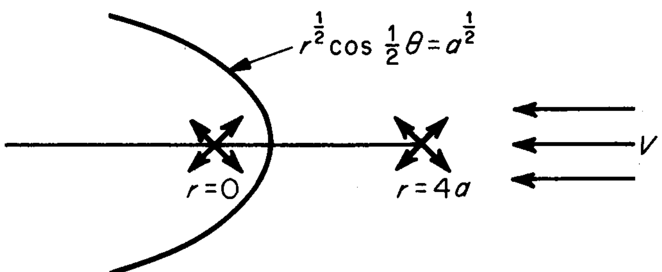


Figure 2.4

To interpret the motion, we have, for large $|z|$, $dw/dz \sim V$, i.e. the flow at infinity behaves like a uniform stream $u = -V$, $v = 0$. The singularities of w coincide with the zeros of $f(z)$ which are (i) $z = 0$,

inside the parabola, (ii) $z = 4a$, a point outside. Near $z = 4a$, put $z = 4a+\delta e^{i\theta}$, where δ is small. Then

$$f(z) = 4a+\delta e^{i\theta}-2\sqrt{(4a^2+a\delta e^{i\theta})} = \tfrac{1}{2}\delta e^{i\theta}+O(\delta^2)$$

and

$$\psi = \operatorname{Im} w = V\operatorname{Im} f(z)-a\lambda\arg f(z) = \tfrac{1}{2}\delta V\sin\theta-\lambda a\theta.$$

For constant δ and variation of θ from 0 to 2π, $\oint d\psi = -2\pi\lambda a$ in which case the singularity at $z = 4a$ is a simple source of strength λa. We ignore the singularity at $z = 0$ inside the parabola. To find an expression for the liquid speed on the parabola we have $dw/dz = Vf'(z)-a\lambda f(z)/f'(z)$, i.e.

$$\frac{dw}{dz} = V\left\{1-\left(\frac{a}{z}\right)^{\frac{1}{2}}\right\}-\frac{a\lambda(1-(a/z)^{\frac{1}{2}})}{z-2(az)^{\frac{1}{2}}}.$$

On the parabola, $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$, $z = re^{i\theta} = a(\sec^2\frac{1}{2}\theta)e^{i\theta}$, hence

$$\frac{dw}{dz} = (1-e^{-\frac{1}{2}i\theta}\cos\tfrac{1}{2}\theta)\left\{V-\frac{\lambda(\cos^2\frac{1}{2}\theta)e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta}-2\cos\frac{1}{2}\theta}\right\}$$
$$= e^{-\frac{1}{2}i\theta}(i\sin\tfrac{1}{2}\theta)(V+\lambda\cos^2\tfrac{1}{2}\theta)$$

so that $q = |dw/dz| = s(V+\lambda-\lambda s^2)$, $s = \sin\frac{1}{2}\theta$. The maximum speed q_m on the parabola occurs when $s = \sqrt{\{(V+\lambda)/3\lambda\}}$, $(dq/ds = 0)$. Since $0 < V < 2\lambda$, $|s| < 1$ in which case θ is real and $q_m = \sqrt{\{4(V+\lambda)^3/27\lambda\}}$. At this speed the pressure will be a minimum $p_m = p_\infty+\frac{1}{2}\rho(V^2-q_m^2) = p_\infty-\rho(V-2\lambda)^2(4V+\lambda)/54\lambda$ by Bernoulli's theorem where p_∞ is the pressure at infinity. To prevent cavitation $p_m > 0$ or

$$p_\infty > \rho(V-2\lambda)^2(4V+\lambda)/54\lambda.$$ □

Problem 2.3 Prove that for incompressible flow in a conservative force field $\partial(\zeta,\psi)/\partial(x,y) = 0$ and deduce that when ζ = constant, the pressure p satisfies

$$(p/\rho)+\tfrac{1}{2}q^2+\Omega-\zeta\psi = \text{constant},$$ where Ω is the field potential.

Solution. By equation 1.17, for steady motion the equations are

$$\begin{aligned} uu_x+vu_y &\equiv \tfrac{1}{2}(q^2)_x-v\zeta = -\Omega_x-\rho^{-1}p_x \\ uv_x+vv_y &\equiv \tfrac{1}{2}(q^2)_y+u\zeta = -\Omega_y-\rho^{-1}p_y \end{aligned} \tag{2.16}$$

where $q^2 = u^2+v^2$, $\zeta = v_x-u_y$ and $(-\Omega_x, -\Omega_y)$ are the bodyforce components for the conservative field. We can eliminate q^2, Ω and p by taking partial derivatives, giving

$$(v\zeta)_y+(u\zeta)_x = 0$$

or, since $u = -\psi_y$, $v = \psi_x$

$$(\zeta\psi_x)_y+(-\zeta\psi_y)_x = 0$$

i.e.

$$\partial(\zeta, \psi)/\partial(x, y) = 0$$

If we multiply the first of equations 2.16 by dx and the second by dy and add we have

$$\tfrac{1}{2}dq^2 + \zeta(u\,dy - v\,dx) = -d\Omega - \rho^{-1}\,dp$$

Integrating, after using the fact that ζ is constant and that $u\,dy - v\,dx = -d\psi$, gives the result

$$\boxed{\tfrac{1}{2}q^2 - \zeta\psi = -\Omega - (dp/\rho) + \text{constant}} \qquad \square \qquad (2.17)$$

Problem 2.4 Liquid in the annular region $a < |z| < b$ has constant vorticity ζ and the liquid outside is at rest. The streamlines are the concentric circles $|z| = r =$ constant with the liquid speed zero on $r = b$ and a constant, V, on $r = a$. Show that $\zeta = 2aV/(a^2 - b^2)$ and deduce that the pressure difference between the two regions at rest is

$$\tfrac{1}{2}\rho V^2\{b^4 - a^4 - 4a^2b^2 \ln(b/a)\}/(b^2 - a^2)^2$$

Solution. Here $\mathbf{q} = q\boldsymbol{\theta}$ where $q = q(r)$, $\boldsymbol{\theta}$ being the transverse unit vector, and $\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = r^{-1}(d/dr)(rq)\mathbf{k}$, where $\mathbf{k}$ is the unit vector normal to the plane. Since $\zeta = \boldsymbol{\zeta}.\mathbf{k}$ is constant,

$$rq = \tfrac{1}{2}\zeta r^2 + \text{constant}$$

But $q = 0$ when $r = b$, hence

$$rq = \tfrac{1}{2}\zeta(r^2 - b^2)$$

Since $q = V$ when $r = a$, the result $\zeta = 2aV/(a^2 - b^2)$ follows.

ψ is a function of r only; therefore,

$$q = \frac{d\psi}{dr} = \tfrac{1}{2}\zeta\left(r - \frac{b^2}{r}\right) \quad \text{and} \quad \psi = \tfrac{1}{4}\zeta r^2 - \tfrac{1}{2}\zeta b^2 \ln r$$

ignoring the irrelevant constant of integration. Using equation 2.17 the pressure p with $\Omega = 0$ is given by

$$p - \rho\zeta\psi + \tfrac{1}{2}\rho q^2 = \text{constant}$$

For $a < r < b$, $p - \rho\zeta^2(\tfrac{1}{4}r^2 - \tfrac{1}{2}b^2 \ln r) + \tfrac{1}{8}\rho\zeta^2(r - b^2/r)^2 =$ constant. Denoting the pressures at $r = a$ and $r = b$ by p_a and p_b respectively

$$p_a - \rho\zeta^2(\tfrac{1}{4}a^2 - \tfrac{1}{2}b^2 \ln a) + \tfrac{1}{8}\rho\zeta^2(a - b^2/a)^2 = p_b - \rho\zeta^2(\tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 \ln b)$$

$$p_b - p_a = \rho\zeta^2\{\tfrac{1}{4}(b^2 - a^2) + \tfrac{1}{8}(b^2 - a^2)^2/a^2 - \tfrac{1}{2}b^2 \ln(b/a)\}$$

$$= \tfrac{1}{2}\rho V^2\{b^4 - a^4 - 4a^2b^2 \ln(b/a)\}/(b^2 - a^2)^2$$

Assuming continuity of pressure across the boundaries the result follows. $\square$

2.3 Hydrodynamic images

Definition. Let $f(z)$ be the complex potential of motion in a liquid and C a simple closed curve enclosing a domain A of the liquid *without including any singularity* of $f(z)$ inside. The hydrodynamic image of $f(z)$ in C is the complex potential $g(z)$ such that $f(z) + g(z)$ is real on C and *all* the singularities of $g(z)$ are contained in A.

Consequently, writing the total complex potential $w = f(z) + g(z)$, $\psi = 0$ on C (i.e. C can represent a rigid boundary) and all the singularities of w outside A coincide with those of $f(z)$.

Case 1. C is the line $\operatorname{Re} z = a$ with A the domain $\operatorname{Re} z < a$. Here,

$$g(z) = \bar{f}(2a - z), \qquad w = f(z) + \bar{f}(2a - z) \tag{2.18}$$

When $a = 0$ and $f(z) = -m \ln(z - z_0)$, $g(z) = -m \ln(z + \bar{z}_0) +$ constant, i.e. the image of a source in $\operatorname{Re} z = 0$ is an equal source at the *image point* $-\bar{z}_0$ of z_0 in the imaginary axis.

Case 2. C is the line $\operatorname{Im} z = b$ with A the domain $\operatorname{Im} z < b$. Here

$$g(z) = \bar{f}(z - 2bi), \qquad w = f(z) + \bar{f}(z - 2bi) \tag{2.19}$$

Case 3. C is the circle $|z| = a$ with A the domain $|z| < a$. Here

$$g(z) = \bar{f}(a^2/z), \qquad w = f(z) + \bar{f}(a^2/z) \tag{2.20}$$

This is the *circle theorem.*

Problem 2.5 Liquid occupies the region $\operatorname{Im} z > 0$ adjacent to a rigid wall along $\operatorname{Im} z = 0$. Motion is due to a uniform stream of magnitude U flowing parallel to the real axis and at $z = ai$ there is a doublet of strength $4a^2\lambda U$ inclined at an angle π to the stream. Show that when $\lambda < 1$ the minimum and maximum speeds on the wall are respectively $(1 - \lambda)U$ and $(1 + 8\lambda)U$. In the case $\lambda = 1$ show that the circles $|z \pm ai| = 2a$ are dividing streamlines.

Solution. The complex potential of the uniform stream flowing parallel to the positive real axis is $-Uz$. Using equation 2.14 with $\alpha = \pi$ the complex potential of the doublet is $-4a^2\lambda U/(z - ai)$. Since the uniform stream contains the rigid boundary $\operatorname{Im} z = 0$ as a streamline, no image is needed for this part of the motion. The image of the doublet, however, is $-4a^2\lambda U/(z + ai)$ by equation 2.19. Consequently, the final complex potential is

$$w(z) = -Uz - 4a^2\lambda U\left(\frac{1}{z - ai} + \frac{1}{z + ai}\right) = -Uz - \frac{8a^2\lambda Uz}{z^2 + a^2}$$

The velocity components are given by

$$-u+\mathrm{i}v = \frac{dw}{dz} = -U+\frac{8a^2U(z^2-a^2)}{(z^2+a^2)^2}$$

On the wall $z = x+0\mathrm{i}$, on which $v = 0$ (proving that the wall *is* a rigid boundary) and

$$u = U-\frac{8a^2\lambda U}{x^2+a^2}+\frac{16a^4\lambda U}{(x^2+a^2)^2} = 4U\left\{\frac{2a^2}{x^2+a^2}-\frac{1}{2}\right\}^2+U(1-\lambda).$$

Since $\lambda < 1$, u is a minimum when $\{2a^2/(x^2+a^2)-\frac{1}{2}\} = 0$, i.e. when $x = \pm a\sqrt{3}$. This minimum is $U(1-\lambda)$. The maximum value of u occurs when $x = 0$ and has a magnitude $4\lambda U(3/2)^2+U(1-\lambda) = (1+8\lambda)U$. When $\lambda = 1$ the streamlines are determined by $\psi = (w-\bar{w})/(2\mathrm{i}) =$ constant $= A$ (say). Substituting,

$$w-\bar{w} = -U\left(z-\bar{z}+\frac{8a^2z}{z^2+a^2}-\frac{8a^2\bar{z}}{\bar{z}^2+a^2}\right) = 2\mathrm{i}A$$

Factorising,

$$(z-\bar{z})\left(\frac{8a^2z\bar{z}-8a^4}{(z^2+a^2)(\bar{z}^2+a^2)}-1\right) = \frac{2\mathrm{i}A}{U}$$

The streamline for which $A = 0$ will *divide* into separate branches on which either $z-\bar{z} = 0$, i.e. $\operatorname{Im} z = 0$, the rigid boundary, or

$$(z^2+a^2)(\bar{z}^2+a^2) = 8a^2z\bar{z}-8a^4$$

i.e.

$$(z\bar{z}-3a^2)^2 = -a^2(z-\bar{z})^2$$

or

$$z\bar{z}-3a^2 = \pm\mathrm{i}a(z-\bar{z})$$

leading to the result

$$(z\pm a\mathrm{i})(\bar{z}\mp a\mathrm{i}) = |z\pm a\mathrm{i}|^2 = 4a^2$$

□

Problem 2.6 Find the complex potential of motion due to a source of strength m at $z = z_0$ ($\operatorname{Re} z_0 > a$) outside a rigid boundary C consisting of the semicircle $|z| = a$, $-\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{1}{2}\pi$ and part of the imaginary axis for which $|\operatorname{Im} z| \geqslant a$. Find an expression for the liquid speed on the semicircle when z_0 is real and equal to ka ($k > 1$) and find its maximum.

Solution. The source alone is represented by the complex potential $f(z) = -m\ln(z-z_0)$. Using equation (2.20), its image in $|z| = a$ is

$$\bar{f}(a^2/z) = -m\ln\{(a^2/z)-\bar{z}_0\} = -m\ln(z-a^2/\bar{z}_0)+m\ln z+A$$

where $A = -m\ln(-\bar{z}_0)$ is a constant which can be ignored. The image

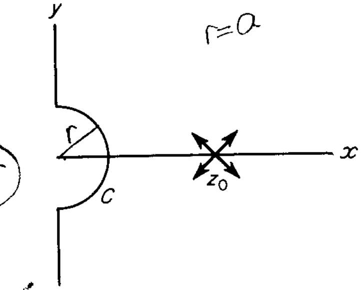


Figure 2.5

system consists of a source of strength m at the inverse point $a^2/\bar{z}_0$ together with an equal sink at the origin. The new complex potential is

$$f_1(z) = -m\ln(z-z_0)-m\ln(z-a^2/\bar{z}_0)+m\ln z$$

The imaginary axis can be made a rigid boundary by introducing the image of $f_1(z)$ in $\operatorname{Re} z = 0$. By equation 2.18, this image is

$$\bar{f}_1(-z) = -m\ln(z+\bar{z}_0)-m\ln(z+a^2/z_0)+m\ln z+\text{constant}$$

giving the final potential of motion as

$$w(z) = f_1(z)+\bar{f}_1(-z) = -m\ln\{(z-z_0)(z+\bar{z}_0)(z-a^2/\bar{z}_0)(z+a^2/z_0)z^{-2}\}$$

This complex potential continues to fulfil the condition that $|z| = a$ is a streamline because, in retaining the sink at the origin in $\bar{f}_1(-z)$, we have ensured that the algebraic sum of the strengths of the sources within the circular boundary is zero so that this semicircle can remain solid.

To find an expression for the speed with $z_0 = ka$ (real), we have

$$w = -m\ln(z^2-a^2k^2)-m\ln(z^2-a^2k^{-2})+2m\ln z$$

so that

$$\frac{dw}{dz} = -\frac{2mz}{z^2-a^2k^2}-\frac{2mz}{z^2-a^2k^{-2}}+\frac{2m}{z}$$

On the semicircle where $z = a\mathrm{e}^{\mathrm{i}\theta}$

$$\frac{dw}{dz} = \frac{2m\,\mathrm{e}^{-\mathrm{i}\theta}}{a}\left(1-\frac{\mathrm{e}^{2\mathrm{i}\theta}}{\mathrm{e}^{2\mathrm{i}\theta}-k^2}-\frac{\mathrm{e}^{2\mathrm{i}\theta}}{\mathrm{e}^{2\mathrm{i}\theta}-k^{-2}}\right)$$

$$= \left(\frac{2mk^2}{a}\right)\frac{\mathrm{e}^{-\mathrm{i}\theta}(\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{-2\mathrm{i}\theta})}{(\mathrm{e}^{2\mathrm{i}\theta}-k^2)(\mathrm{e}^{-2\mathrm{i}\theta}-k^2)} = \frac{4m\mathrm{i}k^2(\sin 2\theta)\,\mathrm{e}^{-\mathrm{i}\theta}}{a(1+k^4-2k^2\cos 2\theta)} = -q\mathrm{e}^{-\mathrm{i}\lambda}$$

where $q = 4mk^2\sin 2\theta/a(1+k^4-2k^2\cos 2\theta) > 0$ for $0 < \theta < \frac{1}{2}\pi$ is the liquid speed inclined at an angle $\lambda = \frac{1}{2}\pi+\theta$ (for $-\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{e}^{-\mathrm{i}(\frac{1}{2}\pi+\theta)} = \mathrm{e}^{-\mathrm{i}\lambda}$) which is tangent to the semicircle. When $-\frac{1}{2}\pi < \theta < 0$ we replace q by $-q$ and λ by $-\lambda$. To find the maximum value of q put $2\theta = \sigma$. Since

$(d/d\sigma)\{\sin\sigma/(1+k^4-2k^2\cos\sigma)\} = 0$ when $\cos\sigma(1+k^4-2k^2\cos\sigma) = 2k^2\sin^2\sigma$, at an extremum $\cos\sigma = 2k^2/(k^4+1)$, $\sin\sigma = \pm(k^4-1)/(k^4+1)$, the angles being real because $k > 1$. The speed has a maximum value $4mk^2/a(k^4-1)$ at points on the semicircle for which

$$\theta = \tfrac{1}{2}\cos^{-1}\{2k^2/(k^4+1)\}. \qquad \square$$

Problem 2.7 Find the complex potential for a source of strength m placed at $z = ai$ between two rigid walls $\operatorname{Im} z = \pm b$, $|b| > |a|$. Express the answer in a closed form when $a = 0$.

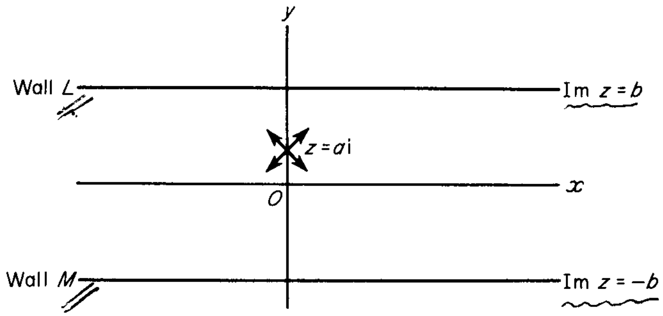


Figure 2.6

Solution. The source at $z = ai$ has a complex potential $f_0(z) = -m\ln(z-ai)$ whose image l_1 in the wall $L(\operatorname{Im} z = b)$ is $-m\ln(z+ai-2bi)$ and its image m_1 in the wall M $(\operatorname{Im} z = -b)$ is $-m\ln(z+ai+2bi)$. This is the first approximation to the flow for m_1 disturbs the condition on L and l_1 disturbs the condition on M. To rectify this, as a second approximation, we add on the image l_2 of m_1 in L, namely $-m\ln(z-ai-4bi)$ and the image m_2 of l_1 in M which is $-m\ln(z-ai+4bi)$. Similarly, in the third approximation we must add the image l_3 of m_2 in L and the image m_3 of l_2 in M, and so on. The nth approximation for the complex potential is, therefore,

$$w_n(z) = -m\ln(z-ai) - m\sum_{k=1}^{n}\ln\{(z-ai)^2+4^2k^2b^2\}$$
$$-m\sum_{k=1}^{n}\ln\{(z+ai)^2+4(2k-1)^2b^2\}$$

All the image singularities are outside the walls and as n increases the compensating images move further from the region of flow. Moreover the magnitude of the difference in fluid velocity between the two successive approximations is

$$\left|\frac{d}{dz}(w_n - w_{n-1})\right| = O(n^{-2})$$

which tends to zero as $n \to \infty$; in this sense the approximations converge. Before taking the limit we add to w_n the constant $m\sum_{k=1}^{k}\ln(4^3k^2(2k-1)^2b^4)$ which does not affect the flow so that

$$w = \lim_{n\to\infty} w_n$$
$$= -m\ln\left\{(z-ai)\prod_{k=1}^{\infty}\left(1+\frac{(z-ai)^2}{4^2k^2b^2}\right)\left(1+\frac{(z+ai)^2}{4(2k-1)^2b^2}\right)\right\}$$

When $a = 0$

$$w = -m\ln\left\{z\prod_{k=1}^{\infty}(1+z^2/4k^2b^2)\right\}$$

Since $z\prod_{n=1}^{\infty}(1+z^2/n^2\pi^2) = \sinh z$, we have, ignoring the constant,

$$w = -m\ln\sinh(\pi z/2b) \qquad \square$$

Problem 2.8 Find the force on the cylinder $|z| = a$ inserted into a flow whose velocity components are $u = Q\cos\alpha - \zeta y$, $v = Q\sin\alpha$, which describe a uniform shear flow superimposed upon a uniform stream.

Solution. Here $v_x - u_y = \psi_{xx}+\psi_{yy}$ = constant $= \zeta$, i.e. for general flow with constant shear ζ, $\psi = \frac{1}{4}\zeta(x^2+y^2)+\psi_0$ where ψ_0 is *any* solution of $\psi_{xx}+\psi_{yy} = 0$. Alternatively, we seek a solution for ψ expressed in the equivalent form

$$\psi = \tfrac{1}{4}\zeta z\bar{z} + \operatorname{Im} f(z) \qquad (2.21)$$

Before inserting the cylinder,

$$u = -\psi_y = Q\cos\alpha - \zeta y, \qquad v = \psi_x = Q\sin\alpha$$

i.e.

$$\psi = Q(x\sin\alpha - y\cos\alpha) + \tfrac{1}{2}\zeta y^2$$

or

$$\psi_0 = \operatorname{Im} f(z) = Q(x\sin\alpha - y\cos\alpha) + \tfrac{1}{4}\zeta(y^2-x^2)$$

so that $f(z) = -Qz\,e^{-i\xi} - \frac{1}{4}i\zeta z^2$ and

$$\psi = \tfrac{1}{4}\zeta z\bar{z} + \operatorname{Im}(-Qz\,e^{-i\alpha} - \tfrac{1}{4}i\zeta z^2) \qquad (2.22)$$

when the cylinder $|z| = a$ is introduced, no image is required for the term $\frac{1}{4}\zeta z\bar{z}$, since it is a constant $\frac{1}{4}\zeta a^2$ on the boundary. By the circle theorem (equation 2.20), the image of the term $\operatorname{Im} f(z)$ is $\operatorname{Im} \bar{f}(a^2/z)$. The final ψ after adding the cylinder to the flow expressed by equation 2.22 is

$$\psi = \tfrac{1}{4}\zeta z\bar{z} + \operatorname{Im}\{-Qz\,e^{-i\alpha} - \tfrac{1}{4}i\zeta z^2 - Q(a^2/z)\,e^{i\alpha} + \tfrac{1}{4}i\zeta(a^2/z)^2\}$$

which is in the form of equation 2.21. In terms of polar coordinates (r, θ) putting $z = r\,e^{i\theta}$

$$\psi = \tfrac{1}{4}\zeta r^2 - Qr\sin(\theta-\alpha) + Q(a^2/r)\sin(\theta-\alpha) - \tfrac{1}{4}\zeta r^2\cos 2\theta + \tfrac{1}{4}\zeta(a^4/r^2)\cos 2\theta$$

The radial component of the velocity $(= -r^{-1}\psi_\theta)$ is zero on $r = a$ and the transverse component $(=\psi_r)$ is

$$q = \tfrac{1}{2}\zeta a(1-2\cos 2\theta) - 2Q\sin(\theta-\alpha) \tag{2.23}$$

The pressure at any point on $r = a$ is p where $p = \text{constant} - \tfrac{1}{2}\rho q^2$ and the force components on the cylinder are (X, Y) given by

$$X = -\int_0^{2\pi} p(\cos\theta)a\,d\theta = \tfrac{1}{2}a\rho\int_0^{2\pi} q^2\cos\theta\,d\theta$$

$$Y = -\int_0^{2\pi} p(\sin\theta)a\,d\theta = \tfrac{1}{2}a\rho\int_0^{2\pi} q^2\sin\theta\,d\theta$$

To evaluate the integrals, by equation 2.23,

$$q = C_0 + C_1\cos\theta + S_1\sin\theta + C_2\cos 2\theta$$

where C_0, C_1, S_1, C_2 are respectively $\tfrac{1}{2}\zeta a$, $2Q\sin\alpha$, $-2Q\cos\alpha$, $-\zeta a$,

$$q^2 = A_0 + \sum_{n=1}^{4}\{A_n\cos n\theta + B_n\sin n\theta\}$$

in which case, using the theory of Fourier coefficients,

$$X = \tfrac{1}{2}a\rho\pi A_1 \quad\text{and}\quad Y = \tfrac{1}{2}a\rho\pi B_1$$

But $A_1 = 2C_0C_1 + C_1C_2 = C_1(2C_0 + C_2) = 0$,

i.e.

$$X = 0$$

and $B_1 = 2C_0S_1 - S_1C_2 = S_1(2C_0 - C_2) = -4Qa\zeta\cos\alpha$

i.e.

$$Y = -2\pi Q\rho a^2\zeta\cos\alpha$$

2.4 Blasius's theorem Given a steady liquid motion with complex potential w in a region containing a rigid boundary C, the force components (X, Y) on an element AB of C joining $A\,(z = z_A)$ to $B\,(z = z_B)$ are given by

$$X - iY = \bar{F} = \tfrac{1}{2}i\rho\int_{AB}(dw/dz)^2\,dz - iP(\bar{z}_B - \bar{z}_A) \tag{2.24}$$

where $P = p + \tfrac{1}{2}\rho q^2 = \text{constant}$. Also the moment M about $z = 0$ of the pressure forces on AB is

$$M = -\tfrac{1}{2}\rho\operatorname{Re}\int_{AB} z(dw/dz)^2\,dz + \tfrac{1}{2}P(|z_B|^2 - |z_A|^2) \tag{2.25}$$

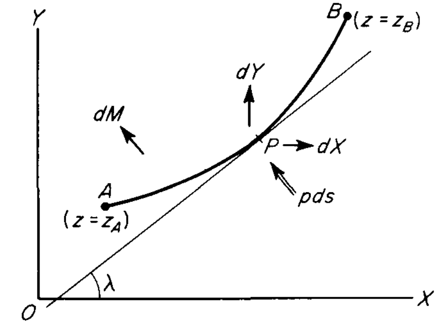


Figure 2.7

To prove this result, consider a point $P \in AB$ (Figure 2.7) where arc $AP = s$ and the tangent to AB at P makes an angle λ with OX. The thrust on ds is $p\,ds$ where p is the liquid pressure at P. The elemental force components on ds are, therefore,

$$dX = -p\,ds\sin\lambda, \qquad dY = p\,ds\cos\lambda$$

i.e.

$$d\bar{F} = dX - i\,dY = -ip\,e^{-i\lambda}\,ds = -ip\,e^{-2i\lambda}\,dz$$

since $dx = ds\cos\lambda$, $dy = ds\sin\lambda$ or $dz = ds\,e^{i\lambda}$ and $d\bar{z} = ds\,e^{-i\lambda}$. By Bernoulli's theorem, $p + \tfrac{1}{2}\rho q^2 = \text{constant} = P$ while on the solid boundary AB, $q\,e^{-i\lambda} = u - iv = dw/dz$ so that

$$p\,e^{-2i\lambda}dz = P\,e^{-2i\lambda}dz - \tfrac{1}{2}\rho(dw/dz)^2\,dz$$
$$= P\,d\bar{z} - \tfrac{1}{2}\rho(dw/dz)^2\,dz$$

Hence, integrating the expression for $d\bar{F}$,

$$X - iY = \bar{F} = -iP\int_{AB} d\bar{z} + \tfrac{1}{2}i\rho\int_{AB}(dw/dz)^2\,dz$$
$$= -iP(\bar{z}_B - \bar{z}_A) + \tfrac{1}{2}i\rho\int_{AB}(dw/dz)^2\,dz$$

The elemental moment about $O\,(z = 0)$ of the force $p\,ds$ is

$$dM = (x\cos\lambda + y\sin\lambda)p\,ds = \operatorname{Re}(p\,e^{-i\lambda}z\,ds) = \operatorname{Re}(p\,e^{-2i\lambda}z\,dz)$$
$$= \operatorname{Re}\{Pz\,d\bar{z} - \tfrac{1}{2}\rho(dw/dz)^2 z\,dz\}$$

Since $\mathrm{Re}(z\,d\bar z) = x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2) = \frac{1}{2}d|z|^2$ the given expression for M follows by integration over the arc AB.

Problem 2.9 Find the force on the quadrant $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ of the circle $z = a\,\mathrm{e}^{\mathrm{i}\theta}$ placed in a uniform stream U whose direction is parallel to the real axis $\theta = 0$ given that the liquid pressure at infinity is Π.

Solution. The complex potential of the uniform stream is $f(z) = -Uz$. From equation 2.20 when the circle $|z| = a$ is added, the complex potential becomes $w = -U(z + a^2/z)$. Using equation 2.24, the force components (X, Y) on the quadrant AB are given by

$$\bar F = X - \mathrm{i}Y = \tfrac{1}{2}\mathrm{i}\rho \int_{AB} U^2(1-a^2/z^2)^2\,dz - \mathrm{i}P(\bar z_B - \bar z_A)$$

$$P = \Pi + \tfrac{1}{2}\rho U^2$$

Putting $z = a\,\mathrm{e}^{\mathrm{i}\theta}$, $z_A = a$, $z_B = a\mathrm{i}$

$$\int_{AB} (1-a^2/z^2)^2\,dz = \int_{\theta=0}^{\theta=\frac{1}{2}\pi} (1-\mathrm{e}^{-2\mathrm{i}\theta})^2 \mathrm{i}a\,\mathrm{e}^{\mathrm{i}\theta}\,d\theta = a\left[\mathrm{e}^{\mathrm{i}\theta} + 2\,\mathrm{e}^{-\mathrm{i}\theta} - \tfrac{1}{3}\mathrm{e}^{-3\mathrm{i}\theta}\right]_0^{\frac{1}{2}\pi}$$

$$= -\tfrac{4}{3}(2+\mathrm{i})a$$

i.e.

$$\bar F = X - \mathrm{i}Y = \tfrac{2}{3}a\rho U^2(1-2\mathrm{i}) - a(1-\mathrm{i})(\Pi + \tfrac{1}{2}\rho U^2)$$

giving

$$X = \tfrac{1}{6}a\rho U^2 - a\Pi, \qquad Y = \tfrac{5}{6}a\rho U^2 - a\Pi \qquad \blacksquare$$

Problem 2.10 Find the force on a closed solid cylinder whose section is a contour C outside which the liquid motion is defined by the complex potential

$$w = Uz\,\mathrm{e}^{-\mathrm{i}\alpha} + \mathrm{i}k\ln z - m\ln(z-a) + \sum_{k=1}^{\infty} \lambda_k z^{-k}$$

where each λ_k is a constant, the infinite sum is convergent, the vortex lies *inside* C and the source lies *outside* C. What is the result when further sources are added outside C?

Solution. By equation 2.24 with $\bar z_A = \bar z_B$ the conjugate complex force $\bar F$ on C is $\frac{1}{2}\mathrm{i}\rho\int_C (dw/dz)^2\,dz$.

Let γ be a small circle $|z-a| = \epsilon$ centred at the source (see Figure 2.8) and let E be a large circle $|z| = R$ enclosing both C and γ. Since there are no singularities in the domain enclosed by E but excluding the interiors of both C and γ, Cauchy's theorem gives

$$\int_{\Gamma} \left(\frac{dw}{dz}\right)^2 dz = 0$$

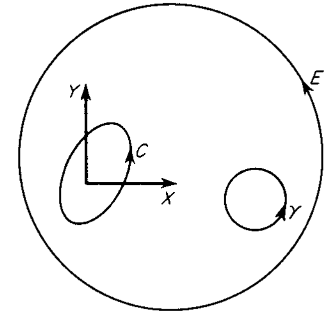


Figure 2.8

where Γ is the total contour $E\downarrow + \gamma\uparrow + C\uparrow$. Hence

$$\int_{C\uparrow} (dw/dz)^2\,dz = \int_{E\uparrow} (dw/dz)^2\,dz - \int_{\gamma\uparrow} (dw/dz)^2\,dz = \bar F/(\tfrac{1}{2}\mathrm{i}\rho)$$

Now

$$\frac{dw}{dz} = U\,\mathrm{e}^{-\mathrm{i}\alpha} + \frac{\mathrm{i}k}{z} - \frac{m}{z-a} - \sum k\lambda_k z^{-k-1}$$

Therefore, on E where $|z|$ is large, $dw/dz = U\,\mathrm{e}^{-\mathrm{i}\alpha} + (\mathrm{i}k - m)/z + O(z^{-2})$ and $(dw/dz)^2 = U^2\,\mathrm{e}^{-2\mathrm{i}\alpha} + 2U\,\mathrm{e}^{-\mathrm{i}\alpha}(\mathrm{i}k-m)z^{-1} + O(z^{-2})$ so that

$$\int_{E\uparrow} \left(\frac{dw}{dz}\right)^2 dz = 4\pi U\mathrm{i}(\mathrm{i}k-m)\,\mathrm{e}^{-\mathrm{i}\alpha}$$

On γ, $-dw/dz = m(z-a)^{-1} + f(z)$ where $f(z)$ is regular within γ and represents the conjugate complex velocity $(u_m - \mathrm{i}v_m)$ when the source is omitted from the flow. Then

$$\int_{\gamma\uparrow} \left(\frac{dw}{dz}\right)^2 dz = \int_{\gamma\uparrow} \left\{\frac{m^2}{(z-a)^2} + \frac{2mf(z)}{z-a} + f^2(z)\right\} dz = 4\pi m\mathrm{i}\,f(a)$$

Hence, $\bar F = X - \mathrm{i}Y = 2\pi\rho(m - \mathrm{i}k)U\,\mathrm{e}^{-\mathrm{i}\alpha} + 2\pi\rho m f(a)$ where $f(a) = u_m - \mathrm{i}v_m$, the conjugate complex velocity at $z = a$ omitting the source there, i.e.

$$X = 2\pi\rho m(U\cos\alpha + u_m) - 2\pi\rho kU\sin\alpha$$
$$Y = 2\pi\rho m(U\sin\alpha + v_m) + 2\pi\rho kU\cos\alpha \qquad (2.26)$$

When further sources are added we need to enclose each of them with a circle γ_m centred on the singularity. The extension to the result is obviously

$$X = 2\pi\rho\sum m(U\cos\alpha + u_m) - 2\pi\rho kU\sin\alpha$$
$$Y = 2\pi\rho\sum m(U\sin\alpha + v_m) + 2\pi\rho kU\cos\alpha \qquad \blacksquare \quad (2.27)$$

Problem 2.11 Using the results of Problem 2.10 find the force on a circle $|z| = a$ due to external sources of strength m_1 and m_2 placed at points $z = b_1$ and $z = b_2$ respectively when both lie on the real axis.

Solution. The complex potential for the sources alone is $f(z) = -m_1 \ln(z-b_1) - m_2(z-b_2)$. Introducing $|z| = a$, the complex potential, using equation 2.20, is

$$w(z) = -m_1 \ln(z-b_1) - m_2 \ln(z-b_2) - m_1 \ln(z-a^2/b_1) - m_2 \ln(z-a^2/b_2) + (m_1+m_2)\ln z$$

so that

$$\frac{dw}{dz} = -\frac{m_1}{z-b_1} - \frac{m_2}{z-b_2} - \frac{m_1 b_1}{b_1 z - a^2} - \frac{m_2 b_2}{b_2 z - a^2} + \frac{m_1+m_2}{z}$$

If (u_1, v_1) are the velocity components at $z = b_1$ excluding the source there

$$-u_1 + \mathrm{i}v_1 = \left(\frac{dw}{dz} + \frac{m_1}{z-b_1}\right)_{z=b_1} = -\frac{m_2}{b_1-b_2} - \frac{m_1 b_1}{b_1^2-a^2} - \frac{m_2 b_2}{b_1 b_2 - a^2} + \frac{m_1+m_2}{b_1}$$

Similarly if (u_2, v_2) are the velocity components at $z = b_2$ excluding its source,

$$-u_2 + \mathrm{i}v_2 = \left(\frac{dw}{dz} + \frac{m_2}{z-b_2}\right)_{z=b_2} = -\frac{m_1}{b_2-b_1} - \frac{m_1 b_1}{b_1 b_2 - a^2} - \frac{m_2 b_2}{b_2^2-a^2} + \frac{m_1+m_2}{b_2}$$

from which $v_1 = v_2 = 0$. Using equation 2.26 with $U = k = 0$, we have $Y = 0$ and

$$X = 2\pi\rho(m_1 u_1 + m_2 u_2)$$
$$= 2\pi\rho a^2 \left\{\frac{m_1^2}{b_1(b_1^2-a^2)} + \frac{m_1 m_2(b_1+b_2)}{b_1 b_2(b_1 b_2 - a^2)} + \frac{m_2^2}{b_2(b_2^2-a^2)}\right\} \quad \square$$

2.5 Orthogonal coordinates A flow in the z-plane is defined by the complex potential $w = \varphi + \mathrm{i}\psi = f(z)$, $z = x + \mathrm{i}y$ so that $\varphi = \varphi(x, y)$, $\psi = \psi(x, y)$. Using the function $z = g(\zeta)$, $\zeta = \xi + \mathrm{i}\eta$ we transform to the ζ-plane with w taking the same values at corresponding points. The new shapes of the streamlines and equipotentials are determined by

$$w = \varphi + \mathrm{i}\psi = f\{g(\zeta)\} = F(\zeta)$$

i.e.

$$\varphi = \varphi(\xi, \eta) = \operatorname{Re} F(\zeta), \qquad \psi = \psi(\xi, \eta) = \operatorname{Im} F(\zeta)$$

Since $dw/dz = (dw/d\zeta)(d\zeta/dz)$ except at points where $dz/d\zeta$ or $d\zeta/dz$ vanish, in general the fluid will have different velocities at the corresponding points of the transformation. The Cauchy–Riemann equations are

(i)	$\xi_x = \eta_y$, $\quad \xi_y = -\eta_x$	from $z = g(\zeta)$	
(ii)	$\varphi_\xi = \psi_\eta$, $\quad \varphi_\eta = -\psi_\xi$	from $w = F(\zeta)$	
(iii)	$\varphi_x = \psi_y$, $\quad \varphi_y = -\psi_x$	from $w = f(z)$	(2.28)

Moreover, when $\nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0$ we have $\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0$ and similarly $\psi_{\xi\xi} + \psi_{\eta\eta} = 0$ except at the singular points of the transformation, namely, the zeros of $dz/d\zeta$ or $d\zeta/dz$.

2.6 Boundary condition on a moving cylinder In Figure 2.9, C represents the cross-section of a cylinder moving with velocity (U, V) and angular velocity Ω referred to fixed axes Oxy. With ds the element of C at a point $P(x, y)$ on it, the boundary condition states that the velocity component of P normal to C equals the liquid velocity component in the same direction, i.e.

$$(U - \Omega y)\frac{dy}{ds} - (V + \Omega x)\frac{dx}{ds} = -\frac{\partial\psi}{\partial s}$$

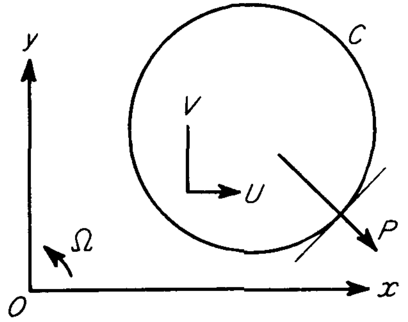


Figure 2.9

By integration along the boundary, on C, ψ satisfies

$$\psi = Vx - Uy + \tfrac{1}{2}\Omega(x^2 + y^2) + \text{constant} \qquad (2.29)$$

2.7 Kinetic energy Using the result in Section 1.11, the kinetic energy per unit thickness of two-dimensional liquid motion is

$$\mathscr{T} = \tfrac{1}{2}\rho \int_{\mathscr{L}} \varphi \nabla\varphi \,.\, \mathbf{ds} \tag{2.30}$$

where $\mathscr{L}$ is the total boundary surrounding the liquid (Figure 2.10), $\mathbf{ds} = ds\,\mathbf{n}$, $\mathbf{n}$ being the unit outward normal to $\mathscr{L}$ and ds an element of $\mathscr{L}$.

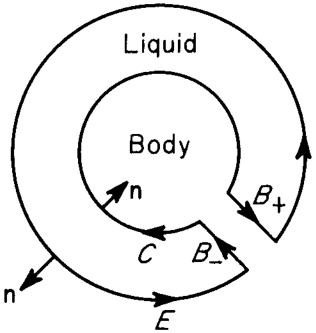


Figure 2.10

The total length $\mathscr{L}$ is made up of the following components

$$\mathscr{L} \equiv \text{body}(C\downarrow) + \text{envelope}(E\uparrow) + \text{bridge out}(B_{+\downarrow}) + \text{bridge in}(B_{-\uparrow})$$

from which the separate contributions to $\mathscr{T}$ are respectively $\mathscr{T}_C, \mathscr{T}_E, \mathscr{T}_{B_+}, \mathscr{T}_{B_-}$.

(i) If φ is univalent (one-valued) then $\mathscr{T}_{B_+} + \mathscr{T}_{B_-} = 0$.

(ii) If φ is *not* univalent then, denoting by φ_+ the value of φ on B_+ etc. we have $\varphi_+ - \varphi_- = 2\pi k$, the circulation, so that $\mathscr{T}_{B_+} + \mathscr{T}_{B_-} = \pi\rho k \int_{B_+} \nabla\varphi \,.\, \mathbf{ds}$.

(iii) If the liquid extends to infinity and is at rest there, then $\mathscr{T}_E = 0$.

(iv) When conditions (i) and (iii) hold together, we have

$$\mathscr{T} = \mathscr{T}_C = \tfrac{1}{2}\rho \int_{C\downarrow} \varphi \frac{\partial \psi}{\partial s}\, ds \quad \left(\nabla\varphi \,.\, \mathbf{n} = \frac{\partial \varphi}{\partial n} = \frac{\partial \psi}{\partial s}\right)$$

$$= \tfrac{1}{2}\rho \int_{C\downarrow} \varphi\, d\psi = -\tfrac{1}{2}\rho \int_{C\downarrow} \psi\, d\varphi \quad \text{(integration by parts)}$$

$$= \tfrac{1}{4}\rho \int_{C\downarrow} (\varphi\, d\psi - \psi\, d\varphi) = -\tfrac{1}{4}\rho \mathrm{i} \int_{C\uparrow} w\, d\bar{w} \tag{2.31}$$

since $w\, d\bar{w} = \tfrac{1}{2} d(\varphi^2 + \psi^2) + \mathrm{i}(\psi\, d\varphi - \varphi\, d\psi)$.

Problem 2.12 The elliptic cylinder $(x/a)^2 + (y/b)^2 = 1$ moves with constant velocity components $(Q\cos\alpha, Q\sin\alpha)$ through a liquid at rest at infinity. Find the stream function of the motion.

Solution. From the transformation $z = c\cosh\zeta$ where $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta$, we have $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$. Eliminating η, $\{x/(c\cosh\xi)\}^2 + \{y/(c\sinh\xi)\}^2 = 1$, i.e. ξ = constant are ellipses. The given ellipse is defined by $\xi = \xi_0$ where $c\cosh\xi_0 = a$, $c\sinh\xi_0 = b$, $c^2 = a^2 - b^2$. Using Sections 2.5 and 2.6 we seek a solution $\psi = \psi(\xi, \eta)$ such that (i) $\psi_{\xi\xi} + \psi_{\eta\eta} = 0$, whilst (ii) on the ellipse $\xi = \xi_0$,

$$\psi = Vx - Uy = Vc\cosh\xi\cos\eta - Uc\sinh\xi\sin\eta$$
$$U = Q\cos\alpha, \qquad V = Q\sin\alpha$$

or $\psi = Qc(\cosh\xi_0 \sin\alpha\cos\eta - \sinh\xi_0\cos\alpha\sin\eta)$ for all η. Finally, (iii) as $|z| \to \infty$, $dw/dz \to 0$ since the liquid is at rest there. However, from the transformation formula, $|z| \to \infty$ corresponds to taking $\xi \to \infty$ (for both $\cos\eta$ and $\sin\eta$ are bounded) and $dw/dz \to 0$ corresponds to $dw/d\zeta \to 0$. Condition (ii) suggests that we seek a solution of (i) of the form $\psi = f(\xi)(A\cos\eta + B\sin\eta)$, A, B constants. Substituting into (i) we find this equation identically satisfied when $f(\xi) = Ce^{\xi} + De^{-\xi}$. To satisfy (iii) we must choose $C = 0$ and when both A and B are arbitrary we may choose $D = 1$ so that $\psi = e^{-\xi}(A\cos\eta + B\sin\eta)$. Finally, to satisfy (ii) putting $\xi = \xi_0$ and equating coefficients of $\cos\eta$ and $\sin\eta$ respectively, $Ae^{-\xi_0} = Qc\cosh\xi_0\sin\alpha$, $Be^{-\xi_0} = -Qc\sinh\xi_0\cos\alpha$ so that the required solution for ψ is

$$\psi = Qce^{-(\xi-\xi_0)}(\cosh\xi_0\sin\alpha\cos\eta - \sinh\xi_0\cos\alpha\sin\eta) \quad \square \tag{2.32}$$

Problem 2.13 Find the kinetic energy of the liquid motion in the preceding problem.

Solution. The kinetic energy is given by formula 2.31. For this we need to evaluate w, complex potential, the imaginary part of which is given by equation 2.32. Now $e^{-\zeta} = e^{-\xi}(\cos\eta - \mathrm{i}\sin\eta)$, i.e. $e^{-\xi}\cos\eta = \mathrm{Im}(\mathrm{i}e^{-\zeta})$, $e^{-\xi}\sin\eta = \mathrm{Im}(-e^{-\zeta})$ so that

$$w = Qce^{\xi_0}(\mathrm{i}e^{-\zeta}\cosh\xi_0\sin\alpha + e^{-\zeta}\sinh\xi_0\cos\alpha)$$
$$= Qce^{-\zeta+\xi_0}(\sinh\xi_0\cos\alpha + \mathrm{i}\cosh\xi_0\sin\alpha)$$
$$= Q(a+b)e^{-\zeta}\sinh(\xi_0 + \mathrm{i}\alpha) \tag{2.33}$$

since $ce^{\xi_0} = c(\cosh\xi_0 + \sinh\xi_0) = a + b$.

The kinetic energy of the liquid is $\mathscr{T} = -\tfrac{1}{4}\rho\mathrm{i}\int_{C\uparrow} w\, d\bar{w}$ where on C $\xi = \xi_0$

and η *increases* from 0 to 2π. Using equation 2.33

$$w\,d\bar{w} = Q^2(a+b)^2\sinh(\xi_0+i\alpha)\sinh(\xi_0-i\alpha)e^{-\zeta}(-e^{-\bar{\zeta}}d\bar{\zeta})$$

where, on C, $\zeta = \xi_0+i\eta$, $d\bar{\zeta} = -i\,d\eta$. Hence,

$$\mathscr{T} = \tfrac{1}{8}\rho Q^2(a+b)^2(\cosh 2\xi_0-\cos 2\alpha)e^{-2\xi_0}\int_0^{2\pi} d\eta$$

$$= \tfrac{1}{4}\pi\rho Q^2(a+b)^2(\cosh 2\xi_0-\cos 2\alpha)e^{-2\xi_0}$$

Using, $a+b = ce^{\xi_0}$, $c^2\cosh 2\xi_0 = c^2(\cosh^2\xi_0+\sinh^2\xi_0) = a^2+b^2$, and $c^2\cos 2\alpha = (a^2-b^2)(\cos^2\alpha-\sin^2\alpha)$, we have, finally,

$$\mathscr{T} = \tfrac{1}{2}\pi\rho Q^2(a^2\sin^2\alpha+b^2\cos^2\alpha) \qquad \square \qquad (2.34)$$

2.8 Rotating cylinders By equation 2.29 the boundary condition on a cylinder rotating with angular velocity Ω is $\psi = \frac{1}{2}\Omega z\bar{z}$. If the equation of C can be expressed in the form $z\bar{z} = f(z)+\bar{f}(\bar{z})$ this boundary condition is satisfied by taking $\psi = \operatorname{Im} w$ where

$$w = i\Omega f(z) \qquad (2.35)$$

When $f'(z)$ has no singularities *inside* C, w will represent the complex potential of liquid motion *inside* C (for in this case there are no singularities of the velocity due to sources or vortices etc.) whereas if $f'(z)$ has no singularities *outside* C, w will represent the motion *outside* C.

Example 1. When $f(z) = \{\frac{1}{2}(a^2-b^2)z^2+a^2b^2\}/(a^2+b^2)$, C, represented by the equation $z\bar{z} = f(z)+\bar{f}(\bar{z})$, is the ellipse $(x/a)^2+(y/b)^2 = 1$.

Example 2. When $f(z) = (4a^3-z^3)/6a$, C has the equation $x^3-3xy^2+3a(x^2+y^2)-4a^3 \equiv (x-y\sqrt{3}+2a)(x+y\sqrt{3}+2a)(x-a) = 0$ the three sides of an equilateral triangle of side $2a\sqrt{3}$ with centroid at $z = 0$.

Problem 2.14 A prism whose section is an equilateral triangle of side $2a\sqrt{3}$ contains liquid and rotates about a generator through a vertex. Find the effective radius of gyration of the liquid.

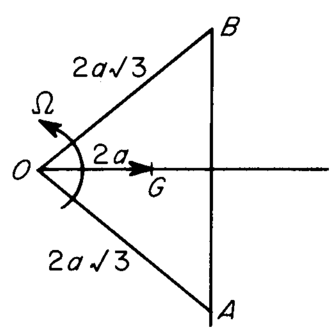


Figure 2.11

Solution. With the centroid G as origin, the equation of the triangular section is defined by $z\bar{z} = f(z)+\bar{f}(\bar{z})$ where $f(z) = (4a^3-z^3)/6a$. Choosing a vertex O as origin where $OG = 2a$, the equation becomes

$$(z-2a)(\bar{z}-2a) = \{4a^3-(z-2a)^3+4a^3-(\bar{z}-2a)^3\}/6a$$

i.e. $z\bar{z} = g(z)+\bar{g}(\bar{z})$ where $g(z) = z^2-(z^3/6a)$.

By Section 2.8 the complex potential of the liquid motion when the prism rotates with angular velocity Ω about O is $w = i\Omega z^2(6a-z)/6a$. Using equation 2.31 and changing the sign because the liquid lies *within* the boundary, the kinetic energy of the liquid is

$$\mathscr{T} = \tfrac{1}{4}\rho i\int_{C\uparrow} w\,d\bar{w} = \tfrac{1}{4}\rho i\Omega^2\int_{C\uparrow}\left(z^2-\frac{z^3}{6a}\right)\left(2\bar{z}-\frac{\bar{z}^2}{2a}\right)d\bar{z}$$

On OA, $z = re^{-\frac{1}{6}\pi i}$; on AB $z = 3a+iy$ and on BO, $z = re^{\frac{1}{6}\pi i}$. We denote these respective contributions to $\mathscr{T}$ by $\mathscr{T}_{OA}$, $\mathscr{T}_{AB}$ and $\mathscr{T}_{BO}$. Writing $\omega = e^{\frac{1}{6}\pi i}$, on OA, $z = r/\omega$, $\bar{z} = r\omega$,

$$\mathscr{T}_{OA} = \tfrac{1}{4}\rho i\Omega^2\int_0^{2a\sqrt{3}}\left(\frac{r^2}{\omega^2}-\frac{r^3}{6a\omega^3}\right)\left(2r\omega-\frac{r^2\omega^2}{2a}\right)\omega\,dr$$

$$= \tfrac{1}{4}\rho i\Omega^2\int_0^{2a\sqrt{3}}\left(2r^3-\frac{r^4}{3a\omega}-\frac{r^4\omega}{2a}+\frac{r^5}{12a^2}\right)dr$$

Similarly, inverting ω and the limits of integration,

$$\mathscr{T}_{BO} = -\tfrac{1}{4}\rho i\Omega^2\int_0^{2a\sqrt{3}}\left(2r^3-\frac{r^4\omega}{3a}-\frac{r^4}{2a\omega}+\frac{r^5}{12a^2}\right)dr$$

Adding,

$$\mathscr{T}_{OA}+\mathscr{T}_{BO} = \tfrac{1}{4}\rho i\Omega^2\left(\frac{1}{6a\omega}-\frac{\omega}{a}\right)\frac{(2a\sqrt{3})^5}{5} = \tfrac{12}{5}\rho a^4\Omega^2\sqrt{3}$$

Since $\omega-\omega^{-1} = 2i\sin\frac{1}{6}\pi = i$. Again

$$\mathscr{T}_{AB} = \tfrac{1}{4}\rho i\Omega^2\int_{-a\sqrt{3}}^{a\sqrt{3}}\left(\tfrac{9}{2}a^2+\tfrac{3}{2}aiy+\tfrac{1}{2}y^2+\frac{iy^3}{6a}\right)\left(\frac{3a}{2}+iy+\frac{y^2}{2a}\right)(-i)\,dy$$

$$= \tfrac{1}{4}\rho\Omega^2\int_{-a\sqrt{3}}^{a\sqrt{3}}\left(\tfrac{27}{4}a^3+\tfrac{3}{2}ay^2+\frac{y^4}{12a}\right)dy$$

(odd powers of y give no contribution)

$$= \tfrac{21}{5}\rho a^4\Omega^2\sqrt{3}$$

Adding, the final $\mathscr{T}$ is $(33/5)\rho a^4\Omega^2\sqrt{3} = \frac{1}{2}mk^2\Omega^2$ where $m = 3\sqrt{3}a^2\rho$ is the liquid mass within C per unit thickness. Hence, k, the effective radius of gyration, is $a\sqrt{(22/5)}$. $\square$

2.9 Conformal mapping. A mapping from the z-plane to the ζ-plane is defined by $\zeta = f(z)$ where $f(z)$ is finite and single valued in some domain A enclosed by a contour C in the z-plane. Any point P within A is transformed uniquely into a definite point Ω of the ζ-plane. We write this as $P \to \Omega$. To C, there corresponds a contour Γ in the ζ-plane enclosing a domain Λ. We shall assume that the mapping is one-to-one between domains so that in the inverse transformation for which $z = g(\zeta)$, $\Omega \to P$ uniquely. This implies conditions (which cannot be proved in the space available here) on the derivative $df/dz = f'(z)$, namely, for all z in A, $f'(z)$ must be finite and nonzero. The zeros of $1/f'(z)$, $f'(z)$ are singularities of the transformation and are normally cut off from the domain.

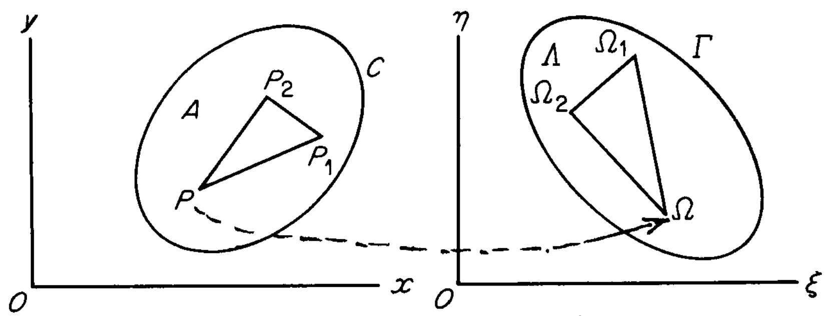


Figure 2.12

Let $P(z)$, $P_1(z_1)$, $P_2(z_2)$ be three neighbouring points in A. Under transformation $P \to \Omega(\zeta)$, $P_1 \to \Omega_1(\zeta_1)$, $P_2 \to \Omega_2(\zeta_2)$, where $\zeta_1 = f(z_1)$ etc. When $|z_1 - z|$ is small,

$$\frac{\zeta_1-\zeta}{z_1-z} = \frac{f(z_1)-f(z)}{z_1-z} = f'(z)+O(|z_1-z|), \quad \frac{\zeta_2-\zeta}{z_2-z} = f'(z)+O(|z_2-z|).$$

Taking moduli and argument, provided $|f'(z)| < \infty$ and $f'(z) \neq 0$ we have to a first order of approximation,

$$\frac{\Omega\Omega_1}{PP_1} = \frac{\Omega\Omega_2}{PP_2} = |f'(z)|$$

$\arg \Omega\Omega_1 - \arg PP_1 = \arg \Omega\Omega_2 - \arg PP_2 = \arg f'(z)$ or $\angle \Omega_1 \Omega\Omega_2 = \angle P_1 P P_2$. Hence, subject to the stated restrictions, the mapping preserves (i) the angles between elemental lines when transformed and (ii) the similarity of corresponding infinitesimal triangles. The term *conformal mapping* is applied to this representation.

Suppose $w = g(\zeta)$ is the complex potential of a liquid motion in the

ζ-plane. We can construct a *corresponding* motion in the z-plane by arranging that w takes the *same (complex) value* at corresponding points P and Ω. The new motion in the z-plane is then described by $w = g\{f(z)\}$. Since $w_\Omega = w_P$ implies $\psi_\Omega = \psi_P$, a streamline in Λ maps into a streamline along the corresponding curve in A. In particular, if Γ were a streamline C would also be a streamline.

If q_ζ is the liquid speed at Ω and q_z the speed at P, we have $q_\zeta^2 = |dw/d\zeta|^2$, $q_z^2 = |dw/dz|^2$ so that

$$\frac{q_z^2}{q_\zeta^2} = \left|\frac{d\zeta}{dz}\right|^2 = |f'(z)|^2 \tag{2.36}$$

except at the singular points of the transformation where either speed is zero. The kinetic energy is preserved in transformation, for if dS_ζ, and dS_z are corresponding elements of area at Ω and P, we have $dS_\zeta/dS_z = (\Omega\Omega_1/PP_1)^2 = |f'(z)|^2$ since the elements are geometrically similar. Using (2.36), $q_\zeta^2\, dS_\zeta = q_z^2\, dS_z$ implying conservation of kinetic energy in transformation.

Problem 2.15 Sources of strength m_0 and m_1 are placed at points $O(z = 0)$ and $P(z = z_1)$ of the z-plane respectively. Examine the corresponding motion in the ζ-plane when $\zeta = z^n$ (n is a positive integer).

Solution. For the source m_1 at $z = z_1$ in the z-plane $\int_{C\uparrow} d\psi = -2\pi m_1$ where C is the circle $|z - z_1| = \epsilon$; ϵ is small. Under the transformation $\zeta = z^n$ this circle becomes the circle $\Gamma: |\zeta - \zeta_1| = \delta$ in the ζ-plane with radius $\delta = O(\epsilon)$ and centre at $\zeta = \zeta_1 = z_1^n$. Moreover, $\int_{C\uparrow} d\omega = \int_{\kappa\uparrow} d\psi$ since Γ corresponds to C and by definition ψ takes the same values at corresponding points of the two planes. The singular points of the transformation are the zeros of $d\zeta/dz = nz^{n-1}$ and $dz/d\zeta = z^{1-n}/n$, i.e. $z = 0 = \zeta$ and the points at infinity in the two planes. When z_1 is finite and nonzero i.e. a nonsingular point of the transformation, *unit* description of C produces *unit* description of Γ. Hence $\int_{\Gamma\uparrow} d\psi = -2\pi m_1$ meaning that a source of strength m_1 at z_1 transforms into a source of equal strength m_1 at $\zeta_1 = z_1^n$.

For the source m_0 at $z = 0$, $\int_{C\uparrow} d\psi = -2\pi m_0$ where C is the circle $|z| = \epsilon$, ϵ is small, for which the corresponding circle Γ is $|\zeta| = \epsilon^n$. Since $z = 0 = \zeta$ is a singularity and $\arg \zeta = n \arg z$, Γ is described n *times* for a *single* circuit of C in which case $n \int_{\Gamma\uparrow} d\psi = -2\pi m_0$ or the corresponding source at $\zeta = 0$ has strength m_0/n. □

Problem 2.16 Liquid in the z-plane is contained in the sector with vertex at $z = 0$ and bounded by lines $\arg z = \pm\frac{1}{2}\pi/n$ (n is a positive integer) and an arc of the circle $|z| = a$. A source of strength $2nm$ is placed at the vertex together with a sink of strength m at $z = b(<a)$ on the real axis. Find an expression for the velocity on the curved boundary.

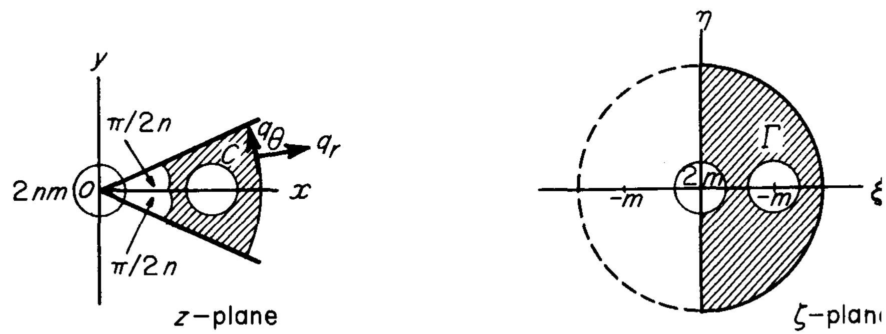


Figure 2.13

Solution. The total volume output into the sector due to the source $2nm$ at 0 is $2\pi(2nm)(1/2n) = 2\pi m$ which balances the input of the sink inside; hence the sector boundary *can be* rigid. Applying the transformation $\zeta = z^n$ where $\arg\zeta = n\arg z$, the lines $\arg z = \pm\frac{1}{2}\pi/n$ become $\arg\zeta = \pm\frac{1}{2}\pi$ and the arc of $|z| = a$ becomes the semicircular arc of the circle $|\zeta| = a^n = c$. The *inside* of the sector is mapped onto the *inside* of the semicircle of radius c in the ζ-plane.

Using the results of Problem 2.15 the source $2nm$ at $z = 0$ is transformed into a source of strength $2nm/n = 2m$ at $\zeta = 0$ and the sink of strength m at $z = b$ is transformed into an equal sink at $\zeta = b^n$. In the ζ-plane, to make $\operatorname{Re}\zeta = 0$ a rigid boundary, we introduce the image sink $-m$ at $\zeta = -b^n$. This makes the complex potential

$$f(\zeta) = -2m\ln\zeta + m\ln(\zeta - b^n) + m\ln(\zeta + b^n) = m\ln(1 - b^{2n}\zeta^{-2}).$$

To find its image in the circle $|\zeta| = c = a^n$ we use the circle theorem (2.20) which gives the final complex potential

$$w = f(\zeta) + \bar{f}(c^2/\zeta) = m\ln(1 - b^{2n}\zeta^{-2}) + m\ln(1 - b^{2n}c^{-4}\zeta^2)$$

In terms of z,

$$w = m\ln\{(1 - b^{2n}z^{-2n})(1 - b^{2n}a^{-4n}z^{2n})\}$$
$$= m\ln\{\lambda - (z/a)^{2n} - (a/z)^{2n}\} + \text{constant}, \quad \text{where } \lambda = (a/b)^{2n} + (b/a)^{2n}$$

Consequently, on the arc, where $z = ae^{i\theta}$,

$$\frac{dw}{dz} = \left.\frac{2nm(a^{2n}z^{-2n-1} - a^{-2n}z^{2n-1})}{\lambda - z^{2n}a^{-2n} - a^{2n}z^{-2n}}\right|_{z=ae^{i\theta}} = \frac{2nme^{-i\theta}}{a}g(\theta)$$

where

$$g(\theta) = \frac{e^{-2ni\theta} - e^{2ni\theta}}{\lambda - e^{2ni\theta} - e^{-2ni\theta}} = -\frac{2i\sin 2n\theta}{\lambda - 2\cos 2n\theta}$$

On writing $dw/dz = -(u - iv) = -(q_r - iq_\theta)e^{-i\theta}$ where q_r and q_θ are respectively the radial and transverse components of the liquid velocity we find that

$$q_r = 0, \qquad q_\theta = -\frac{4nm\sin 2n\theta}{a(\lambda - 2\cos 2n\theta)}, \qquad \lambda = (a/b)^{2n} + (b/a)^{2n}$$ □

2.10 Joukowski transformation $\zeta = z + c^2/z$ or $(\zeta - 2c)/(\zeta + 2c) = \{(z - c)/(z + c)\}^2$ is called the *Joukowski* transformation mapping function. The singularities are $z = 0, \pm c, \infty$ at which $\zeta = \infty, \pm 2c, \infty$. The inverse transformation is $z = \frac{1}{2}\{\zeta \pm \sqrt{(\zeta^2 - 4c^2)}\}$ so that *one* value of ζ corresponds to *two* values of z. If we choose the positive sign associated with the square root then for large ζ, $z \sim \zeta$ so that infinities in the two planes will correspond. Choosing the negative sign, $z \sim 2c^2/\zeta$, i.e. the infinity of the ζ plane will transform into the neighbourhood of $z = 0$.

Case 1. If C : circle $z = fe^{i\theta}$ $(f > c)$ then Γ : ellipse $(\xi/a)^2 + (\eta/b)^2 = 1$; $a = f + c^2/f$, $b = f - c^2/f$.

Case 2. If C : circle $z = ce^{i\theta}$ then Γ : straight line $\xi = 2c\cos\theta$, $\eta = 0$.

Case 3. If C : circle $|z - z_0| = r$ where $|c - z_0| = r$, i.e. $z = c$ lies on C and $|-c - z_0| < r$, i.e. $z = -c$ lies inside C then Γ : aerofoil section with a cusp at $\zeta = 2c$.

2.11 Kutta condition In Case 3 of the last section the liquid speed q_r at the cusp $\zeta = 2c$ where $d\zeta/dz = 0$ will be infinite (see equation 2.36) unless the speed $q = |dw/dz|$ at the corresponding point $z = c$ on C is zero. Normally it is possible to achieve this condition by introducing a circulation about C and adjusting its strength accordingly.

Problem 2.17 Find the force on the symmetrical Joukowski aerofoil section formed from the circle C : $|z - \lambda| = c + \lambda$, λ real and positive due to an incompressible air stream with velocity components $(-U\cos\alpha, U\sin\alpha)$.

Solution. Here $z = -c$ lies on C and $z = c$ lies within C (Figure 2.14). With $z = (c + \lambda)e^{i\theta} + \lambda$ the Joukowski transformation $\zeta = \xi + i\eta = z + c^2/z$

produces the symmetrical profile with a cusp at $\zeta = -2c$

$$\zeta = \xi + i\eta = (c+\lambda)e^{i\theta} + \lambda + \frac{c^2\{(c+\lambda)e^{-i\theta}+\lambda\}}{\{(c+\lambda)e^{i\theta}+\lambda\}\{(c+\lambda)e^{-i\theta}+\lambda\}}$$

i.e. $\xi = \{(c+\lambda)\cos\theta + \lambda\}\{1+f(\theta)\}$, $\quad \eta = (c+\lambda)(\sin\theta)\{1-f(\theta)\}$,
where $f(\theta) = c^2/\{(c+\lambda)^2+\lambda^2+2\lambda(c+\lambda)\cos\theta\}$.

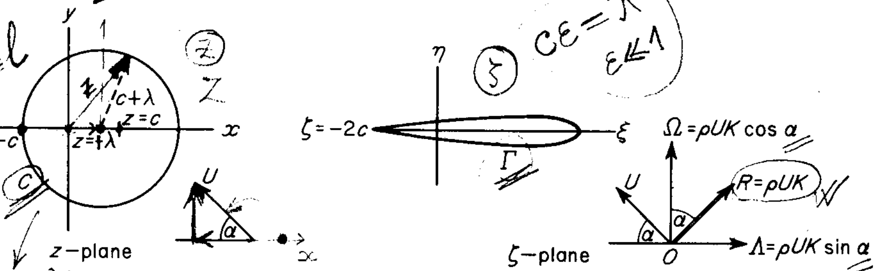


Figure 2.14

To make the infinities in the two planes correspond we choose the inverse transformation as $z = \frac{1}{2}\{\zeta + \sqrt{(\zeta^2-4c^2)}\}$; then $z \sim \zeta$ for large $|z|$ or $|\zeta|$. In particular the uniform stream in the z-plane with complex potential $Ue^{i\alpha}z$ for large z behaves like $Ue^{i\alpha}\zeta$, i.e. the uniform stream has the same magnitude and direction in both planes. To find the flow past the circle in the z-plane we represent it by the equation $|Z| = c+\lambda$ where $Z = z-\lambda$. By the circle theorem (2.20) and adding a circulation term $(iK/2\pi)\ln Z$, the complex potential is

$$w(Z) = Ue^{i\alpha}(Z+\lambda) + Ue^{-i\alpha}\{(c+\lambda)^2Z^{-1}+\lambda\} + (iK/2\pi)\ln Z$$

Hence,

$$w(z) = Ue^{i\alpha}z + Ue^{-i\alpha}\frac{(c+\lambda)^2}{z-\lambda} + \left(\frac{iK}{2\pi}\right)\ln(z-\lambda) + \text{constant}$$

To satisfy the Kutta condition,

$$\left.\frac{dw}{dz}\right|_{z=-c} = 0 = Ue^{i\alpha} - Ue^{-i\alpha} - \frac{iK}{2\pi(\lambda+c)}$$

i.e. $K = 4\pi U(c+\lambda)\sin\alpha$, which determines the strength of the circulation.

The flow in the ζ-plane is found by eliminating z using $z = \frac{1}{2}\{\zeta + \sqrt{(\zeta^2-4c^2)}\}$. The complex force F on this aerofoil is given by the Blasius formula (2.24). Writing $F = \Lambda + i\Omega$, we have

$$\bar{F} = \Lambda - i\Omega = \tfrac{1}{2}\rho i \int_{\Gamma} \left(\frac{dw}{d\zeta}\right)^2 d\zeta = \tfrac{1}{2}\rho i \int_{C} \left(\frac{dw}{dz}\right)^2 \frac{dz}{d\zeta}\, dz$$

Since there are no singularities outside C we can deform this circle into the circle E defined by $|z| = R$ where R is large. For large $|z|$

$$\frac{dw}{dz} = Ue^{i\alpha} + \frac{iK}{2\pi z} + O(|z|^{-2}) \quad \text{and} \quad \frac{dz}{d\zeta} = 1 + O(|z|^{-2})$$

$$\bar{F} = \Lambda - i\Omega = \tfrac{1}{2}\rho i \int_E \left(Ue^{i\alpha} + \frac{iK}{2\pi z} + O(|z|^{-2})\right) dz = \tfrac{1}{2}\rho i(2\pi i)\left(\frac{UiK}{\pi}\right)e^{i\alpha}$$

so that

$$\Lambda = \rho UK\sin\alpha, \qquad \Omega = \rho UK\cos\alpha$$

which give a resultant lift force R of magnitude ρUK perpendicular to the uniform stream. □

2.12 The Schwarz–Christoffel transformation Here the boundary of a polygon in the z-plane with vertices $z_1, z_2, \ldots, z_n$ and internal angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$ is mapped onto $\operatorname{Im}\zeta = 0$ by

$$\frac{dz}{d\zeta} = L(\zeta-\xi_1)^{\alpha_1-1}(\zeta-\xi_2)^{\alpha_2-1}\ldots(\zeta-\xi_n)^{\alpha_n-1} \tag{2.37}$$

where the vertices of the polygon become the points $\xi_1, \xi_2, \ldots, \xi_n$, respectively on $\operatorname{Im}\zeta = 0$. Furthermore when the polygon is simple its interior is mapped onto $\operatorname{Im}\zeta > 0$.

Problem 2.18 Liquid streams in the region $\operatorname{Im} z > 0$ with velocity U parallel to the real axis. Assuming that the real axis is solid together with that part of the imaginary axis for which $0 \leqslant \operatorname{Im} z \leqslant a$, find the complex potential of motion.

Solution Referring to Figure 2.15, $A_1A_2A_3$ is an isosceles triangle of height a lying in $\operatorname{Im} z \geqslant 0$ with its base A_1A_3 on the real axis, $A_1O = OA_3$ where O is the origin, $\angle A_1A_2A_3 = (2-\alpha_2)\pi$, and $\angle A_2A_1A_3 = \angle A_2A_3A_1 = (1-\alpha_1)\pi$.

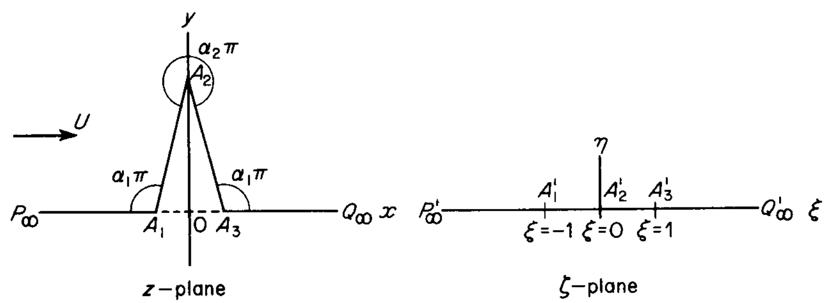


Figure 2.15

The domain of the actual liquid motion is the region $\operatorname{Im} z > 0$ outside the triangle whose base is then allowed to shrink to zero into coincidence with the origin O, i.e. $\alpha_1 \to \frac{1}{2}$ and $\alpha_2 \to 2$. This domain is mapped by the Schwarz–Christoffel transformation onto $\operatorname{Im}\zeta > 0$ where $A_1'(\zeta = -1)$, $A_2'(\zeta = 0)$, $A_3'(\zeta = 1)$ are the points on $\operatorname{Im}\zeta = 0$ corresponding respectively to the limits of the points $A_1\,(z = 0-)$, $A_2\,(z_2 = ai)$, $A_3\,(z = 0+)$. By equation 2.37 the mapping function is

$$dz/d\zeta = L(\zeta-1)^{-\frac{1}{2}}\zeta(\zeta+1)^{-\frac{1}{2}}$$

which, integrated, gives

$$z = L(\zeta^2-1)^{\frac{1}{2}}+M, \quad L, M \text{ are constants.}$$

Since $\zeta = 1$ when $z = 0+$, $M = 0$ and $\zeta = 0$ when $z = ai$ gives $L = a$. Hence $z = a(\zeta^2-1)^{\frac{1}{2}}$ and $\zeta = (a^2+z^2)^{\frac{1}{2}}/a$. For large $|\zeta|$, $z \sim a\zeta$ so that $w = -Uz$, the uniform stream in the z-plane, becomes $w = -Ua\zeta$ in the ζ-plane or the streaming speed here is aU. Moreover, since $\operatorname{Im}\zeta = 0$ is a rigid boundary $w = -Ua\zeta$. In terms of z, $w = -U(z^2+a^2)^{\frac{1}{2}}$. □

2.13 Impulsive motion The impulsive pressure ϖ at any point of a liquid set in motion impulsively from rest is $\rho\varphi$ where $\mathbf{q} = -\operatorname{grad}\varphi$. Since $\operatorname{div}\mathbf{q} = 0$ by the equation of continuity, $\nabla^2\varphi = 0$. Therefore, in polar coordinates, $\varphi = \sum_{n=1}^{N} \varphi_n(r,\theta)$ where

$$\varphi_n(r,\theta) = (A_n \cos n\theta + B_n \sin n\theta)r^n + (C_n \cos n\theta + D_n \sin n\theta)r^{-n}.$$

A_n, B_n, C_n, D_n are constants.

Problem 2.19 A circular cylinder $|z| = a$ lies at rest in a liquid which is set in motion from rest with the velocity potential φ defined in Section 2.13. Show that the component of the impulsive liquid thrust (per unit thickness) on the cylinder in the direction $\theta = \alpha$ is $-\pi\rho a\varphi_1(a,\alpha)$.

Solution. The impulsive thrust on an element $a\,d\theta$ of the cylinder at $P(z = ae^{i\theta})$ is $-\rho a\varphi(a,\theta)\,d\theta$ along OP where O is the centre $z = 0$. If I, J are the components of the total thrust parallel to the real and imaginary axes respectively, we have, on integration, $I+iJ$

$$= -\rho a \int_0^{2\pi} (\cos\theta + i\sin\theta)\,\varphi(a,\theta)\,d\theta$$

$$= -\rho a \int_0^{2\pi} (\cos\theta + i\sin\theta) \sum_{n=1}^{N} \{(A_n \cos n\theta + B_n \sin n\theta)a^n + (C_n \cos n\theta + D_n \sin n\theta)a^{-n}\}\,d\theta$$

Inverting the order of summation with integration and using

$$\int_0^{2\pi} \cos n\theta \sin\theta\,d\theta = 0 = \int_0^{2\pi} \sin n\theta \cos\theta\,d\theta \quad \text{for all } n$$

$$\int_0^{2\pi} \cos n\theta \cos\theta\,d\theta = 0 = \int_0^{2\pi} \sin n\theta \sin\theta\,d\theta \quad \text{for all } n \text{ except } n = 1$$

$$\int_0^{2\pi} \cos^2\theta\,d\theta = \pi = \int_0^{2\pi} \sin^2\theta\,d\theta$$

we have,

$$I+iJ = -\pi\rho a(A_1 a + iB_1 a + C_1 a^{-1} + iD_1 a^{-1})$$

The impulsive thrust component along $\theta = \alpha$ is, therefore,

$$I\cos\alpha + J\sin\alpha = -\pi\rho a\{(A_1 a + C_1 a^{-1})\cos\alpha + (B_1 a + D_1 a^{-1})\sin\alpha\}$$

$$= -\pi\rho a\varphi_1(a,\alpha) \quad \square \quad (2.38)$$

Problem 2.20 Liquid of density ρ lies at rest in the annular region external to the uniform cylinder $|z| = a$ of mass m and internal to the uniform shell $|z| = b > a$ of mass M. (Both masses are measured per unit thickness perpendicular to the z-plane.) The inner cylinder is suddenly given a velocity $(u, 0)$ and at the same time the outer shell is given a velocity $(0, v)$. Show that the liquid motion is initially of the form

$$\varphi = (A\cos\theta + B\sin\theta)r + (C\cos\theta + D\sin\theta)/r.$$

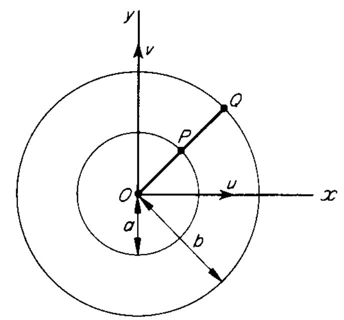


Figure 2.16

Solution. The given φ satisfies $\nabla^2\varphi = 0$. At $P(z = ae^{i\theta})$ the boundary condition for the initial motion is

$$-\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = -A\cos\theta - B\sin\theta + (C\cos\theta + D\sin\theta)a^{-2} = u\cos\theta,$$

for all θ. Hence,

$$-A + Ca^{-2} = u, \qquad -B + Da^{-2} = 0$$

Similarly at $Q(z = be^{i\theta})$ the boundary condition is

$$-\left.\frac{\partial\varphi}{\partial r}\right|_{r=b} = -A\cos\theta - B\sin\theta + (C\cos\theta + D\sin\theta)b^{-2} = v\sin\theta,$$

for all θ. Hence,

$$-A + Cb^{-2} = 0, \qquad -B + Db^{-2} = v$$

Referring to the general expression for φ given in Section 2.13, only the $\varphi_1(r,\theta)$ so chosen can possibly satisfy these boundary conditions. Solving for the four unknown constants and writing $b^2 - a^2 = \Delta$ we see that all conditions are satisfied by the given φ with

$$A = ua^2/\Delta, \qquad B = -vb^2/\Delta, \qquad C = ua^2b^2/\Delta,$$
$$D = -va^2b^2/\Delta \qquad \square \qquad (2.39)$$

Problem 2.21 Find the external impulses which must be applied to produce the motion of Problem 2.20.

Solution. Let I_1, J_1 be the external impulse components (per unit thickness) applied to the cylinder $|z| = a$. The corresponding components of the impulsive liquid thrust, using equation 2.38, are $\{-\pi\rho a\varphi(a,0), -\pi\rho a\varphi(a,\frac{1}{2}\pi)\}$. The impulsive equations of motions of the inner cylinder are

$$I_1 - \pi\rho a\varphi(a,0) = mu, \qquad J_1 - \pi\rho a\varphi(a,\tfrac{1}{2}\pi) = 0$$

Using equation 2.39,

$$I_1 = mu + \pi\rho a(Aa + Ca^{-1}) = mu + \pi\rho ua^2\{(a^2+b^2)/(b^2-a^2)\}$$
$$J_1 = \pi\rho a(Ba + Da^{-1}) = -2\pi\rho vab^2/(b^2-a^2)$$

Similarly if I_2, J_2 are the external impulsive components on $z = b$, we have

$$I_2 = \pi\rho b\varphi(b,0) = \pi\rho b(Ab + Cb^{-1}) = 2\pi\rho ua^2b/(b^2-a^2)$$
$$J_2 = Mv + \pi\rho b\varphi(b,\tfrac{1}{2}\pi) = Mv - \pi\rho vb^2\{(a^2+b^2)/(b^2-a^2)\} \qquad \square$$

EXERCISES

1. Given a complex potential $w = -m\ln\{(z^2-b^2)(b^2z^2-a^4)/(b^2z^2)\}$, $b > a$, show that $\psi = 0$ when $x = 0$, $y = 0$, or $x^2+y^2 = a^2$ where $z = x+iy$. Interpret the motion, express dw/dz in a closed form and hence show that the magnitude of the liquid speed at $z = ae^{i\theta}$ is $|(2m/a)\sin 2\theta/(\cos 2\theta - \lambda)|$ where $2a^2b^2\lambda = a^4+b^4$.

2. A solid cylinder $|z| \leqslant a$ is placed in a liquid whose velocity potential of two-dimensional motion is originally $\lambda(x^2+x-y^2)$. Show that the force on the cylinder is $4\pi\rho a^2\lambda^2$.

3. Prove that the impulse on the solid cylinder $|z| = a$ due to the sudden application of a source of strength m at a point $z = b$ outside on the real axis is $2\pi m\rho^2/b$ away from the source.

4. Obtain the relations between the constants a, h, b, A, H, B, in order that

$$u = \frac{ax^2+2hxy+by^2}{(x^2+y^2)^2} \qquad v = \frac{Ax^2+2Hxy+By^2}{(x^2+y^2)^2}$$

may represent a possible liquid motion. Show by any means that this motion is also irrotational, and the streamlines are circles.

Chapter 3

Two-Dimensional Unsteady Flow

3.1 Fundamentals In this chapter we adopt the same assumptions (ii), (iii), and (iv) defined in the first paragraph of Section 2.1 of the previous chapter, but we replace (i) by the new condition that flow is, in general, *unsteady*. Consequently, all or some quantities are *time dependent*. Again suffixes are used to denote partial differentiation. The main features of flow are:

The *velocity vector* is

$$\mathbf{q} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j} + 0\mathbf{k} \tag{3.1}$$

The *equation of continuity*, since ρ is constant, is the same as in the case of steady flow, i.e. $u_x + v_y = 0$.

The *equations of motion* are derived from the results of Section 1.7. For two-dimensional motion we have $\mathbf{q} \wedge \boldsymbol{\zeta} = (u\mathbf{i} + v\mathbf{j}) \wedge \zeta\mathbf{k} = \zeta v\mathbf{i} - \zeta u\mathbf{j}$ whilst $\nabla\chi = \chi_x\mathbf{i} + \chi_y\mathbf{j}$ where $\chi = p/\rho + \frac{1}{2}(u^2+v^2) + \Omega$. Hence

$$u_t + \chi_x = \zeta v, \qquad v_t + \chi_y = -\zeta u \tag{3.2}$$

are one form of the equations.

Eliminating χ from these equations using $(\chi_x)_y = (\chi_y)_x$ we have $(\zeta v)_y + (\zeta u)_x - u_{yt} + v_{xt} = 0$. Using the equation of continuity and $\zeta = v_x - u_y$ we arrive at the result

$$\zeta_t + u\zeta_x + v\zeta_y \equiv D\zeta/Dt = 0 \tag{3.3}$$

This means that vorticity following (i.e. attached to) any point which moves with the liquid remains invariant. In particular, a point vortex for which $w = ik\ln(z - z_0)$ will, if free (i.e. not *tied* on a boundary), move with the liquid particle associated with the point z_0. This principle is illustrated in the next problem.

Problem 3.1 Discuss the motion of two vortex filaments in a uniform stream U.

Solution. Choose the real axis parallel to the uniform stream and let the vortex filaments of strengths k_1 and k_2 occupy the points $A_1(z = z_1)$ and $A_2(z = z_2)$ respectively at time $t = 0$. The complex potential at this instant is

$$w = -Uz + ik_1\ln(z - z_1) + ik_2\ln(z - z_2)$$

Note that this satisfies $-\int_{\gamma_1} d\varphi = 2\pi k_1$ where γ_1 is any *small* circle centre A_1 and $-\int_{\gamma_2} d\varphi = 2\pi k_2$ where γ_2 is any *small* circle centre A_2. The particle at A_1 which *carries* the vorticity k_1 will have a velocity (u_1, v_1) induced in it by the uniform stream and by the vortex A_2 i.e.

$$u_1 - iv_1 = -\frac{d}{dz}(w - ik_1\ln(z - z_1))_{z=z_1} = U - \frac{ik_2}{z_1 - z_2}. \tag{3.4}$$

u_1, v_1 are not constant since z_1 and z_2 will vary with time. Similarly, the velocity (u_2, v_2) induced in A_2 is $U - ik_1/(z_2 - z_1)$. Hence

$$k_1(u_1 - iv_1) + k_2(u_2 - iv_2) = (k_1 + k_2)U$$

or

$$k_1u_1 + k_2u_2 = (k_1+k_2)U \quad \text{and} \quad k_1v_1 + k_2v_2 = 0.$$

Since $u_1 + iv_1 = \dot{z}_1 (\equiv dz_1/dt)$ and $u_2 + iv_2 = \dot{z}_2$ then $k_1\dot{z}_1 + k_2\dot{z}_2 = (k_1+k_2)U$. If z_G is the centre of gravity associated with masses k_1, k_2 at z_1 and z_2 respectively, provided $k_1 + k_2 = 0$, $z_G = (k_1z_1 + k_2z_2)/(k_1+k_2)$. Therefore, $\dot{z}_G = U$, i.e. the centre of gravity moves with constant velocity of magnitude U parallel to the real axis. It should be emphasised that the liquid velocity *under* z_G is *not* $\dot{z}_G$ but $(dw/dz)_{z=z_G}$. To find the positions of the vortices at any instant we use equation 3.4 *et seq.*, giving

$$\dot{\bar{z}}_1 = u_1 - iv_1 = U - \frac{ik_2}{z_1 - z_2} \quad \text{and} \quad \dot{\bar{z}}_2 = u_2 - iv_2 = U - \frac{ik_1}{z_2 - z_1}$$

Writing $z_1 - z_2 = re^{i\theta}$, substituting and taking conjugates we have

$$\dot{z}_1 = U + i(k_2/r)e^{i\theta}, \qquad \dot{z}_2 = U - i(k_1/r)e^{i\theta}$$

from which

$$\dot{z}_1 - \dot{z}_2 = (d/dt)(re^{i\theta}) = (\dot{r} + ir\dot{\theta})e^{i\theta} = i(k_1+k_2)e^{i\theta}/r$$

i.e. $\dot{r} = 0$ or $r = A_1A_2 =$ constant, and $\dot{\theta} = (k_1 + k_2)/r^2 =$ constant $= \omega$, so that $\theta = \omega t + \alpha$ where $\theta = \alpha$ when $t = 0$. Again, $\dot{z}_2 = U - i(k_1/r)e^{i(\omega t+\alpha)}$. Given $z_2 = \zeta$ at $t = 0$, integration leads to the result

$$z_2 = \zeta + Ut - (k_1/r\omega)e^{i\alpha}(e^{i\omega t} - 1)$$

Also

$$z_1 = z_2 + re^{i(\omega t+\alpha)}.$$

In the special case when $k_1 + k_2 = 0$, referred to as a *vortex couple*, it follows from the above analysis that $\dot{r} = 0 = \dot{\theta}$, i.e. both r and θ are constant so that A_1A_2 has constant velocity of translation only. Denoting the constant θ by α we have $\omega = 0$ and $\dot{z}_2 = U - i(k_1/r)e^{i\alpha}$ from which $z_2 = \zeta + (U - ik_1e^{i\alpha}/r)t$, $z_1 = z_2 + re^{i\alpha}$. Both vortices will remain at rest if $rU = ik_1e^{i\alpha}$ or $\cos\alpha = 0$ ($\alpha = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$) and $\sin\alpha = -rU/k_1$ ($\alpha = \frac{1}{2}\pi \Rightarrow rU = -k_1$, $\alpha = \frac{3}{2}\pi \Rightarrow rU = k_1$). □

Problem 3.2 A sink of strength m is fixed at the origin $r = 0$ whilst two vortices of strengths k and $-k$ are free to move in the liquid. If (r_1, θ_1) and (r_2, θ_2) denote their respective polar coordinates of position at any instant, deduce that $r_1^2 - r_2^2 =$ constant. If initially $r_1 = r_2$ deduce the equation of the path of the vortex k.

Solution. The complex potential of the motion at any instant is

$$w = ik\ln(z-z_1) - ik\ln(z-z_2) + m\ln z, \qquad z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}$$

The velocity of the vortex at z_1 has components u_1, v_1 where

$$-u_1 + iv_1 = \frac{d}{dz}(w - ik\ln(z-z_1))_{z=z_1} = -\frac{ik}{z_1 - z_2} + \frac{m}{z_1}$$

But $u_1 + iv_1 = \dot{z}_1 = (d/dt)(r_1 e^{i\theta_1}) = e^{i\theta_1}(\dot{r}_1 + ir_1\dot{\theta}_1)$. Hence,

$$u_1 - iv_1 = e^{-i\theta_1}(\dot{r}_1 - ir_1\dot{\theta}_1) = \frac{ik}{r_1 e^{i\theta_1} - r_2 e^{i\theta_2}} - \frac{m}{r_1 e^{i\theta_1}}$$

$$= \frac{ike^{-i\theta_1}}{(r_1 - r_2 e^{i\lambda})}\cdot\frac{(r_1 - r_2 e^{-i\lambda})}{(r_1 - r_2 e^{-i\lambda})} - \frac{me^{-i\theta_1}}{r_1}, \qquad \lambda = \theta_2 - \theta_1$$

Dividing throughout by $e^{-i\theta_1}$ followed by separating real and imaginary parts.

$$\dot{r}_1 = -\frac{m}{r_1} - \frac{kr_2}{R^2}\sin\lambda, \qquad r_1\dot{\theta}_1 = \frac{k}{R^2}(r_2\cos\lambda - r_1),$$

$$R^2 = r_1^2 + r_2^2 - 2r_1 r_2\cos\lambda$$

Similarly, replacing k by $-k$ and interchanging r_1 with r_2 and θ_1 with θ_2 so that λ becomes $-\lambda$.

$$\dot{r}_2 = -\frac{m}{r_2} - \frac{kr_1}{R^2}\sin\lambda, \qquad r_2\dot{\theta}_2 = -\frac{k}{R^2}(r_1\cos\lambda - r_2)$$

It follows that $r_1\dot{r}_1 - r_2\dot{r}_2 = 0$, i.e. $r_1^2 - r_2^2 = \text{constant} = A$. If $r_1 = r_2$ initially, $A = 0$ or $r_1 = r_2$ permanently and $R^2 = 2r_1^2(1-\cos\lambda)$. Hence

$$r_1\dot{\theta}_1 = kr_1(\cos\lambda - 1)/[2r_1^2(1-\cos\lambda)] \quad \text{or} \quad \dot{\theta}_1 = -\tfrac{1}{2}kr_1^{-2}$$

Similarly $\dot{\theta}_2 = \frac{1}{2}kr_2^{-2} = -\dot{\theta}_1$ so that $\theta_1 + \theta_2 = \text{constant} = 2\alpha$ (say). We can now eliminate both r_2 and θ_2 terms from the above equations giving

$$\dot{r}_1 = -\frac{m}{r_1} - \frac{k\sin\lambda}{2r_1(1-\cos\lambda)}, \qquad \dot{\theta}_1 = \tfrac{1}{2}kr_1^{-2}, \qquad \lambda = \theta_2 - \theta_1 = 2(\alpha - \theta_1)$$

from which

$$\frac{\dot{r}_1}{r_1\dot{\theta}_1} = \frac{1}{r_1}\left(\frac{dr_1}{d\theta_1}\right) = \frac{2m}{k} + \cot\tfrac{1}{2}\lambda = \frac{2m}{k} - \cot(\theta_1 - \alpha)$$

Integrating, the path (r_1, θ_1) of the vortex k is

$$\ln r_1 = (2m\theta_1/k) - \ln\sin(\theta_1 - \alpha) = \text{constant}$$

i.e. $r_1\sin(\theta_1 - \alpha)e^{-2m\theta_1/k} = \text{constant}$. □

Problem 3.3 n vortices each of strength k are placed at equal intervals along the circumference of a circle of radius a. Show that the configuration rotates as a rigid system with angular velocity $\frac{1}{2}(n-1)k/a^2$.

Solution. The complex potential w for n vortices at $z = z_r = ae^{2r\pi i/n}$ is $w = ik\sum_{r=1}^{n}\ln(z - z_r) = ik\ln\prod_{r=1}^{n}(z - ae^{2r\pi i/n}) = ik\ln(z^n - a^n)$. The velocity of the vortex at $z = a$ (i.e. $z = z_n$) is determined from

$$w^* = w - ik\ln(z-a) = ik\ln\{(z^n - a^n)/(z-a)\}$$
$$= ik\ln(z^{n-1} + z^{n-2}a + \ldots + a^{n-1})$$

The velocity components u, v satisfy

$$-u + iv = \left.\frac{dw^*}{dz}\right|_{z=a} = ik\left(\frac{(n-1)z^{n-2} + (n-2)z^{n-3}a + \ldots + a^{n-2}}{z^{n-1} + z^{n-2} + \ldots + a^{n-1}}\right)_{z=a}$$

$$= ik\left(\frac{\frac{1}{2}n(n-1)}{na}\right) = \frac{ik(n-1)}{2a}$$

i.e. $u = 0$, $v = \frac{1}{2}(n-1)k/a$ so that the vortex moves tangentially to $|z| = a$ with speed v or angular velocity $\frac{1}{2}(n-1)k/a^2$. By symmetry the other vortices move with this same speed tangential to the circle and therefore the system of vortices moves as a rigid system on the circle $|z| = a$ with angular velocity $\frac{1}{2}(n-1)k/a^2$. □

Problem 3.4 At time $t = 0$, n vortices each of strength k occupy positions $z = z_r = ae^{2\pi ri/n}$ $(r = 1, 2, \ldots, n)$ whilst a similar set of n vortices are placed at points $z = z_s = be^{i(\alpha + 2\pi s/n)}$ $(s = 1, 2, \ldots, n)$. Show that members of the second set will remain equidistant from $z = 0$ and determine the initial values of db/dt and $d\alpha/dt$.

Solution. A vortex at z_s has the tangential velocity $\frac{1}{2}(n-1)k/b$ induced in it by the members of its own circle as proved in the previous problem. The vortices on $z = a$ will also induce a velocity calculated from the complex potential $w = ik\ln(z^n - a^n)$. Denoting this contribution by its radial and transverse components q_r and q_θ respectively, we have, using equation 2.11,

$$\left.\frac{dw}{dz}\right|_{z=z_s} = \frac{nikz_s^{n-1}}{z_s^n - a^n} = -(q_r - iq_\theta)e^{-i\theta}, \qquad z_s = be^{i\theta}, \qquad \theta = \alpha + 2\pi s/n$$

Since $e^{2\pi si} = 1$,

$$q_r + iq_\theta = \frac{nik\bar{z}_s^{n-1}e^{-i\theta}}{\bar{z}_s^n - a^n} = \frac{nikb^{n-1}e^{-ni\alpha}}{b^n e^{-ni\alpha} - a^n} = \frac{nikb^{n-1}(b^n - a^n e^{-ni\alpha})}{R^2}$$

where $R^2 = (b^n e^{-ni\alpha} - a^n)(b^n e^{ni\alpha} - a^n) = a^{2n} - 2a^n b^n \cos n\alpha + b^{2n}$. Since $q_r + iq_\theta$ is independent of s, each vortex of the ring has the *same induced velocity*, and will, therefore, remain equidistant from O ($z = 0$). This distance, however, changes with time t and initially

$$\frac{db}{dt} = q_r = \mathrm{Re}\left\{\frac{nikb^{n-1}(b^n - a^n e^{-ni\alpha})}{R^2}\right\} = -\frac{nkb^{n-1}a^n \sin n\alpha}{R^2}.$$

Again,

$$\frac{d\alpha}{dt} = \frac{q_\theta}{b} + \frac{(n-1)k}{2b^2} = \frac{1}{b}\mathrm{Im}\left\{\frac{nikb^{n-1}(b^n - a^n e^{-ni\alpha})}{R^2}\right\} + \frac{(n-1)k}{2b^2}$$
$$= \frac{nkb^{n-2}(b^n - a^n \cos n\alpha)}{R^2} + \frac{(n-1)k}{2b^2}$$

□

Problem 3.5 A thin plate of width $2a$ is placed perpendicular to a uniform stream of magnitude U. Assuming the absence of cavitation, show that two vortices of strengths k and $-k$ can remain at rest downstream of the plate and find their positions given that the liquid speed is nowhere infinite.

Solution. Choosing the z-plane as the region of flow with the stream parallel to the positive real axis and the plate lying in the imaginary axis between $z = ai$ and $z = -ai$, the flow in the absence of vortices is represented by $w = -U(z^2 + a^2)^{\frac{1}{2}}$ as solved in Problem 2.18. By symmetry, the vortices $-k$, $+k$ will lie at the image points z_0 and $\bar{z}_0$ respectively, so that we need only consider the flow in $\mathrm{Im}\, z > 0$. Referring to Figure 2.15, provided z_0 is not a singular point, the hydrodynamic image of the vortex $-k$ at z_0 is an equal vortex $-k$ at the corresponding point ζ_0. In order that the real axis in the ζ-plane remains a solid boundary in correspondence with the solid boundary in the z-plane, we must introduce an image vortex of strength k at $\bar{\zeta}_0$. The complex potential of motion in the ζ-plane is then

$$w = -Ua\zeta - ik\ln(\zeta - \zeta_0) + ik\ln(\zeta - \bar{\zeta}_0).$$

The vortex at ζ_0 will remain at rest if

$$\frac{d}{d\zeta}\{-Ua\zeta + ik\ln(\zeta - \bar{\zeta}_0)\}_{\zeta=\zeta_0} = 0 = -Ua + \frac{ik}{\zeta_0 - \bar{\zeta}_0}$$

Writing $\zeta_0 = \xi_0 + i\eta_0$ this becomes $2Ua\eta_0 = k$. When this condition is satisfied the vortex $-k$ at z_0 in the z-plane will also be stationary since ζ_0 is a nonsingular point. Again by Problem 2.18 the mapping function is $z = a(\zeta^2 - 1)^{\frac{1}{2}}$ where $dz/d\zeta = 0$ when $\zeta = 0$. The velocity will be infinite at the point $z = ai$ unless $dw/d\zeta = 0$ when $\zeta = 0$ (Kutta condition—Section 2.11). This implies $-Ua + ik\zeta_0^{-1} - ik\bar{\zeta}_0^{-1} = 0$ or $aU(\zeta_0^2 + \eta_0^2) = 2k\eta_0$. With $k = 2Ua\eta_0$ we have $\xi_0 = k\sqrt{3}/(2Ua)$, $\eta_0 = k/(2Ua)$ from which $z_0 = a(\zeta_0^2 - 1)^{\frac{1}{2}}$ is determined. Writing $z_0 = x_0 + iy_0$ we have

$$z_0^2 = (x_0 + iy_0)^2 = a^2(\xi_0^2 - \eta_0^2 - 1 + 2i\xi_0\eta_0)$$

i.e.

$$x_0^2 - y_0^2 = a^2(\xi_0^2 - \eta_0^2 - 1), \qquad x_0 y_0 = a^2 \xi_0 \eta_0$$

from which

$$x_0^2 = \tfrac{1}{2}a^2\{(E^2 + 4\xi_0^2\eta_0^2)^{\frac{1}{2}} + E\}, \qquad y_0^2 = \tfrac{1}{2}a^2\{(E^2 + 4\xi_0^2\eta_0^2)^{\frac{1}{2}} - E\}$$

where $E \equiv \xi_0^2 - \eta_0^2 - 1$. Substituting for ξ_0 and η_0 the result follows. □

Problem 3.6 Liquid lying at rest in $\mathrm{Re}\, z > 0$ is bounded by rigid walls coincident with the real and imaginary axes and $\mathrm{Im}\, z = \pi/\lambda$ ($\lambda > 0$). Show that a vortex of strength k can remain at rest at a point midway between the two parallel walls. Find this point and evaluate the force on the wall $\mathrm{Re}\, z = 0$ in this case. (Assume the stagnation pressure is p_0.)

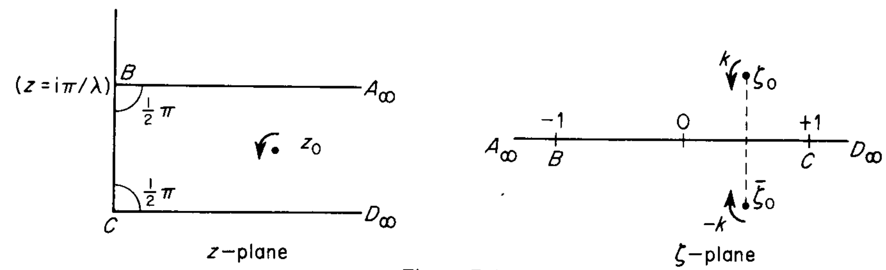


Figure 3.1

Solution. Referring to Figure 3.1, we use the Schwarz–Christoffel transformation of Section 2.12 to map the region $A_\infty BCD_\infty$ occupied by the liquid in the z-plane onto the upper half of the ζ-plane with $B(z = i\pi/\lambda)$ and $C(z = 0)$ represented by $\zeta = -1$ and $+1$ respectively on $\mathrm{Im}\,\zeta = 0$. By equation 2.37, with $\alpha_1 = \alpha_2 = \frac{1}{2}\pi$, $\zeta_1 = -1$, $\zeta_2 = 1$ we have

$$\frac{dz}{d\zeta} = L(\zeta + 1)^{\frac{1}{2}-1}(\zeta - 1)^{\frac{1}{2}-1} = L(\zeta^2 - 1)^{-\frac{1}{2}}$$

Integrating, $z = L\cosh^{-1}\zeta + M$ or $\zeta = \cosh\{(z - M)/L\}$ where L, M are constants. Since $\zeta = 1$ when $z = 0$, $M = 0$ and with $\zeta = -1$ when

$z = i\pi/\lambda - 1 = \cosh(i\pi/\lambda L) = \cos(\pi/\lambda L)$ or $L = 1/\lambda$ giving $\zeta = \cosh\lambda z$. A vortex of strength k placed at a nonsingular point z_0 (say) in the z-plane transforms into a vortex of the same strength at the corresponding point $\zeta = \zeta_0 = \cosh\lambda z_0$ of the ζ-plane. Since $A_\infty BCD_\infty$ is a rigid boundary so is $\operatorname{Im}\zeta = 0$. Therefore we must insert the image vortex of strength $-k$ at $\bar{\zeta}_0$ in the ζ-plane so that the complex potential of liquid motion is $w = ik\ln(\zeta-\zeta_0) - ik\ln(\zeta-\bar{\zeta}_0)$ in the ζ-plane and in the z-plane where $\zeta = \cosh\lambda z$, $w = ik\ln(\cosh\lambda z - \cosh\lambda z_0) - ik\ln(\cosh\lambda z - \cosh\lambda\bar{z}_0)$. The velocity components u, v induced in the vortex at z_0 by the image system are given by

$$-u+iv = \frac{d}{dz}[w - ik\ln(z-z_0)]_{z=z_0}$$

$$= ik\lim_{z\to z_0}\left(\frac{\lambda\sinh\lambda z}{\cosh\lambda z - \cosh\lambda z_0} - \frac{1}{z-z_0} - \frac{\lambda\sinh\lambda z}{\cosh\lambda z - \cosh\lambda\bar{z}_0}\right)$$

$$= ik\lim_{\epsilon\to 0}\left(\frac{\lambda\sinh(\lambda z_0+\epsilon)}{\cosh(\lambda z_0+\epsilon) - \cosh\lambda z_0} - \frac{\lambda}{\epsilon} - \frac{\lambda\sinh(\lambda z_0+\epsilon)}{\cosh(\lambda z_0+\epsilon) - \cosh\lambda\bar{z}_0}\right),$$

$$= ik\lambda\left(\coth\lambda z_0 - \frac{\sinh\lambda z_0}{\cosh\lambda z_0 - \cosh\lambda\bar{z}_0}\right)$$

A point midway between the parallel walls can be represented by $z_0 = (\alpha + \frac{1}{2}\pi i)/\lambda$ for which $\cosh\lambda z_0 = i\sinh\alpha$ and $\sinh\lambda z_0 = i\cosh\alpha$ leading to

$$-u+iv = ik\lambda(\tanh\alpha - \tfrac{1}{2}\coth\alpha)$$

Hence, $u = 0$ and the vortex will remain stationary provided $\tanh^2\alpha = \frac{1}{2}$. Since $\alpha > 0$, $\sinh\alpha = 1$, $\cosh\alpha = \sqrt{2}$, $\alpha = \ln(\sqrt{2}+1)$.

The velocity (u, v) on BC is given by

$$-u+iv = \frac{dw}{dz} = ik\lambda\sinh\lambda z\left(\frac{1}{\cosh\lambda z - \cosh\lambda z_0} - \frac{1}{\cosh\lambda z - \cosh\lambda\bar{z}_0}\right)$$

with, $z = iy$. Since $\cosh\lambda z_0 = \cosh(\alpha + \frac{1}{2}\pi i) = i\sinh\alpha = i$, $\cosh\lambda z = \cos\lambda y$, etc.

$$-u+iv = -k\lambda\sin\lambda y\left(\frac{1}{\cos\lambda y - i} - \frac{1}{\cos\lambda y + i}\right) = -\frac{2k\lambda i\sin\lambda y}{\cos^2\lambda y + 1}$$

or $u = 0$, $v = -2k\lambda\sin\lambda y/(\cos^2\lambda y + 1)$. By Bernoulli's equation the pressure p on BC is $p_0 - \frac{1}{2}\rho v^2$ where p_0 is the stagnation pressure. The

thrust P on BC is, therefore,

$$P = \int_0^{\pi/\lambda}\left\{p_0 - \frac{2\rho k^2\lambda^2\sin^2\lambda y}{(\cos^2\lambda y+1)^2}\right\}dy = \pi\frac{p_0}{\lambda} - 2\rho k^2\lambda\int_0^\pi \frac{\sin^2\theta}{(\cos^2\theta+1)^2}d\theta,$$

$$\theta = \lambda y$$

To evaluate the integral consider

$$I(a) = \int_0^\pi \frac{d\theta}{1+a\cos^2\theta} = 2\int_0^\infty \frac{dt}{1+a+t^2} = \frac{\pi}{(1+a)^{\frac{1}{2}}}, \qquad t = \tan\theta$$

and

$$-\frac{\partial I(a)}{\partial a} = \int_0^\pi \frac{\cos^2\theta\, d\theta}{(1+a\cos^2\theta)^2} = \frac{\pi}{2(1+a)^{\frac{3}{2}}}$$

Hence

$$\int_0^\pi \frac{\sin^2\theta\, d\theta}{(1+\cos^2\theta)^2} = \int_0^\pi \frac{d\theta}{(1+\cos^2\theta)} - 2\int_0^\pi \frac{\cos^2\theta\, d\theta}{(1+\cos^2\theta)^2} = \frac{\pi}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

ie

$$P = \frac{\pi}{\lambda}\left(p_0 - \frac{\rho k^2\lambda^2}{\sqrt{2}}\right) \qquad \square$$

3.2 Pressure and forces in unsteady flow When flow is unsteady, the pressure $p = p(\mathbf{r}, t)$ is no longer given by Bernoulli's equation. Instead, the pressure is determined by integrating the equation of motion $D\mathbf{q}/Dt = \mathbf{F} - \operatorname{grad} p/\rho$. Assuming that $\mathbf{F} = -\operatorname{grad}\Omega$ and curl $\mathbf{q} = \mathbf{0}$, in which case φ exists with $\mathbf{q} = -\operatorname{grad}\phi$, we have

$$p/\rho = (\partial\varphi/\partial t) - \tfrac{1}{2}q^2 - \Omega + A(t)$$

where $A(t)$ is an arbitrary function.

Problem 3.7 Find the stream function ψ and pressure p due to a vortex field in which the vorticity $\zeta = 0$ for all $|z| = r > a$, and $\zeta = r + 2a$ for $r < a$. Discuss the incidence of cavitation within the vortex.

Solution. Since $\psi = \psi(r)$, we have $\nabla^2\psi \equiv \psi'' + \psi'/r = \zeta$ where $\psi' \equiv d\psi/dr$ etc. For $r < a$, $\psi = \psi_1$ where $\psi_1'' + \psi_1'/r = (r\psi_1')'/r = r + 2a$. Integrating

$$r\psi_1' = \tfrac{1}{3}r^3 + ar^2 + c, \qquad c = \text{constant}$$

For finite speed $(=\psi_1')$ at $r = 0$, $c = 0$, i.e. $\psi_1' = \frac{1}{3}r^2 + ar$ and $\psi_1 = \frac{1}{9}r^3 + \frac{1}{2}ar^2 + e$, $e = $ constant. For $r > a$, $\psi = \psi_0$ where $(r\psi_0')' = 0$. Integrating, $r\psi_0' = $ constant $= f$ and

$$\psi_0 = f\ln r + g, \qquad g = \text{constant}$$

For no slip at $r = a$, $\psi_0' = \psi_1'$, i.e. $\frac{1}{3}a^3 + a^3 = \frac{4}{3}a^3 = f$. Also for continuity in ψ (the absence of sources on $r = a$), $\frac{1}{9}a^3 + \frac{1}{2}a^3 + e = f\ln a + g$. Choosing

$g+f\ln a = 0$, we have $e = -11a^3/18$. Hence, when

$$r > a, \quad \psi_0 = \tfrac{4}{3}a^3 \ln(r/a)$$

and for

$$r < a, \quad \psi_1 = (2r^3+9ar^2-11a^3)/18$$

Next we determine the pressure. Outside the vortex where $r > a$, φ exists and equals $-\frac{4}{3}a^3 \arg(z/a)$. Using Section 3.2, the pressure p_0 is given by $p_0 = \rho\{(\partial\varphi/\partial t)-\frac{1}{2}q^2+A(t)\}$, $q = \psi_0' = \frac{4}{3}a^3/r$, $\partial\varphi/\partial t = 0$. Given that as $r \to \infty$, $p \to p_\infty$ = constant, $A(t) = 0$, i.e. for

$$r > a, \quad p_0 = p_\infty - 8\rho a^6/9r^2$$

When $r < a$, φ does not exist in which case we find the pressure p_1 directly from the equation of motion. An element at a distance $r(<a)$ having a speed $q = \psi_1'$ moves in a circle of radius r with acceleration q^2/r towards $r = 0$. Hence,

$$\frac{D\mathbf{q}}{Dt} = -\frac{q^2\mathbf{r}}{r^2} = -\frac{1}{\rho}\nabla p_1 = -\frac{1}{\rho}\left(\frac{dp_1}{dr}\right)\frac{\mathbf{r}}{r}, \qquad q = \psi_1' = \tfrac{1}{3}r^2+ar$$

i.e.

$$\frac{1}{\rho}\left(\frac{dp_1}{dr}\right) = \frac{q^2}{r} = r(\tfrac{1}{3}r+a)^2$$

Integrating,

$$p_1 = \tfrac{1}{36}\rho(r^4+8ar^3+18a^2r^2)+h, \qquad h = \text{constant}$$

For continuity of pressure across the boundary $r = a$

$$p_0 = p_\infty - \tfrac{8}{9}\rho a^4 = \tfrac{3}{4}\rho a^4 + h$$

which determines h, leading to

$$p_1 = p_\infty + \tfrac{1}{36}\rho(r^4+8ar^3+18a^2r^2-59a^4)$$

Finally, to consider the incidence of cavitation within the vortex, we need to evaluate the minimum pressure p_m for $r < a$. We note that $dp_1/dr = \rho r(r+3a)^2/9$ and $d^2p_1/dr^2 = \frac{1}{3}\rho(r+a)(r+3a)$ from which p_1 is a minimum at $r = 0$. This minimum value is $p_m = p_\infty - 59\rho a^4/36$. To prevent cavitation within $r < a$, $p_m > 0$, or, $p_\infty > 59\rho a^4/36$. If this condition is not fulfilled, cavitation of radius $R < a$ could occur when $p_1 = 0$ at $r = R\,(<a)$ which is a root of the equation $R^2(R^2+8aR+18a^2) = 59a^4 - 36p_\infty/\rho$. The vortex will be *completely hollow* if $R = a$, i.e. provided $p_\infty = 11\rho a^4/18$. □

Problem 3.8 Find the pressure on a circular cylinder $|z| = a$ due to an external vortex of strength $3k$ placed at a point $z = 2a$ on the real axis given that the total circulation about the cylinder is $4\pi k$.

Solution. The complex potential of the vortex $3k$ at $z = 2a$ is $3ik\ln(z-2a)$. After inserting $|z| = a$, the complex potential is

$$3ik\ln(z-2a) - 3ik\ln\{(a^2/z)-2a\}$$
$$= 3ik\ln(z-2a) - 3ik\ln(z-\tfrac{1}{2}a) + 3ik\ln z - 3ik\ln(-2a)$$

by the circle theorem (2.20). This result gives zero circulation about the circle since the algebraic sum of the vortices inside, i.e. $-3k$ at $z = \frac{1}{2}a$ and $3k$ at $z = 0$, is zero. To produce the required circulation we add the term $2ik\ln z$ which does not upset the rigid boundary condition on $z = a$. Ignoring the constant term the complex potential w_0 at time $t = 0$ is then

$$w_0 = 3ik\ln(z-2a) - 3ik\ln(z-\tfrac{1}{2}a) + 5ik\ln z$$

This expression does not persist because the vortex at $z = 2a$ has a velocity (u, v) induced in it by the image vortices. We have

$$-u+iv = \frac{d}{dz}(w_0 - 3ik\ln(z-2a))_{z=2a} = -\frac{3ik}{2a-\frac{1}{2}a}+\frac{5ik}{2a} = \frac{ik}{2a}.$$

Hence, $u = 0$ and $v = k/2a$ or the vortex at $z = 2a$ moves in a circle of radius $2a$ centred at $z = 0$ with angular velocity $\omega = k/4a^2$. At time $t \geqslant 0$, its position in the plane is defined by $z = 2ae^{i\omega t}$ and therefore the complex potential of this external vortex for $t \geqslant 0$ is $3ik\ln(z-2ae^{i\omega t})$. Applying the circle theorem to find its image in $|z| = a$ and adding the circulation term $2ik\ln z$, the final complex potential for *all* t is

$$w = 3ik\ln(z-2ae^{i\omega t}) - 3ik\ln\{(a^2/z)-2ae^{-i\omega t}\} + 2ik\ln z$$
$$= 3ik\ln(z-2ae^{i\omega t}) - 3ik\ln(z-\tfrac{1}{2}ae^{i\omega t}) + 5ik\ln z$$

ignoring the term independent of z. Putting $t = 0$ we obtain w_0 again. The pressure $p = \rho\{(\partial\varphi/\partial t)-\frac{1}{2}q^2\}+A(t)$. q is determined by $|dw/dz|$ where

$$\frac{dw}{dz} = \frac{3ik}{z-2a\,e^{i\omega t}} - \frac{3ik}{z-\frac{1}{2}a\,e^{i\omega t}} + \frac{5ik}{z}$$

Also

$$\frac{\partial\varphi}{\partial t} = \mathrm{Re}\,\frac{\partial w}{\partial t} = \mathrm{Re}\left(\frac{6k\omega a\,e^{i\omega t}}{z-2ae^{i\omega t}} - \frac{3k\omega a\,e^{i\omega t}}{2z-ae^{i\omega t}}\right)$$

As $|z| \to \infty$, $q = |dw/dz| \to 0$, $\partial\varphi/\partial t \to 0$ so that $A(t) = \rho p_\infty$ where p_∞ is the pressure at infinity. Since we are evaluating the pressure on the circle when the vortex is in its initial position, we can put $t = 0$ and $z = ae^{i\theta}$ in which case,

$$\frac{\partial\varphi}{\partial t} = 3k\omega\,\mathrm{Re}\left(\frac{2}{e^{i\theta}-2} - \frac{1}{2e^{i\theta}-1}\right) = -\frac{9k\omega}{n^2}$$

where $n^2 = e^{-i\theta}(2-e^{i\theta})(2e^{i\theta}-1) = 5-4\cos\theta$ (na is the distance between $z = ae^{i\theta}$ and $z = 2a$).

$$\frac{dw}{dz} = \frac{3ik}{a}\left(\frac{1}{e^{i\theta}-2}-\frac{2}{2e^{i\theta}-1}\right)+\frac{5ike^{-i\theta}}{a} = \frac{ike^{-i\theta}}{a}\left(5-\frac{9}{n^2}\right)$$

Hence

$$q^2 = \frac{k^2}{a^2}\left(5-\frac{9}{n^2}\right)^2 \quad \text{and since } \omega = k/4a^2, \quad \frac{\partial\varphi}{\partial t} = -\frac{9k^2}{4n^2a^2}$$

we have, finally,

$$p = p_\infty+\rho\left(\frac{\partial\varphi}{\partial t}-\tfrac{1}{2}q^2\right)$$

$$= p_\infty-\frac{\rho k^2}{4a^2}\left\{\frac{9}{n^2}+2\left(5-\frac{9}{n^2}\right)^2\right\}, \qquad n^2 = 5-4\cos\theta$$ □

3.3 Paths of liquid particles These are found by integrating the velocity $\mathbf{q} = d\mathbf{r}/dt$ to find the position vector $\mathbf{r}$ of a specific particle, from which t, the time, can then be eliminated to find its path. The next two problems illustrate the method.

Problem 3.9 Find the paths of particles for uniform streaming past a circular cylinder with circulation.

Solution. Choosing the real axis parallel to the uniform stream U and the circular cylinder represented by the equation $|z| = a$ the complex potential w of liquid motion is

$$w = U(z+a^2/z)+ik\ln z$$

where k is the strength of circulation. The velocity components u, v are found from

$$\frac{dw}{dz} = -u+iv = U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z}$$

For this solution it is easier to work in polar coordinates (r, θ). Therefore, we put $z = re^{i\theta}$ so that $u-iv = \dot{\bar{z}} = (\dot{r}-ir\dot{\theta})e^{-i\theta}$, i.e.

$$-(\dot{r}-ir\dot{\theta})e^{-i\theta} = U\left(1-\frac{a^2e^{-2i\theta}}{r^2}\right)+\frac{ike^{-i\theta}}{r}$$

or

$$\dot{r}-ir\dot{\theta} = -U\left(e^{i\theta}-\frac{a^2e^{-i\theta}}{r^2}\right)-\frac{ik}{r}$$

Separating real and imaginary parts,

$$\dot{r} = -U(1-a^2/r^2)\cos\theta, \qquad r\dot{\theta} = U(1+a^2/r^2)\sin\theta+k/r$$

To eliminate t, we have

$$\frac{r\dot{\theta}}{\dot{r}} = r\frac{d\theta}{dr} = -\frac{U(r^2+a^2)\sin\theta+kr}{U(r^2-a^2)\cos\theta}$$

Rearranging to give

$$\sin\theta\left(1+\frac{a^2}{r^2}\right)dr+\left(r-\frac{a^2}{r}\right)\cos\theta\,d\theta = -\frac{k}{Ur}dr$$

the equation is an exact differential whose integral, a particle path, is

$$\left(r-\frac{a^2}{r}\right)\sin\theta+\frac{k}{U}\ln r = \text{constant} = A$$

This, of course, is simply the family of streamlines $\psi = \operatorname{Im} w = \text{constant} = UA$. □

Problem 3.10 In the preceding problem find expressions for the time t taken by a particle in moving along the cylinder from $z = ae^{i\alpha}$ to $z = ae^{i\theta}$.

Solution. Using the results of the last problem, since $r = \text{constant} = a$ we have $a\dot{\theta} = 2U\sin\theta+k/a$. Therefore,

$$t = \int_\alpha^\theta \frac{a^2\,d\theta}{2Ua\sin\theta+k}$$

or putting $\tan\frac{1}{2}\theta = x$, $\tan\frac{1}{2}\alpha = x_0$, $2Ua/k = \lambda$,

$$t = \frac{2a^2}{k}\int_{x_0}^{x}\frac{dx}{(x+\lambda)^2+1-\lambda^2}$$

There are three cases to consider dependent upon the value of λ.

Case 1. $\lambda = 2Ua/k < 1$; large circulation.

Putting $1-\lambda^2 = \omega^2$,

$$t = \frac{2a^2}{k\omega}\left[\tan^{-1}\left(\frac{x+\lambda}{\omega}\right)\right]_{x_0}^{x}$$

$$= \frac{2a^2}{k\omega}\left\{\tan^{-1}\left(\frac{x+\lambda}{\omega}\right)-\tan^{-1}\left(\frac{x_0+\lambda}{\omega}\right)\right\}$$

$$= \frac{2a^2}{k\omega}\tan^{-1}\left\{\frac{(x-x_0)\omega}{xx_0+\lambda(x+x_0)+1}\right\}$$

$$= \frac{2a^2}{k\omega}\tan^{-1}\left\{\frac{\omega(\tan\frac{1}{2}\theta-\tan\frac{1}{2}\alpha)}{1+\tan\frac{1}{2}\theta\tan\frac{1}{2}\alpha+\lambda(\tan\frac{1}{2}\theta+\tan\frac{1}{2}\alpha)}\right\}$$

Case 2. $\lambda = 2Ua/k = 1$.

Here

$$\frac{Ut}{a} = \int_{x_0}^{x} \frac{dx}{(x+1)^2} = \left[-\frac{1}{x+1} \right]_{x_0}^{x}$$

$$= \frac{x-x_0}{(x+1)(x_0+1)} = \frac{\tan\frac{1}{2}\theta - \tan\frac{1}{2}\alpha}{(\tan\frac{1}{2}\theta+1)(\tan\frac{1}{2}\alpha+1)}$$

provided $\tan\frac{1}{2}\alpha + 1 \neq 0$, i.e. $\alpha \neq \frac{3}{2}\pi$. The point $z = a\,e^{i\frac{3}{2}\pi}$ is a point of stagnation of the flow since $\dot{\theta} = 0$ here. We must *not allow* $\tan\frac{1}{2}\theta + 1 = 0$ either. This means that the time to or from a point of liquid stagnation is infinite.

Case 3. $\lambda = 2Ua/k > 1$; small circulation.

Putting $\lambda^2 - 1 = \Omega^2$

$$t = \frac{a^2}{k\Omega} \int_{x_0}^{x} \left(\frac{1}{x+\lambda-\Omega} - \frac{1}{x+\lambda+\Omega} \right) dx$$

$$= \frac{a^2}{k\Omega} \ln \left\{ \left(\frac{\tan\frac{1}{2}\theta + \lambda - \Omega}{\tan\frac{1}{2}\theta + \lambda + \Omega} \right) \left(\frac{\tan\frac{1}{2}\alpha + \lambda + \Omega}{\tan\frac{1}{2}\alpha + \lambda - \Omega} \right) \right\}$$

provided neither $\tan\frac{1}{2}\theta + \lambda \pm \Omega$ nor $\tan\frac{1}{2}\alpha + \lambda \pm \Omega$ vanishes. From $\tan\frac{1}{2}\theta + \lambda \pm \Omega = 0$ we find that

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta} = -\frac{2(\lambda\pm\Omega)}{2\lambda(\lambda\pm\Omega)} = -\frac{1}{\lambda} = -\frac{k}{2Ua}$$

Since $\lambda > 1$, θ is real. Assuming both k and U are positive, $\theta = \pi + \beta$, or $-\beta$ where $\beta = \sin^{-1}\{k/(2Ua)\}$. These are the points of stagnation on the cylinder. Hence t, the time, is finite provided the liquid particle is not entering or leaving (when $\tan\frac{1}{2}\alpha + \lambda \pm \Omega = 0$) either point of stagnation.

□

3.4 Surface waves To examine the characteristics of small constant-amplitude waves propagated over the surface of still water of constant depth h choose the axis OX in the horizontal bed (Figure 3.2) and let $P(x, h+\eta)$ be a point on the wave profile at any time t where $\eta = \eta(x, t)$. In the case of a simple monochromatic wave of amplitude a, wavelength $\lambda = 2\pi/m$ and period $\tau = 2\pi/n$ propagated with speed $c = n/m$, η will be of the form $\eta = a\cos(mx - nt)$. Let $\mathscr{R}$ denote the fluid region and $R(x, y)$ a typical point within; $\mathscr{R}$ is bounded by the profile $\mathscr{W}$ and bed $y = 0$. We assume that due to the surface wave (i) liquid motion in $\mathscr{R}$ is irrotational with velocity potential $\varphi = \varphi(x, y, t)$, (ii) the pressure on the profile $\mathscr{W}$ is a constant Π for all x and t, (iii) η, φ and speed $q = |\operatorname{grad}\varphi|$ are all small, (iv) the profile slope is everywhere small.

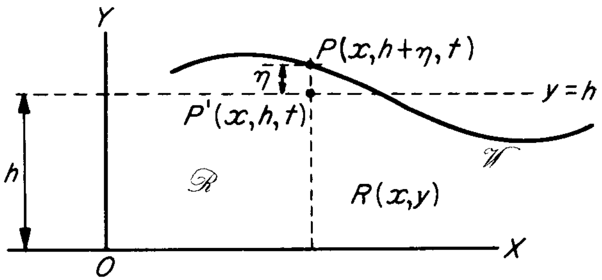


Figure 3.2

On the profile $\mathscr{W}$ applying (i) and (ii) to the pressure equation (1.18) with ρ = constant, $\Omega = gy = g(h+\eta)$,

$$\frac{\Pi}{\rho} - \frac{\partial\varphi}{\partial t} + \tfrac{1}{2}q^2 + g(h+\eta) = A(t)$$

where $A(t)$ is arbitrary. By (iii) we neglect q^2. Also we may replace φ by $\varphi - B(t)$ where $dB/dt = A(t) - gh - \Pi/\rho$ since the velocity field $\mathbf{q} = -\operatorname{grad}\varphi$ due to either φ is the same. In terms of the modified φ,

$$-\frac{\partial\varphi(x, h+\eta, t)}{\partial t} + g\eta = 0 \quad \text{for all} \quad P \in \mathscr{W}$$

When η is small

$$\phi(x, h+\eta, t) = \varphi(x, h, t) + \eta\left[\frac{\partial\varphi(x, y, t)}{\partial y}\right]_{y=h} + \dots$$

Invoking (iv), the above surface condition on the *unknown* profile $\mathscr{W}$ can be replaced by an equivalent condition

$$-\frac{\partial\varphi}{\partial t}(x, h, t) + g\eta = 0 \tag{3.5}$$

evaluated on the still water surface $y = h$, i.e. for all P' (x, h, t). Using (iv), $\partial\eta/\partial t$ is approximately $-\partial\varphi(x, h, t)/\partial y$. Hence,

$$\frac{\partial^2\varphi}{\partial t^2} = g\frac{\partial\eta}{\partial t} = -g\frac{\partial\varphi}{\partial y}, \qquad \varphi \equiv \varphi(x, h, t) \tag{3.6}$$

is the final (equivalent) surface condition applied at ʾoints $P'(x, h, t)$. Using suffixes to denote partial derivatives, motion is solved by

For all t and $R(x, y) \in \mathscr{R}$, $\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} = 0$ (3.7a)

Condition 3.6 (3.7b)

Normal velocity $-\partial\varphi/\partial y = 0$ when $y = 0$ for all x, t, (3.7c)

By equation 3.5, $\eta = \partial\varphi(x, h, t)/g\partial t$ for all x, t. (3.7d)

Problem 3.11 Using the above theory, find the speed of propagation of a monochromatic surface wave of wavelength λ over still water of uniform depth h. Deduce the complex potential of motion *relative* to the wave.

Solution. We assume that $\eta = a\sin(mx-nt)$ for which $\lambda = 2\pi/m$ and the speed $c = n/m$. Using (3.7d), for all x, t,

$$\frac{\partial\varphi(x,h,t)}{\partial t} = g\eta = ag\sin(m-nt) \tag{3.8}$$

from which we deduce that φ has the form $\varphi = f(y)\cos(mx-nt)$. Using (3.7a), $\nabla^2\varphi = (f_{yy}-m^2f)\cos(mx-nt) = 0$ for all x, t, implies $f_{yy}-m^2f = 0$ or $f = A\cosh my + B\sinh my$ where A and B are arbitrary constants. Invoking condition (3.7c) we must have $B = 0$, giving $\varphi = A\cosh my \times \cos(mx-nt)$ and, by equation 3.8, $nA\cosh mh = ag$. Finally, the surface condition (3.6) gives the speed for, when $y = h$, for all x, t, $\partial^2\varphi/\partial t^2 = -n^2A\cosh mh\cos(mx-nt) = -g(\partial\varphi/\partial y) = -gmA\sinh mh\cos(mx-nt)$

i.e. $$c^2 = (n/m)^2 = (g/m)\tanh mh, \qquad m = 2\pi/\lambda \tag{3.9}$$

To find the complex potential of relative motion we have $\varphi = A\cosh my\cos(mx-nt) = \operatorname{Re} w$ where $w = A\cos(mz-nt)$ or $w = A\cos m(z-ct)$ with $A = ac\operatorname{cosech} mh$ from equations 3.9. Replacing $z-ct$ *by* z and imposing a velocity $-c\mathbf{i}$ on the whole system, the axes and wave profile are brought to rest whilst the liquid has a velocity $-c\mathbf{i}$. Choosing the new origin $z = 0$ in the *free* surface, the required complex potential becomes

$$w = cz + ac\operatorname{cosech} mh\cos m(z+ih) \tag{3.10}$$

Moreover it is easily verified that the corresponding ψ is

$$\psi = cy - ac\operatorname{cosech} mh\sinh m(y+h)\sin mx$$

so that the bed $y = -h$ is the streamline $\psi = -ch$ and neglecting a^2 (compared with a) the surface $y = a\sin mx$ corresponds to $\psi = 0$. □

Problem 3.12 A liquid of density ρ_1 fills the region $0 \leqslant y \leqslant h_1$ and flows with velocity $U_1\mathbf{i}$ over an immiscible liquid of density $\rho_2(>\rho_1)$ which fills the region $-h_2 \leqslant y \leqslant 0$ flowing with velocity $U_2\mathbf{i}$. Assuming that rigid walls lie along $y = h_1$ and $y = -h_2$, find an expression for the speed of propagation of a small surface wave at the interface of the two liquids.

Solution. We assume the wave has an elevation $\eta = a\sin(mx-nt)$ above the interface $y = 0$. Following the method of the last problem we superimpose on the whole system a velocity $-c\mathbf{i}$ which reduces the wave

to rest and changes the streaming velocities to $(U_1-c)\mathbf{i}$, $(U_2-c)\mathbf{i}$. Using equation 3.10, the complex potential of the lower liquid in $-h_2 \leqslant y \leqslant 0$ is

$$w_2 = -(U_2-c)z - a(U_2-c)\operatorname{cosech} mh_2\cos m(z+ih_2)$$

where $\eta = a\sin mx$ is the streamline $\psi_2 = 0$. The liquid speed is q_2 where

$$q_2^2 = \left(\frac{dw_2}{dz}\right)\left(\frac{d\bar{w}_2}{d\bar{z}}\right) = (U_2-c)^2\{1-ma\operatorname{cosech} mh_2\sin m(z+ih_2)\} \times \{1-ma\operatorname{cosech} mh_2\sin m(\bar{z}-ih_2)\}$$
$$= (U_2-c)^2\{1-2ma\operatorname{cosech} mh_2\cosh m(y+h_2)\sin mx\} + O(a^2)$$

Neglecting a^2, at the interface $y = 0$ the speed is q_{20} where

$$q_{20}^2 = (U_2-c)^2(1-2ma\coth mh_2\sin mx)$$

We find the corresponding result for the upper liquid by writing U_1 for U_2 and $-h_1$ for h_2, i.e. the speed q_{10}^2 is given by

$$q_{10}^2 = (U_1-c)^2(1+2ma\coth mh_1\sin mx)$$

Since the pressure must be continuous across the interface for all x

$$p_{10} = \text{constant} - \tfrac{1}{2}\rho_1q_{10}^2 - \rho_1ga\sin mx$$
$$= \text{constant} - \tfrac{1}{2}\rho_2q_{20}^2 - \rho_2ga\sin mx = p_{20}$$

The coefficients of $\sin mx$ on either side must equate in which case

$$m\rho_1(U_1-c)^2\coth mh_1 + m\rho_2(U_2-c)^2\coth mh_2 = g(\rho_2-\rho_1)$$

the required result. □

EXERCISES

1. A two-dimensional vortex of strength k is placed at the origin $z = 0$ in a liquid confined between the two parallel walls $\operatorname{Im} z = \pm\frac{1}{2}a$. Show that the complex potential of the liquid motion due to the vortex is $w = ik\ln\tanh(\pi z/2a)$. Prove that the vortex will remain at rest and that the streamline which passes through the point $z = \alpha ai$ $(0 < \alpha < \frac{1}{2})$ will intersect the real axis at the points $z = \pm(a/\pi)\ln(\tan\alpha\pi + \sec\alpha\pi)$.

2. A vortex of circulation $2\pi k$ is at rest at the point $z = a\sec\alpha$ (α is real and $0 < \alpha < \frac{1}{2}\pi$) in the presence of the circular boundary $|z| = a$, around which there is a circulation $2\pi k'$. Show that $k' = k\cot^2\alpha$. Prove that there are two stagnation points on the boundary $z = ae^{i\theta}$ symmetrically placed about the real axis in the quadrants nearest to the external vortex given by $2\cos\theta = \cos\alpha\,(3-\cos^2\alpha)$, and deduce that θ is real.

3. Three vortex filaments, each of strength k are symmetrically placed inside a fixed circular cylinder of radius a, and pass through the corners of an equilateral triangle of side $\frac{1}{2}a\sqrt{3}$. Given that there is no circulation in the liquid apart from the effect of the vortices, prove that they will revolve about the axis of the cylinder with angular velocity $88k/21a^2$.

4. Find the complex potential for vortices of strengths k and $-k$ at ζ_0 and $-\bar{\zeta}_0$ respectively outside the cylinder $|\zeta| = c$ at rest in uniform streaming motion of magnitude U parallel to the imaginary axis when there is no circulation about the cylinder. Apply the transformation $iz = \zeta + c^2/\zeta$, $c = \frac{1}{2}a$, to show that the complex potential of motion due to vortices $\pm k$ at z_0, $\bar{z}_0$ behind a plate of length $2a$ is

$$w = -U\sigma(z) + ik\ln\{(\sigma(z)-\sigma(z_0))/(\sigma(z)-\sigma(\bar{z}_0))\}, \quad \sigma(z) = (z^2+a^2)^{\frac{1}{2}}$$

Deduce that in order that the vortices remain at rest $z_0^2\sigma(z_0)$ must be purely imaginary.

5. A source, of strength m, move along the axis OX with velocity U relative to the undisturbed liquid. Show that the equations to the paths of the liquid particles can be expressed in the form $x-Ut = R\cos\theta$, $y = R\sin\theta$, $R = \{(x-Ut)^2+y^2\}^{\frac{1}{2}} = (U\sin\theta)/\dot{\theta} = r(\theta-\alpha)/\sin\theta$, $\sin^2\theta = r(\theta-\alpha)\dot{\theta}$ where $r = m/U^2$. Prove that

$$t/r = -(\theta-\alpha)\cot\theta + \ln\sin\theta + \text{constant}$$

Chapter 4

Three-Dimensional Axisymmetric Flow

4.1 Fundamentals We assume that (i) density ρ is constant, (ii) the geometry of any immersed obstacle together with the flow is symmetrical about OX, the axis of symmetry, (iii) using cylindrical coordinates (x, ϖ, θ) where $x = x$, $y = \varpi\cos\theta$, $z = \varpi\sin\theta$, the flow is independent of θ, i.e. $\partial/\partial\theta \equiv 0$.

The *velocity vector* is $\mathbf{q} = q_x\mathbf{i} + q_\varpi\boldsymbol{\varpi} + q_\theta\boldsymbol{\theta}$ where $\mathbf{i}, \boldsymbol{\varpi}, \boldsymbol{\theta}$, are the unit vectors parallel to OX and perpendicular to OX in the radial and transverse directions respectively. These components q_x, q_ϖ and q_θ are functions of x and ϖ only.

In any source-free region $\mathscr{R}_s$ the *equation of continuity* is

$$0 = \operatorname{div}\mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi q_\varpi). \tag{4.1}$$

The *stream function* $\psi = \psi(x, \varpi)$ exists in $\mathscr{R}_s$ by virtue of equation 4.1 and satisfies

$$\varpi q_\varpi = \frac{\partial\psi}{\partial x}, \quad \varpi q_x = -\frac{\partial\psi}{\partial\varpi} \tag{4.2}$$

ψ is called the *Stokes* stream function.

Alternatively, we have $\mathbf{q} = \operatorname{curl}(-\psi\boldsymbol{\theta})$ where ψ is a component of a vector potential function.

The *vorticity*

$$\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \frac{1}{\varpi}\begin{vmatrix} \mathbf{i} & \boldsymbol{\varpi} & \varpi\boldsymbol{\theta} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial\varpi} & 0 \\ q_x & q_\varpi & q_\theta \end{vmatrix} \tag{4.3}$$

When $q_\theta = 0$, $\boldsymbol{\zeta} = \zeta\boldsymbol{\theta}$ where

$$\zeta = \frac{\partial q_\varpi}{\partial x} - \frac{\partial q_x}{\partial\varpi} = \frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) = \frac{1}{\varpi}E(\psi) \tag{4.4}$$

and

$$E \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}$$

When flow is irrotational $\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \mathbf{0}$, for which φ exists with

$\mathbf{q} = -\operatorname{grad} \varphi$. By equation 4.4, ψ satisfies the equation

$$E(\psi) = 0 = \frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial x}\right)+\frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right) \tag{4.5}$$

The flux of volume flow Q across the curved surface formed by the complete rotation of the meridian plane curve joining $A(a, b)$ to $P(x, \varpi)$ in Figure 4.1 (where ds is an element of AP and is inclined to OX at an angle λ) is

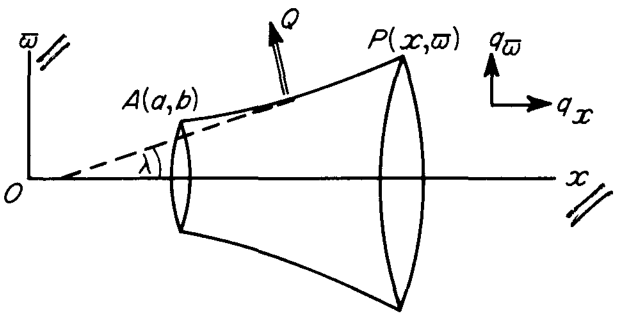


Figure 4.1

$$\begin{aligned} Q &= \int_A^P (q_\varpi \cos\lambda - q_x \sin\lambda) 2\pi\varpi \, ds \\ &= 2\pi \int_A^P \left(\frac{\partial \psi}{\partial x}\frac{dx}{ds}+\frac{\partial \psi}{\partial \varpi}\frac{d\varpi}{ds}\right) ds \\ &= 2\pi \int_A^P d\psi = 2\pi(\psi_P - \psi_A) \end{aligned}$$

where Q is taken as positive measured in the sense right to left for an observer at A looking towards P. When P lies on the stream surface through A, $Q = 0$ or $\psi_P = \psi_A$.

To find the velocity in terms of ψ let $P(x, \varpi)$ and $Q(x+\delta x, \varpi+\delta\varpi)$ be neighbouring points belonging to surfaces ψ = constant and $\psi+\delta\psi$ = constant and respectively. The volume flux across the surface (a conical frustrum) formed by the revolution about OX of $PQ = \delta s$ is $2\pi(\psi+\delta\psi-\psi) = 2\pi\,\delta\psi$. Referring to Figure 4.2 the area of this frustrum is $2\pi\varpi\,\delta s$ so that if q_n is the average normal component of velocity across PQ we have

$$2\pi\varpi\,\delta s\, q_n + O(\delta s^2) = 2\pi\,\delta\psi$$

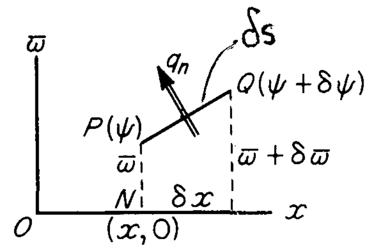


Figure 4.2

i.e.

$$q_n = \lim_{\delta s \to 0} \frac{1}{\varpi}\frac{\delta\psi}{\delta s} = \frac{1}{\varpi}\frac{\partial\psi}{\partial s} = \text{normal velocity component}$$

Putting

$$\delta s \equiv \delta\varpi, \; q_n \equiv -q_x = \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}; \quad \delta s \equiv \delta x, \; q_n \equiv q_\varpi = \frac{1}{\varpi}\frac{\partial\psi}{\partial x};$$

which are the relations (4.2).

4.2 Spherical polar coordinates In terms of cylindrical coordinates (x, ϖ). The polar coordinates (r, θ) are defined by $x = r\cos\theta$, $\varpi = r\sin\theta$. In axisymmetrical flow we write the velocity $\mathbf{q} = q_r\mathbf{r}+q_\theta\boldsymbol{\theta}$, $\mathbf{r}, \boldsymbol{\theta}$ being unit vectors in the radial and transverse directions. These components can be expressed in terms of ψ using equation 4.5. With $\delta s \equiv r\,\delta\theta$, $-q_n \equiv q_r = -r^{-2}(\operatorname{cosec}\theta)(\partial\psi/\partial\theta)$ and with $\delta s \equiv \delta r$, $q_n \equiv q_\theta = r^{-1}(\operatorname{cosec}\theta)\times(\partial\psi/\partial r)$. If a velocity potential φ exists then $\mathbf{q} = -\operatorname{grad}\varphi$ giving $q_r = -\partial\varphi/\partial r$, and $q_\theta = -\partial\varphi/r\partial\theta$. Replacing the variable θ by μ where $\mu = \cos\theta$, we have

$$\begin{aligned} r^2 q_r &= -r^2\varphi_r = \psi_\mu \\ r\sin\theta\, q_\theta &= (1-\mu^2)\varphi_\mu = \psi_r \end{aligned} \tag{4.6}$$

where the suffixes attached to φ and ψ only denote partial differentiation. Eliminating ψ the equation for φ is $(r^2\varphi_r)_r + \{(1-\mu^2)\varphi_\mu\}_\mu = 0$. A solution is

$$\varphi = \{Ar^n + Br^{-n-1}\}P_n(\mu), \qquad P_n(\mu) = \frac{1}{2^n n!}\left(\frac{d^n}{d\mu^n}\right)(\mu^2-1)^n \tag{4.7}$$

where n is a positive integer, A and B are constants and $P_n(\mu)$ is a Legendre polynomial of order n. The equation for ψ is $(1-\mu^2)\psi_{\mu\mu}+r^2\psi_{rr} = 0$ with a solution satisfying equations 4.6 and 4.7 given by

$$\psi = \{A(n+1)^{-1}r^{n+1} - Bn^{-1}r^{-n}\}(1-\mu^2)\,dP_n(\mu)/d\mu \tag{4.8}$$

4.3 Elementary results

4.3.1 Uniform stream Given $q = Ui$ = constant everywhere, $\varphi = -Ux$ and by equations 4.2 $\partial\psi/\partial\varpi = -U$, $\partial\psi/\partial x = 0$ or $\psi = -\frac{1}{2}U\varpi^2$ (ignoring the constant of integration).

4.3.2 Point source A *point source* at $\mathbf{r} = \mathbf{0}$ has constant volume output $4m\pi$, i.e. of *strength m*. Here $\int_s \mathbf{q}\,.\,d\mathbf{S} = 4m\pi$ for *all* simple surfaces S enclosing $\mathbf{r} = \mathbf{0}$. Choosing S as the sphere $|\mathbf{r}| = r$ = constant, on which, by symmetry, $q_r = q_r(r)$, $q_\theta = 0$ we have $4m\pi = 4\pi r^2 q_r$. From equations 4.6,

$$r^2 q_r = m = -r^2\varphi_r = \psi_\mu$$

$$r\sin\theta\, q_\theta = 0 = (1-\mu^2)\varphi_\mu = \psi_r$$

Solving,

$$\varphi = m/r \quad\text{and}\quad \psi = m\mu = mx/r = m(\mathbf{i}\,.\,\mathbf{r})/r$$

ψ of course exists everywhere *except at the origin* $\mathbf{r} = \mathbf{0}$. If the source is placed at a point $\mathbf{r} = \mathbf{a}$ instead, we have

$$\varphi = m/|\mathbf{r}-\mathbf{a}|, \qquad \psi = m\mathbf{i}\,.\,(\mathbf{r}-\mathbf{a})/|\mathbf{r}-\mathbf{a}| \tag{4.9}$$

The resultant flow flow is now axisymmetric about an axis through $\mathbf{r} = \mathbf{a}$.

4.3.3 Doublet A *doublet* whose strength and axial direction is given by $\boldsymbol{\mu}$ at $\mathbf{r} = \mathbf{0}$ is defined as the combination of a sink of strength m at $\mathbf{r} = \mathbf{0}$ with a source of equal strength at $\mathbf{r} = \epsilon\boldsymbol{\mu}$ where $\epsilon \to 0$ and $m \to \infty$ with $m\epsilon = 1$. Using equations 4.9

$$\varphi = \lim_{m\to\infty,\,\epsilon\to 0}\left\{\frac{m}{|\mathbf{r}|} - \frac{m}{|\mathbf{r}-\epsilon\boldsymbol{\mu}|}\right\}$$

$$= \lim_{m\to\infty,\,\epsilon\to 0}(-\epsilon\boldsymbol{\mu}\,.\,\nabla)\frac{m}{r} = -(\boldsymbol{\mu}\,.\,\nabla)\frac{1}{r} = \frac{\boldsymbol{\mu}\,.\,\mathbf{r}}{r^3} \tag{4.10}$$

(Note that $d\mathbf{r}\,.\,\nabla f = df(\mathbf{r}) = f(\mathbf{r}+d\mathbf{r}) - f(\mathbf{r})$ and the direction of $d\mathbf{r}$ is $\hat{\boldsymbol{\mu}}$). Since, by equations 4.9, when $\varphi = 1/r$, $\psi = x/r$, the corresponding solution for ψ is,

$$\psi = -(\boldsymbol{\mu}\,.\,\nabla)\frac{x}{r} = -\boldsymbol{\mu}\,.\,\nabla\left(\frac{x}{r}\right) = \boldsymbol{\mu}\,.\,\frac{(r^2\mathbf{i} - x\mathbf{r})}{r^3}$$

$$= [\boldsymbol{\mu}, \mathbf{r}, \mathbf{i}\wedge\mathbf{r}]/r^3 \tag{4.11}$$

4.4 Butler's sphere theorem

4.4 Butler's sphere theorem $\psi_0(r,\mu)$ is the stream function of an axisymmetric irrotational flow devoid of rigid boundaries and $\psi_0 = 0$ at $\mathbf{r} = \mathbf{0}$. If a rigid sphere $|\mathbf{r}| = a$ is introduced into this flow and none of

its singularities is covered by the sphere, the new motion is represented by the stream function

$$\psi = \psi_0(r,\mu) - (r/a)\psi_0(a^2/r,\mu) \tag{4.12}$$

Denoting the image stream function $-(r/a)\psi_0(a^2/r,\mu)$ by ψ_1 there are four steps to establish in the proof:

1 Since motion is irrotational, by Section 4.2, ψ_0 satisfies $(1-\mu^2)\psi_{\mu\mu} + r^2\psi_{rr} = 0$ in which case ψ_1 also satisfies this condition for irrotationality. (The verification is left to the reader.)

2 $\psi(a,\mu) = 0$ for all μ, i.e. the sphere $|\mathbf{r}| = a$ *is* a stream surface.

3 Since r and a^2/r are inverse points with respect to $|\mathbf{r}| = a$ all singularities of ψ_1 *lie inside* $|\mathbf{r}| = a$ given those of ψ_0 lie outside, i.e. no new singularities are introduced *into the fluid* by the image system.

4 With ψ_0 regular in $|\mathbf{r}| < a$ and $\psi_0 = 0$ when $\mathbf{r} = \mathbf{0}$ we have $\psi_0 = O(r)$ for small r so that $\psi_1 = O(1/r) +$ constant, or by equations 4.6, the velocity at infinity due to ψ_1 is $O(1/r^3)$ which tends to zero as $r \to \infty$. Moreover, the volume flux across the sphere at infinity is $O(1/r)$ which vanishes as $r \to \infty$.

The verification is therefore established.

Problem 4.1 Find the liquid speed on a solid sphere due to an external source.

Solution. We assume that a source of strength m lies at $\mathbf{r} = b\mathbf{i}$ outside the sphere $|\mathbf{r}| = a$. The axis OX is the axis of symmetry. Using equations 4.9 the stream function of this source is $\psi_s = m(x-b)/\{(x-b)^2+\varpi^2\}^{\frac{1}{2}}$. However, near $\mathbf{r} = \mathbf{0}$ this expression behaves like $-m$. Consequently, to ensure the correct conditions at infinity in Butler's theorem we must arrange that $\psi = 0$ at $\mathbf{r} = \mathbf{0}$ by adding a constant m to ψ_s. In the notation of Section 4.4 with $\mu = \cos\theta$, $x = r\mu$, $\varpi = r(1-\mu^2)^{\frac{1}{2}}$,

$$\psi_0 = m + \psi_s = m + m(r\mu - b)/(r^2 - 2br\mu + b^2)^{\frac{1}{2}}$$

Hence, using equation 4.12, the final stream function in the presence of the sphere is

$$\psi = m - \frac{mr}{a} + \frac{m(r\mu - b)}{(r^2-2br\mu+b^2)^{\frac{1}{2}}} - \frac{mr(a^2\mu - br)}{a(a^4 - 2a^2br\mu + b^2r^2)^{\frac{1}{2}}}$$

The radial component of the liquid velocity on $|\mathbf{r}| = a$ must be zero so that the required speed is simply the transverse component $q_\theta = r^{-1}\times$ cosec $\theta\psi_r$. Now, writing $(r^2-2br\mu+b^2)^{\frac{1}{2}} = A(r)$, $(a^4-2a^2br\mu+b^2r^2)^{\frac{1}{2}} = B(r)$

$$\psi_r = -\frac{m}{a} + \frac{m\mu}{A(r)} + \frac{m(r\mu-b)(b\mu-r)}{A^3(r)} - \frac{m(a^2\mu-2br)}{aB(r)} - \frac{mr(a^2\mu-br)(a^2b\mu-b^2r)}{aB^3(r)}$$

When $r = a$, $A(a) = (a^2-2ab\mu+b^2)^{\frac{1}{2}}$, $B(a) = aA(a)$, so that

$$\psi_r|_{r=a} = -\frac{m}{a}+\frac{2mb}{aA(a)}+\frac{m(b^2-a^2)(a\mu-b)}{aA^3(a)}, \qquad b-a \leqslant A(a) \leqslant b+a$$

giving the speed q_θ on $r = a$ as

$$q_\theta = \frac{m}{a^2(1-\mu^2)^{\frac{1}{2}}}\left\{\frac{b(3a^2+b^2)-a\mu(3b^2+a^2)}{A^3(a)}-1\right\}, \qquad \mu = \cos\theta \qquad \square$$

Problem 4.2 Show that $\psi = \frac{1}{2}U\varpi^2\{1-2a/[x+\sqrt{(x^2+\varpi^2)}]\}$ is a possible stream function in irrotational motion and deduce that it represents streaming motion past a paraboloid. Given that the pressure at infinity is zero show that the pressure at any point P(x, ϖ) of the surface varies inversely with $2a-x$.

Solution. By equation 4.5, ψ must satisfy the equation

$$\frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right)=0$$

Since $\{x+\sqrt{(x^2+\varpi^2)}\}^{-1} = \{\sqrt{(x^2+\varpi^2)}-x\}/\varpi^2$, the given ψ can be written as $\psi = \frac{1}{2}U\varpi^2 - Ua\{\sqrt{(x^2+\varpi^2)}-x\}$. The first term which represents a uniform stream is an obvious solution of the equation and so is Uax. Again, writing $\sqrt{(x^2+\varpi^2)} = \chi$, $\chi_x = x(x^2+\varpi^2)^{-\frac{1}{2}}$, $\chi_\varpi = \varpi(x^2+\varpi^2)^{-\frac{1}{2}}$ so that

$$\frac{\partial}{\partial x}\left(\frac{1}{\varpi}\frac{\partial\chi}{\partial x}\right)+\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\chi}{\partial\varpi}\right) = \frac{1}{\varpi}\{(x^2+\varpi^2)^{-\frac{1}{2}}-x^2(x^2+\varpi^2)^{-\frac{3}{2}}\} - \varpi(x^2+\varpi^2)^{-\frac{3}{2}} = 0$$

in which case χ and Uax are solutions, i.e. the given ψ *is a possible stream function.*

To interpret the motion we have $\psi = 0$ when $\varpi = 0$, the axis, or $x+\sqrt{(x^2+\varpi^2)} = 2a$ so that $x^2+\varpi^2 = (2a-x)^2$ or $\varpi^2 = -4a(x-a)$, which is the equation of a paraboloid with focus at $(0,0)$ and vertex at $(a,0)$. Moreover, when $\sqrt{(x^2+\varpi^2)}$ is large, $\psi \sim \frac{1}{2}U\varpi^2$, i.e. ψ behaves like a uniform stream $-U\mathbf{i}$ at infinity. The motion, therefore, is a uniform stream of magnitude U moving parallel to the axis and towards the vertex of the paraboloid defined by the above equation.

The pressure p is given by $p+\frac{1}{2}\rho q^2 =$ constant where from the evaluations at infinity, the constant is $\frac{1}{2}\rho U^2$. On the paraboloid where $\varpi^2 = -4a(x-a)$,

$$q_\varpi = \frac{1}{\varpi}\frac{\partial\psi}{\partial x} = -\frac{Ua}{\varpi}\{x(x^2+\varpi^2)^{-\frac{1}{2}}-1\} = \frac{2Ua}{\varpi}\left(\frac{a-x}{2a-x}\right)$$

and

$$q_x = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = -U+Ua(x^2+\varpi^2)^{-\frac{1}{2}} = -U\left(\frac{a-x}{2a-x}\right)$$

so that

$$q^2 = q_\varpi^2+q_x^2 = \left(\frac{a-x}{2a-x}\right)^2\left(1+\frac{4a^2}{\varpi^2}\right) = \frac{a-x}{2a-x}$$

Thus

$$p = \frac{1}{2}\rho U^2-\frac{1}{2}\rho U^2\left(\frac{a-x}{2a+x}\right) = \frac{1}{2}\rho a\frac{U^2}{2a-x} \qquad \square$$

Problem 4.3 A uniform straight-line sink of total volume flux input $12\pi a^2 U$ lying on OX between $x = -a$ to $x = 0$ is followed by a compensating uniform line source of total output $12\pi a^2 U$ stretching from $x = 0$ to $x = a$. If a uniform stream $-U\mathbf{i}$ is introduced, show that the resultant flow is equivalent to a streaming motion past a solid of revolution of length $4a$ which is symmetrical about the equatorial plane $x = 0$. Prove also that the radius of the equatorial section is σa where σ is a root of $\sigma^3(\sigma+24) = 144$ and deduce that the liquid speed on the equator is $U(18+\sigma)/(12+\sigma)$.

Solution. For the line source of total output $12\pi a^2 U$ in $0 \leqslant x \leqslant a$, the strength per unit length is $\lambda = 12\pi a^2 U/(4\pi a) = 3Ua$. By equations 4.9 the stream function $d\psi$ for an element of this source of length $d\xi$ at $x = \xi$ is $\lambda(x-\xi)\{(x-\xi)^2+\varpi^2\}^{-\frac{1}{2}}\,d\xi$. Hence integrating, the stream function ψ_+ due to the line source is

$$\psi_+ = \int_0^a \lambda(x-\xi)\{x-\xi)^2+\varpi^2\}^{-\frac{1}{2}}\,d\xi = -\lambda[\{x-\xi)^2+\varpi^2\}^{\frac{1}{2}}]_0^a$$
$$= \lambda\{(x^2+\varpi^2)^{\frac{1}{2}}-((x-a)^2+\varpi^2)^{\frac{1}{2}}\}, \qquad \lambda = 3Ua$$

Similarly the stream function due to the line sink is

$$\psi_- = \lambda[\{(x-\xi)^2+\varpi^2\}^{\frac{1}{2}}]_{-a}^0 = \lambda\{(x^2+\varpi^2)^{\frac{1}{2}}-((x+a)^2+\varpi^2)^{\frac{1}{2}}\}, \qquad \lambda = 3Ua$$

adding to the stream function $\frac{1}{2}U\varpi^2$ of the uniform stream the final ψ is

$$\psi = \frac{1}{2}U\varpi^2+\psi_++\psi_-$$
$$= \frac{1}{2}U\varpi^2+3Ua\{2(x^2+\varpi^2)^{\frac{1}{2}}-((x+a)^2+\varpi^2)^{\frac{1}{2}}-((x-a)^2+\varpi^2)^{\frac{1}{2}}\}$$

It is important to realise that all the square roots must be given their positive values, e.g. when $\varpi = 0$

for $x > a$, $\quad \psi = 3Ua\{2(x)-(x+a)-(x-a)\} = 0$

for $x < -a$, $\quad \psi = 3Ua\{2(-x)-(-x-a)-(a-x)\} = 0$

for $-a < x < a$, $\quad \psi = 3Ua\{2|x|-(a+x)-(a-x)\} = 6Ua\{|x|-a\} \neq 0$

$\psi = 0$ is a *dividing* stream surface giving the axis of symmetry as one branch (provided $|x| > a$) and a closed surface as another. If (x, ϖ) lies on this surface so does $(-x, \varpi)$, i.e. the surface is symmetrical about the plane $x = 0$. The equation of the body is,

$$\psi/\tfrac{1}{2}U\varpi^2 = 0 = 1+6a\{2(x^2+\varpi^2)^{\frac{1}{2}}-((x+a)^2+\varpi)^{\frac{1}{2}}-((x-a)^2+\varpi^2)^{\frac{1}{2}}\}/\varpi^2$$

It meets the axis at the value of x for which $\varpi = 0$. For small ϖ and $x > a$ we have

$$0 = 1+\frac{6a}{\varpi^2}\left\{2x\left(1+\frac{\varpi^2}{2x^2}\right)-(x+a)\left(1+\frac{\varpi^2}{2(x+a)^2}\right)\right.$$
$$\left.-(x-a)\left(1+\frac{\varpi^2}{2(x-a)^2}\right)+O(\varpi^4)\right\}$$

In the limit as $\varpi \to 0$, x satisfies

$$0 = 1+6a\left(\frac{1}{x}-\frac{1}{2(x+a)}-\frac{1}{2(x-a)}\right)$$

or $x(x^2-a^2) = 6a^3$, giving one real root $x = 2a$, which is a half length of the body. Alternatively, we could find this point from the condition that it must be a point of liquid stagnation, i.e. $q_x = -\varpi^{-1}\,\partial\psi/\partial\varpi = 0$ when $\varpi = 0$ and $x = 2a$.

At $x = 0$, the radius of this equatorial section is σa. From the body equation putting $\varpi = \sigma a$ with $x = 0$,

$$0 = 1+6a\{2\,\sigma a - a(1+\sigma^2)^{\frac{1}{2}} - a(1+\sigma^2)^{\frac{1}{2}}\}/(\sigma^2 a^2)$$

i.e.

$$(1+\sigma^2)^{\frac{1}{2}}-\sigma = \tfrac{1}{12}\sigma^2$$

or

$$\sigma^3(\sigma+24) = 144$$

By symmetry, $q_\varpi = \varpi^{-1}\,\partial\psi/\partial x = 0$ at $x = 0$ so that the equatorial liquid speed is (in the negative sense),

$$q = -(q_x)_{x=0} = \frac{1}{\varpi}\left|\frac{\partial\psi}{\partial\varpi}\right|_{x=0} = U+3Ua\left\{\frac{2}{\varpi}-\frac{1}{(a^2+\varpi^2)^{\frac{1}{2}}}-\frac{1}{(a^2+\varpi^2)^{\frac{1}{2}}}\right\},$$
$$\varpi = a\sigma$$
$$= U+6U\left\{\frac{1}{\sigma}-\frac{1}{(1+\sigma^2)^{\frac{1}{2}}}\right\}$$

But $(1+\sigma^2)^{\frac{1}{2}} = \sigma(1+\frac{1}{12}\sigma)$, hence

$$\frac{q}{U} = 1+6\left(\frac{1}{\sigma}-\frac{12}{\sigma(12+\sigma)}\right) = \frac{18+\sigma}{12+\sigma}$$ □

Problem 4.4 In the case of steady axisymmetric motion show that the vorticity ζ and stream function ψ satisfy the Jacobian $\partial(\psi, \zeta/\varpi)/\partial(x, \varpi) = 0$. Deduce that there exists a solution $\zeta = A\varpi$ with $\psi = \{B+A(x^2+\varpi^2)/10\}\varpi^2$ where A and B are constants. Interpret this solution when $B = -a^2A/10$.

Solution. From equation 1.17 of Section 1.7 for steady flow

$$\mathbf{q}\wedge\boldsymbol{\zeta} = \nabla\chi \quad \text{where } \chi = \int \rho^{-1}\,dp+\tfrac{1}{2}q^2+\Omega$$

so that $\operatorname{curl}(\mathbf{q}\wedge\boldsymbol{\zeta}) = \operatorname{curl}\operatorname{grad}\chi = \mathbf{0}$. In the case of axisymmetric motion $\mathbf{q} = q_x\mathbf{i}+q_\varpi\boldsymbol{\varpi}$. From equation 4.3,

$$\boldsymbol{\zeta} = \operatorname{curl}\mathbf{q} = \zeta\boldsymbol{\theta} = \left(\frac{\partial q_\varpi}{\partial x}-\frac{\partial q_x}{\partial\varpi}\right)\boldsymbol{\theta}$$

so that

$$\mathbf{q}\wedge\boldsymbol{\zeta} = \zeta q_\varpi\,\mathbf{i}-\zeta q_x\,\boldsymbol{\varpi}$$

and

$$\operatorname{curl}(\mathbf{q}\wedge\boldsymbol{\zeta}) \equiv \frac{1}{\varpi}\begin{vmatrix}\mathbf{i} & \boldsymbol{\varpi} & \varpi\boldsymbol{\theta}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial\varpi} & 0\\ \zeta q_\varpi & -\zeta q_x & 0\end{vmatrix} = \mathbf{0}$$

Thus

$$\frac{\partial}{\partial x}(\zeta q_x)+\frac{\partial}{\partial\varpi}(\zeta q_\varpi) = 0 \quad\text{or}\quad -\frac{\partial}{\partial x}\left(\frac{\zeta}{\varpi}\frac{\partial\psi}{\partial\varpi}\right)+\frac{\partial}{\partial\varpi}\left(\frac{\zeta}{\varpi}\frac{\partial\psi}{\partial x}\right) = 0$$

which is

$$\frac{\partial(\psi, \zeta/\varpi)}{\partial(x, \varpi)} = 0$$

The Jacobian is obviously zero when $\zeta/\varpi = \text{constant} = A$. Using equation 4.4 ψ must then satisfy the equation

$$E(\psi) \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right)\psi = A\varpi^2$$

Substituting $\psi = C(x^2\varpi^2+\varpi^4)$ where C is constant as a trial solution, we have $E(\psi) = C(2\varpi^2+12\varpi^2-2x^2-4\varpi^2) = 10C\varpi^2$. Hence, choosing $C = A/10$, we have a *particular integral*. Substituting $\psi = B\varpi^2$, $E(\psi) = 0$, so that $\psi = \{B+A(x^2+\varpi^2)/10\}\varpi^2$ *is* a solution for arbitrary constants A and B. Choosing $B = -a^2A/10$ and writing $x = r\cos\theta$, $\varpi = r\sin\theta$, we have $\psi = -(A/10)(a^2-r^2)r^2\sin^2\theta$ for which $\psi = 0$ when $r = a$. Consequently, a vortex for which $\zeta = A\varpi = Ar\sin\theta$ can be contained within the sphere $r = a$. This is known as *Hill's spherical vortex*. □

Problem 4.5 An inviscid incompressible liquid moves irrotationally with velocity potential $\varphi_0 = Ux$ where U is constant. Verify that the perturbation in φ_0 when a sphere $r = a$ is introduced with its centre at the origin $\mathbf{r} = \mathbf{0}$ is $\frac{1}{2}U(a/r)^3x$. If the sphere is divided into two hemispheres by a plane passing through the axis OX prove that the force on either portion due to the liquid pressure is $\{(11/32)\rho U^2-p_\infty\}\pi a^2$ where p_∞ is the liquid pressure at infinity.

Solution. If φ_1 denotes the perturbation potential then $\varphi = \varphi_1+\varphi_0$ must satisfy the conditions (i) $\nabla^2\varphi = 0$ in the liquid, (ii) $\partial\varphi/\partial r = 0$ when $r = a$, (iii) $\mathbf{q} = -\operatorname{grad}\varphi = -U\mathbf{i}$ as $r \to \infty$.

Since $\varphi_0 = Ux = Ur\cos\theta$, $\nabla^2\varphi_0 = 0$, $\partial\varphi_0/\partial r = U\cos\theta$, $\operatorname{grad}\varphi_0 = -U\mathbf{i}$. Hence we seek a solution φ_1 of $\nabla^2\varphi_1 = 0$ where $\operatorname{grad}\varphi_1 \to 0$ as $r \to \infty$ and $\partial\varphi_1/\partial r = -U\cos\theta$ when $r = a$ for all θ. Using equations 4.7 the first two conditions are fulfilled by choosing $\varphi_1 = \Sigma B_n r^{-n-1} \times P_n(\cos\theta)$. Finally, to satisfy the boundary condition on the sphere, we must have $n = 1$ and $2B_1 = Ua^3$ giving $\varphi_1 = \frac{1}{2}Ua^3r^{-2}\cos\theta$, as the result.

On the sphere, $\mathbf{r} = a(\cos\theta\mathbf{i}+\sin\theta\cos\omega\,\mathbf{j}+\sin\theta\sin\omega\mathbf{k})$, $dS = a^2\sin\theta \times d\omega\, d\theta$. We choose the hemisphere for which $z = \mathbf{r}.\mathbf{k} \geqslant 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \omega \leqslant \pi$. By symmetry the force on this hemisphere is $-Z$ along the z-axis and is given by $Z = -\iint p\mathbf{n}.\mathbf{k}\, dS$ evaluated over the surface where p is the liquid pressure and $\mathbf{n} = \mathbf{r}/a$ the unit outward normal to the surface at the element dS. Using Bernoulli's equation we have $p+\frac{1}{2}\rho q^2 = p_\infty+\frac{1}{2}\rho U^2 = P$ where q is the liquid speed on $r = a$, i.e.

$$q = q_\theta = (r^{-1}\partial\varphi/\partial\theta)_{r=a} = -\tfrac{3}{2}U\sin\theta,$$

from $\varphi = U\cos\theta(r+\frac{1}{2}a^3r^{-2})$. Hence

$$Z = -a^{-1}\iint (P-\tfrac{1}{2}\rho q^2)\mathbf{r}.\mathbf{k}\, dS$$

$$= -a^2\int_{\omega=0}^{\pi}\int_{\theta=0}^{\pi}(P-\tfrac{9}{8}\rho U^2\sin^2\theta)\sin^2\theta\sin\omega\, d\theta\, d\omega$$
$$= 4a^2\int_0^{\frac{1}{2}\pi}(\tfrac{9}{8}\rho U^2\sin^2\theta-P)\sin^2\theta\, d\theta = \pi a^2(\tfrac{27}{32}\rho U^2-P)$$
$$= \pi a^2(\tfrac{27}{32}\rho U^2-p_\infty-\tfrac{1}{2}\rho U^2) = \pi a^2(\tfrac{11}{32}\rho U^2-p_\infty)$$ □

Problem 4.6 A sphere with centre O and radius a moves through an infinite inviscid liquid of constant density ρ at rest at infinity. The velocity of O at any instant t is $V(t)\mathbf{i}$ where $\mathbf{i}$ is a unit vector in a fixed direction. Show that when there are no body forces, the pressure p at a point P on the sphere whose position vector is $\mathbf{r}$ referred to the centre O is given by

$$p = p_0 - \tfrac{1}{8}\rho V^2(5a^2 - 9x^2 - 18\lambda x)/a^2$$

where $x = \mathbf{r}.\mathbf{i}$, and $\lambda = (2/9)(a^2/V^2)(dV/dt)$, p_0 being the pressure at infinity. Obtain conditions that ensure the absence of cavitation on the sphere.

Solution. Using the result proved in the previous problem and denoting the position vector of the centre O referred to a fixed origin by $\mathbf{R}$, the velocity potential of the liquid motion is

$$\varphi = \frac{\frac{1}{2}a^3\mathbf{V}.(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^3} \quad \text{where } \mathbf{V} = V\mathbf{i}$$

The pressure anywhere in the liquid is $p = \rho\{(\partial\varphi/\partial t)-\frac{1}{2}q^2+A(t)\}$ where $A(t)$ is some function of time to be determined. Writing $\mathbf{r}-\mathbf{R} = \mathbf{d}$, $|\mathbf{d}| = d$

$$\mathbf{q} = -\operatorname{grad}\varphi = \tfrac{1}{2}a^3\{d^{-3}\mathbf{V}-3(\mathbf{V}.\mathbf{d})d^{-5}\mathbf{d}\}$$
$$\partial\varphi/\partial t = \tfrac{1}{2}a^3\{(\dot{\mathbf{V}}.\mathbf{d})d^{-3}-\mathbf{V}^2d^{-3}+3(\mathbf{V}.\mathbf{d})^2d^{-5}\}$$

As $r = |\mathbf{r}| \to \infty$, $d \to \infty$ so that both $q = |\mathbf{q}|$ and $\partial\varphi/\partial t \to 0$ giving $A(t) = p_0/\rho$. Moreover, on the sphere $|\mathbf{d}| = a$ or $\mathbf{d} = a\mathbf{n}$ where $|\mathbf{n}| = 1$,

$$\mathbf{q} = \tfrac{1}{2}\{\mathbf{V}-3(\mathbf{V}.\mathbf{n})\mathbf{n}\}, \qquad q^2 = \mathbf{q}.\mathbf{q} = \tfrac{1}{4}\{\mathbf{V}^2+3(\mathbf{V}.\mathbf{n})^2\}$$

and $\partial\varphi/\partial t = \frac{1}{2}\{a\dot{\mathbf{V}}.\mathbf{n}-\mathbf{V}^2+3(\mathbf{V}.\mathbf{n})^2\}$. Therefore,

$$(p-p_0)/\rho = \tfrac{1}{2}\{a\dot{\mathbf{V}}.\mathbf{n}-\mathbf{V}^2+3(\mathbf{V}.\mathbf{n})^2\}-\tfrac{1}{8}\{\mathbf{V}^2+3(\mathbf{V}.\mathbf{n})^2\}$$
$$= \tfrac{1}{2}a\dot{\mathbf{V}}.\mathbf{n}-\tfrac{5}{8}\mathbf{V}^2+\tfrac{9}{8}(\mathbf{V}.\mathbf{n})^2$$

a result which is true for all $\mathbf{V}$ and $\dot{\mathbf{V}}$. In the given problem $\mathbf{V}$ and $\dot{\mathbf{V}}$ have the same fixed direction. Writing $\mathbf{V} = V\mathbf{i}$, $\dot{\mathbf{V}} = \dot{V}\mathbf{i}$, $\mathbf{n}.\mathbf{i} = x/a$ we have with $\dot{V} = 9\lambda V^2/2a^2$, the result

$$(p-p_0)/\rho = -\tfrac{1}{8}V^2(5a^2-9x^2-18\lambda x)/a^2$$

To ensure no cavitation the pressure p must be positive everywhere. To find the minimum value of p on the sphere we have, differentiating $dp/dx = 9V^2(x+\lambda)/4a^2$, $d^2p/dx^2 = 9V^2/4a^2$. Provided that $|\lambda| \leqslant a$, p is a minimum p_m when $x = -\lambda$ and $p_m = p_0 - \frac{1}{8}\rho V^2(5a^2+9\lambda^2)/a^2$, which is positive provided $p_0 > \rho(45V^4+4a^2\dot{V}^2)/72V^2$. When $|\lambda| \geqslant a$, p is a minimum p_m when $x = -a$, with $p_m = p_0 - \frac{1}{4}\rho V^2(9\lambda - 2a)/a$ which is positive when $p_0 > \frac{1}{2}\rho(a\dot{V} - V^2)$. □

4.5 Impulsive motion The method is illustrated in the following problem and solution.

Problem 4.7 Liquid at rest is bounded externally by a spherical shell $|\mathbf{r}| = b$ and internally by the spherical shell $|\mathbf{r}| = a < b$. If the shells are instantaneously given velocities $\mathbf{V}$ and $\mathbf{U}$ respectively, verify that the resulting irrotational motion is described *instantaneously* by a velocity potential φ of the form $\boldsymbol{\mu}\,.\,\mathbf{r} + (\boldsymbol{\lambda}\,.\,\mathbf{r})|\mathbf{r}|^{-3}$. Prove also that the impulse experienced by the internal shell from the liquid is $-\frac{4}{3}\pi\rho(a^3\boldsymbol{\mu} + \boldsymbol{\lambda})$, and write down an expression for the external impulse which must be applied to this shell of mass M to produce the motion.

Solution. The given φ *does* satisfy Laplace's equation $\nabla^2\varphi = 0$ since the first term $\boldsymbol{\mu}\,.\,\mathbf{r}$ corresponds to a uniform stream and the second to a doublet whose strength and direction is $\boldsymbol{\lambda}$. The liquid velocity $\mathbf{q}$ determined from φ is

$$\mathbf{q} = -\operatorname{grad}\varphi = -\{\boldsymbol{\mu} + \boldsymbol{\lambda} r^{-3} - 3(\boldsymbol{\lambda}\,.\,\mathbf{r})\mathbf{r}r^{-5}\}, \qquad r = |\mathbf{r}|$$

The boundary condition on the shell $|\mathbf{r}| = a$ is $\mathbf{n}\,.\,\mathbf{q} = \mathbf{n}\,.\,\mathbf{U}$ where $\mathbf{n}$ is the unit normal to the shell. Since $\mathbf{n} = \mathbf{r}/a$ the equivalent condition is $\mathbf{r}\,.\,\mathbf{q} = \mathbf{r}\,.\,\mathbf{U}$ for all $\mathbf{r}$ when $|\mathbf{r}| = a$, i.e.

$$\mathbf{r}\,.\,\mathbf{q}\big|_{r=a} = -\boldsymbol{\mu}\,.\,\mathbf{r} + 2\boldsymbol{\lambda}\,.\,\mathbf{r}a^{-3} = \mathbf{r}\,.\,\mathbf{U} \quad \text{for all } \mathbf{r}$$

This is satisfied when $-\boldsymbol{\mu} + 2\boldsymbol{\lambda}a^{-3} = \mathbf{U}$. Similarly, the boundary condition on the outer shell is $\mathbf{r}\,.\,\mathbf{q} = \mathbf{r}\,.\,\mathbf{V}$ for all $\mathbf{r}$ when $|\mathbf{r}| = r = b$. This is satisfied when $-\boldsymbol{\mu} + 2\lambda b^{-3} = \mathbf{V}$. Solving, we have,

$$\boldsymbol{\mu} = (a^3\,\mathbf{U} - b^3\mathbf{V})/(b^3 - a^3), \qquad \lambda = \tfrac{1}{2}a^3b^3(\mathbf{U} - \mathbf{V})/(b^3 - a^3)$$

This solution is of course only instantaneously true for the symmetry is instantly destroyed by the ensuing motion.

The impulse of the liquid on the internal shell is

$$\mathbf{I} = -\oint_{r=a} \rho\varphi\, d\mathbf{S} = -\rho\oint_{r=a}\{\boldsymbol{\mu}\,.\,\mathbf{r} + \boldsymbol{\lambda}\,.\,\mathbf{r}a^{-3}\}\, d\mathbf{S}, \qquad d\mathbf{S} = \mathbf{n}\; dS, \qquad \mathbf{n} = \mathbf{r}/a$$

The simplest way to evaluate the surface integral is to convert it into a

volume integral using the Gauss divergence theorem. We have

$$\oint_{r=a} (\boldsymbol{\mu}\,.\,\mathbf{r})\, d\mathbf{S} = \int_V \nabla(\boldsymbol{\mu}\,.\,\mathbf{r})\, d\tau = \int_V \boldsymbol{\mu}\, d\tau = \tfrac{4}{3}\pi a^3\boldsymbol{\mu}$$

Hence

$$\mathbf{I} = -\rho\{\tfrac{4}{3}\pi a^3\boldsymbol{\mu} + \tfrac{4}{3}\pi\boldsymbol{\lambda}\} = -\tfrac{4}{3}\pi\rho(a^3\boldsymbol{\mu} + \boldsymbol{\lambda})$$

If $\mathbf{J}$ is the external impulse we have

$$\mathbf{J} + \mathbf{I} = M\mathbf{U}$$

i.e.

$$\mathbf{J} = M\mathbf{U} + \frac{4\pi\rho a^3}{3(b^3 - a^3)}\{a^3\mathbf{U} - b^3\mathbf{V} + \tfrac{1}{2}(\mathbf{U} - \mathbf{V})b^3\}$$

$$= M\mathbf{U} + \frac{2\pi\rho a^3}{3(b^3 - a^3)}\{(2a^3 + b^3)\mathbf{U} - 3b^3\mathbf{V}\}$$

□

4.6 Miscellaneous examples In conclusion two examples will be considered in which motion of the liquid is not necessarily axisymmetric.

Problem 4.8 Given a closed geometrical surface S within a moving liquid show that the integral $\mathbf{H} = \int_S \frac{1}{2}\mathbf{q}^2\, d\mathbf{S} - \mathbf{q}(d\mathbf{S}\,.\,\mathbf{q})$ has the same value for every surface S' reconcilable with S. Interpret $\mathbf{H}$ when S is a fixed solid surface. Also, evaluate $\mathbf{H}$ for a sphere S centred on $\mathbf{r} = \mathbf{0}$, where there is a source of output $4\pi m$, given that (i) the sphere encloses no other singularity and (ii) the liquid velocity at $\mathbf{r} = \mathbf{0}$ due to all other effects excluding the source is $\mathbf{U}$.

Solution. We shall assume that S' is a nonintersecting surface reconcilable with S and V is the volume enclosed between. By Gauss's theorem and its extension we have, integrating over the *total* surface $S + S'$ enclosing V,

$$\int_{S+S'} \tfrac{1}{2}\mathbf{q}^2\, d\mathbf{S} - \mathbf{q}(d\mathbf{S}\,.\,\mathbf{q}) = \int_V [\nabla\tfrac{1}{2}\mathbf{q}^2 - \{\mathbf{q}(\nabla\,.\,\mathbf{q}) + (\mathbf{q}\,.\,\nabla)\mathbf{q}\}]\, d\tau$$

$$= \int_V [\mathbf{q}\wedge(\nabla\wedge\mathbf{q}) - \mathbf{q}(\nabla\,.\,\mathbf{q})]\, d\tau$$

Since S and S' are reconcilable surfaces, no singularities of the liquid motion exist within V. In the absence of vortices and sources $\nabla\wedge\mathbf{q} = \mathbf{0}$ and $\nabla\,.\,\mathbf{q} = 0$ respectively. Consequently, since the volume integral is zero, the *total* surface integral is zero so that the component integrals of $\mathbf{H}$ over S and S' are equal.

From Bernoulli's equation for steady motion, $p + \frac{1}{2}\rho q^2 = P$ where

P is a constant. For any closed surface S, $\int_S P\,\mathbf{dS} = 0$, hence we can write $\rho\mathbf{H} = -\int_S [p\,\mathbf{dS} + \rho\mathbf{q}(\mathbf{dS}\,.\,\mathbf{q})]$. For any solid surface, $-\int_S p\,\mathbf{dS}$ is the force $\mathbf{F}$ on it whilst $\mathbf{q}\,.\,\mathbf{dS} = 0$ by the boundary condition when S is a fixed surface. Hence in this case $\mathbf{H} = \mathbf{F}/\rho$.

On a sphere $|\mathbf{r}| = \epsilon$ we can write the liquid velocity $\mathbf{q} = m\epsilon^{-2}\mathbf{n} + \mathbf{U} + \mathbf{O}(\epsilon)$ where $\mathbf{n}$ is the unit outward normal to S the sphere, the first term being the source velocity and the third term, $O(\epsilon)$, the correction to $\mathbf{U}$ due to the evaluation on S. Again $\mathbf{q}^2 = m^2\epsilon^{-4} + U^2 + 2m\epsilon^{-2}\mathbf{n}\,.\,\mathbf{U} + O(\epsilon^{-1})$ so that

$$\mathbf{H} = \int\{\tfrac{1}{2}m^2\epsilon^{-4} + \tfrac{1}{2}U^2 + m\epsilon^{-2}(\mathbf{n}\,.\,\mathbf{U})\}\,\mathbf{dS}$$

$$-\int\{m\epsilon^{-2}\mathbf{n} + \mathbf{U} + \mathbf{O}(\epsilon)\}\{m\epsilon^{-2}\mathbf{n}\,.\,\mathbf{dS} + \mathbf{U}\,.\,\mathbf{dS} + \mathbf{O}(\epsilon)\,\mathbf{dS}\}$$

$$= \tfrac{1}{2}(m^2\epsilon^{-4} + U^2)\int\mathbf{dS} - \mathbf{U}\int\mathbf{U}\,.\,\mathbf{dS} - m^2\epsilon^{-4}\int(\mathbf{n}\,.\,\mathbf{dS})\mathbf{n}$$

$$+ m\epsilon^{-2}\{\int(\mathbf{n}\,.\,\mathbf{U})\,\mathbf{dS} - \int\mathbf{n}(\mathbf{U}\,.\,\mathbf{dS}) - \mathbf{U}\int\mathbf{n}\,.\,\mathbf{dS}\} + \mathbf{O}(\epsilon)$$

where the integrals are taken over the surface of the sphere $|\mathbf{r}| = \epsilon$. For any closed surface S, $\int\mathbf{dS} = \mathbf{0}$, $\int\mathbf{U}\,.\,\mathbf{dS} = 0$. Also, $\int(\mathbf{n}\,.\,\mathbf{dS})\mathbf{n} = \int dS\,\mathbf{n} = \int\mathbf{dS} = \mathbf{0}$. For the given S, $\int\mathbf{n}\,.\,\mathbf{dS} = \int dS = 4\pi\epsilon^2$. Again $\int(\mathbf{n}\,.\,\mathbf{U})\,\mathbf{dS} - \int\mathbf{n}(\mathbf{U}\,.\,\mathbf{dS}) = \int\mathbf{U}(\mathbf{dS}\wedge\mathbf{n}) = \mathbf{0}$ since $\mathbf{dS} = dS\mathbf{n}$. Hence,

$$\mathbf{H} = -m\epsilon^{-2}\mathbf{U}4\pi\epsilon^2 + \mathbf{O}(\epsilon) = -4\pi m\mathbf{U} + \mathbf{O}(\epsilon)$$

However, by the result proved in the first part of the problem, $\mathbf{H}$ must be independent of ϵ, i.e. the term $\mathbf{O}(\epsilon)$ must be identically zero or $\mathbf{H} = -4\pi m\mathbf{U}$. □

Problem 4.9 Using the results of the previous problem prove that the force on a sphere $|\mathbf{r}| = a$ due to an external point source of output $4\pi m$ placed at $\mathbf{r} = b\mathbf{i}$ is $4\pi\rho m^2 a^3\mathbf{i}/b(b^2 - a^2)^2$.

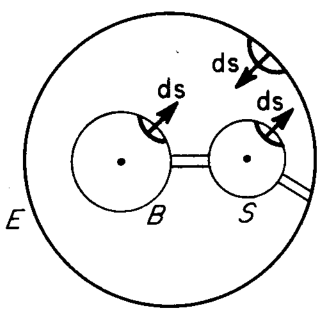


Figure 4.3

Solution. In Figure 4.3, B denotes the solid sphere $|\mathbf{r}| = a$, S the sphere $|\mathbf{r} - b\mathbf{i}| = \epsilon$ of small radius ϵ enclosing the source of output $4\pi m$ at $\mathbf{r} = b\mathbf{i}$ and E is an enclosing envelope $|\mathbf{r}| = R$ where R is large. If V is the volume of liquid internal to E and external to both B and S, there are no singularities of the liquid motion within V. By the extension to Gauss's theorem we have

$$\int_V [\mathbf{q}\wedge(\nabla\wedge\mathbf{q}) - \mathbf{q}(\nabla\,.\,\mathbf{q})]\,d\tau = \mathbf{0} = \{\int_E + \int_B + \int_S\}[\tfrac{1}{2}\mathbf{q}^2\,\mathbf{dS} - \mathbf{q}(\mathbf{dS}\,.\,\mathbf{q})]$$

or

$$\mathbf{0} = \mathbf{H}_E + \mathbf{H}_B + \mathbf{H}_S$$

Using the results of Problem 4.8, $\mathbf{H}_B = \mathbf{F}/\rho$ where $\mathbf{F}$ is the required force on B and $\mathbf{H}_S = -4\pi m\mathbf{U}$ where $\mathbf{U}$ is the velocity at $\mathbf{r} = b\mathbf{i}$ due to all effects excepting the source there, i.e. $\mathbf{U}$ is the velocity at $\mathbf{r} = b\mathbf{i}$ due to the image of the source in the sphere B. Hence

$$\mathbf{F} = \rho\mathbf{H}_B = 4\pi m\rho\mathbf{U} - \rho\mathbf{H}_E$$

We now show that $\mathbf{H}_E \to \mathbf{0}$ when $R \to \infty$. For large R the velocity $\mathbf{q}$ at a point $\mathbf{r} = \mathbf{R}$ on E due the source at $\mathbf{r} = b\mathbf{i}$ is of the form $\mathbf{q} = m\mathbf{R}/R^3 + \mathbf{O}(R^{-3})$. Since the image in the solid sphere B can have no *resultant* source inside (i.e. the sum of the source and sink there must be zero or else there will be a flow across B), the $\mathbf{q}$ due to this image system will behave, at most, like $\mathbf{O}(R^{-3})$ on E. On the envelope we may, therefore, write $\mathbf{q} = m\mathbf{R}/R^3 + \mathbf{O}(R^{-3})$ and since $\mathbf{dS} = \mathbf{O}(R^2)$, $\mathbf{H}_E = \mathbf{O}(R^{-2})$ at most. (In fact $\mathbf{H}_E = \mathbf{O}(R^{-3})$). It appears that $\mathbf{H}_E \to \mathbf{0}$ as $R \to \infty$ so that $\mathbf{F} = 4\pi m\rho\mathbf{U}$. To evaluate $\mathbf{U}$ we use Problem 4.1 from which the image of the source in $r = a$ has a Stokes stream function given by

$$\psi_1 = \psi - \psi_0 = -\frac{mr}{a} - \frac{mr(a^2\mu - br)}{a(a^4 - 2a^2br\mu + b^2r^2)^{\frac{1}{2}}}$$

where $\mu = \cos\theta$. Now $\mathbf{U} = \mathbf{i}(q_r)_{r=b,\,\mu=1}$ where

$$q_r = \frac{1}{r^2}\frac{\partial\psi}{\partial\mu} = -\frac{m}{ar}\frac{d}{d\mu}\left\{\frac{a^2\mu - br}{(b^2r^2 - 2a^2br\mu + a^4)^{\frac{1}{2}}}\right\}$$

$$= -\frac{m}{ar}\left\{\frac{a^2(b^2r^2 - 2a^2br\mu + a^4) + (a^2\mu - br)a^2br}{(b^2r^2 - 2a^2br\mu + b^2r^2)^{\frac{3}{2}}}\right\}$$

When $\mu = 1$ and $r = b$, remembering that we choose the positive value to the square root, since $b > a$,

$$q_r = -\frac{m}{ab}\left\{\frac{a^2(b^2 - a^2)^2 - (b^2 - a^2)a^2b^2}{(b^2 - a^2)^3}\right\} = \frac{ma^3}{b(b^2 - a^2)^2}$$

so that finally we have the result for the force as

$$\mathbf{F} = 4m\pi\rho\mathbf{U} = \frac{4m^2a^3\pi\rho\mathbf{i}}{b(b^2-a^2)^2} \qquad \square$$

EXERCISES

1. Show that $\psi = \varpi^2(Ar^2+Bx)r^{-5}$ is a possible Stokes stream function. Given that $0 < \epsilon \ll 1$ verify that $\psi = \frac{1}{2}U\varpi^2(r^5-r^2+3\epsilon x)r^{-5}$ represents, to a first approximation in ϵ, the streaming motion past an ellipsoid of revolution.

2. A source and sink of equal strengths m are placed on the axis OX a distance $2a$ apart. A uniform stream of magnitude $8m/(9a^2)$ flows parallel to the axis from source to sink. Show that the flow corresponds to a streaming motion past a solid of revolution which is symmetrical about an equatorial section and is of length $4a$. Deduce that the radius of the equatorial section is $a\lambda^{\frac{1}{2}}$ where λ is a root of the equation $4\lambda^2(1+\lambda) = 81$ and prove that the liquid speed at the equator is $1+2\lambda^3/81$ times the magnitude of the uniform stream.

3. Prove that a solid sphere of radius a moving with velocity $U\mathbf{i}\sin\omega t$ through a liquid otherwise at rest experiences a resistance $\frac{2}{3}\pi\rho\omega a^3U\cos\omega t$.

4. A doublet $\mu\mathbf{i}$ is placed at $S(\mathbf{r} = b\mathbf{i})$ in the presence of a fixed sphere $|\mathbf{r}| = a$. Find the stream function and show that the speed at a point P on the surface of the sphere is

$$3\mu r^{-5}(b^2-a^2)\sin\theta$$

where $r = SP$ and $\theta = \angle SOP$.

Deduce that the resultant force on the sphere is

$$24\mu^2\rho\pi a^3 b(b^2-a^2)^{-4}$$

towards S.

Table 1. List of the main symbols used

$\mathbf{i}, \mathbf{j}, \mathbf{k}$,	constant unit vectors parallel to fixed Cartesian axes, OX, OY, OZ respectively
$\mathbf{r}$	$= x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ position vector
$\mathbf{q}$	$= u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$ velocity vector
$\boldsymbol{\zeta}$	$= \text{curl}\,\mathbf{q}$ vorticity vector
p	pressure
ρ	fluid density
γ	adiabatic constant
a	$= \sqrt{(dp/d\rho)}$ acoustic speed
$\mathscr{R}$	fluid space
$\mathscr{R}_s$	source-free fluid region
$\mathscr{R}_s^*$	source field of $\mathscr{R}$
$\mathscr{R}_v$	vortex-free fluid region
$\mathscr{R}_v^*$	vortex field of $\mathscr{R}$
$\mathscr{R}_{sv}$	source-free and vortex-free region, i.e. intersection of $\mathscr{R}_s$ with $\mathscr{R}_v$
$\in$	belongs to
$\subset$	contains
φ	velocity potential where $\mathbf{q} = -\text{grad}\,\varphi$
ψ	stream function
D/Dt	mobile operator

Table 2. Some useful results in vector calculus

Gauss's theorem:

$$\int_S \mathbf{F}.\mathbf{dS} = \int_V \nabla.\mathbf{F}\,d\tau \qquad \text{or} \qquad \int_S \varphi\,\mathbf{dS} = \int_V \nabla\varphi\,d\tau \qquad \square$$

(Surface S encloses volume V, $d\tau$ is an element of V, $\mathbf{dS}$ is an elemental vector area outward from V)

Extension to Gauss's theorem:

$$\int_S \mathbf{G}(\mathbf{F}.\mathbf{dS}) = \int_V [\mathbf{G}(\nabla.\mathbf{F})+(\mathbf{F}.\nabla)\mathbf{G}]\,d\tau$$

Green's theorem:

$$\int_V \nabla\varphi.\nabla\psi\,d\tau = \int_S \psi\nabla\varphi.\mathbf{dS} - \int_V \psi\nabla^2\varphi\,d\tau$$

Stokes' theorem:

$$\int_S \text{curl}\,\mathbf{F}.\mathbf{dS} = \int_{\mathscr{C}} \mathbf{F}.\mathbf{dr} \qquad \square$$

(S is a surface spanning $\mathscr{C}$)

$\text{grad}\,\varphi = \varphi_x\mathbf{i}+\varphi_y\mathbf{j}+\varphi_z\mathbf{k}$ $\quad (\varphi_x \equiv \partial\varphi/\partial x$ etc.), $\quad \text{grad}\,\varphi.\mathbf{dr} = d\varphi$

$\text{grad}\,\varphi\psi = \varphi\,\text{grad}\,\psi+\psi\,\text{grad}\,\varphi$, $\quad \text{grad}\,f(r) = f'(r)\mathbf{r}/r$

$\text{div}\,\mathbf{q} = u_x+v_y+w_z$, $\quad (u = \mathbf{i}.\mathbf{q},\ u_x = \partial u/\partial x$ etc.), $\quad \text{div}\,\mathbf{r} = 3$

$\text{div}\,\varphi\mathbf{F} = \varphi\,\text{div}\,\mathbf{F}+\mathbf{F}.\text{grad}\,\varphi$, $\quad (\mathbf{q}.\nabla)\mathbf{q} = \nabla(\frac{1}{2}\mathbf{q}^2)-\mathbf{q}\wedge\text{curl}\,\mathbf{q}$

$\nabla\wedge(\mathbf{F}\wedge\mathbf{G}) = (\mathbf{G}.\nabla)\mathbf{F}-(\mathbf{F}.\nabla)\mathbf{G}-\mathbf{G}(\nabla.\mathbf{F})+\mathbf{F}(\nabla.\mathbf{G})$

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